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Pseudo-Anosov Maps and Pairs of Filling Simple Closed Geodesics on Riemann Surfaces

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Abstract. Let S be a Riemann surface of finite area with at least one puncture x. Let $a \subset S$ be a simple closed geodesic. In this paper, we show that for any pseudo-Anosov map f of S that is isotopic to the identity on $S \cup \{x\}$, the pair $(a, f^m(a))$ of geodesics fills S for $m \geq 3$. We also study the cases of $0 < m \leq 2$ and show that if $(a, f^2(a))$ does not fill S, then there is only one geodesic b such that b is disjoint from both a and $f^2(a)$. In fact, $b = f(a)$ and $\{a, f(a)\}$ forms the boundary of an x-punctured cylinder on S. As a consequence, we show that if a and $f(a)$ are not disjoint, then $(a, f^m(a))$ fills S for any $m \geq 2$.

1. Introduction

In an important paper [9], Thurston proved that there exist pseudo-Anosov maps on a hyperbolic surface S that are obtained from products of Dehn twists along two simple closed geodesics. Let a and b be simple closed geodesics on S. The pair (a, b) is said to fill S if every component of $S\backslash\{a, b\}$ is a polygon or an once punctured polygon, which is equivalent to that the union $a \cup b$ intersects every non-trivial closed curve on S, where a curve is non-trivial if it neither bounds a disk nor an once punctured disk. Let t_a and t_b denote the positive Dehn twists along a and b, respectively. By [9] we know that $t_a \circ t_b^{-1}$ represents a pseudo-Anosov mapping class whenever (a, b) fills S. See [9] and [4] for a detailed account of pseudo-Anosov maps.

Let S be an analytically finite Riemann surface with type (p, n), where p is the genus and n is the number of punctures of S. Assume that $3p - 3 + n > 0$. Let $f : S \to S$ be a pseudo-Anosov map and let $a \subset S$ be a simple closed geodesic. Denote by $f^m(a)$ the geodesic homotopic to the image curve of a under the map f^m. It is well-known [4] that

$$\mathscr{S} = \{f^m(a) : m \in \mathbf{Z}\}$$

fills S (in the sense that $S\backslash\mathscr{S}$ consists of polygons or once punctured polygons). Later, Fathi [3] showed that a finite subset of $\mathscr{S}$ fills S. It is natural to ask if any pair of elements of $\mathscr{S}$ also fills S. Unfortunately, the answer to this question is "no". In fact, Wang–Wu–Zhou [10]

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showed that for any two non-separating non-isotopic simple closed geodesics a, b on S, there is a pseudo-Anosov map f such that $f(a) = b$.

By contrast, in [7], Masur–Minsky showed that there is an integer $K = K(f)$, independent of the choice of a, such that $(a, f^m(a))$ fills S for all integers $m \geq K$. To determine the smallest possible integer m with this property, Farb–Leininger–Margalit [5] considered the curve complex $\mathscr{C} = \mathscr{C}(S)$ on S that is equipped with the path metric $d_{\mathscr{C}}$, then they introduced the asymptotic translation length μ on $\mathscr{C}$ defined by $\mu = \lim_{m\to\infty} \inf d_{\mathscr{C}}(a, f^m(a))/m$, which is strictly larger than zero and is independent of the choice of a, and showed that if m is the smallest integer so that $m\mu > 2$, then $(a, f^k(a))$ fills S for all $k \geq m$.

In this paper, we study the similar problem on a surface S that contains at least one puncture x. Write $\tilde{S} = S \cup \{x\}$. Let $\mathscr{F}$ be the set of pseudo-Anosov maps of S that are isotopic to the identity on $\tilde{S}$. Kra [6] proved that $\mathscr{F}$ is non-empty and contains infinitely many elements. Let $f \in \mathscr{F}$ and let $F : [0, 1] \times \tilde{S} \to \tilde{S}$ denote the isotopy between f and the identity as x is filled in. Then $\tilde{c} = F(t, x)$, $t \in [0, 1]$, is an oriented filling closed curve passing through x in the sense that $\tilde{c}$ intersects every simple closed geodesic. Note that the curve $\tilde{c}$ is defined on $\tilde{S}$, not on S.

For a simple closed geodesic $a \subset S$, we denote by $\tilde{a}$ the simple closed geodesic (maybe trivial) on $\tilde{S}$ homotopic to a when a is viewed as a curve on $\tilde{S}$. Let $K = K(f)$ be the smallest integer such that $(a, f^m(a))$ fills S whenever $m \geq K$.

THEOREM 1.1. *For an element $f \in \mathscr{F}$ and a simple closed geodesic $a \subset S$, we have $K \leq 3$ and the inequality is sharp. If $\tilde{a}$ is non-trivial and intersects $\tilde{c}$ more than once, then $K \leq 2$. If $\tilde{a}$ is trivial, then $K = 1$.*

Due to Kra [6], it is well understood that every oriented filling closed curve $\tilde{c}$ determines a conjugacy class of pseudo-Anosov maps on S and one in the class, denoted by $f_{\tilde{c}}$, is constructed by pushing the point x along $\tilde{c}$ until returning to its original position. We may parameterize $\tilde{c} = \tilde{c}(t)$ for $0 \leq t \leq 1$, which is consistent with the orientation of $\tilde{c}$, so that $\tilde{c}(0) = x$.

Assume that $\tilde{c}$ is so chosen that there is an x-punctured cylinder $P \subset S$ such that $\tilde{c}$ traverses P only once and that a is the boundary component of P lying on the left side of P with respect to the orientation of $\tilde{c}$. Let a_0 denote the other boundary component of P. In this case, the map $f_{\tilde{c}}$ satisfies the condition that $f_{\tilde{c}}(a) = a_0$. See [16] for an exposition. Since $f_{\tilde{c}}$ is a homeomorphism of S, $f_{\tilde{c}}(a)$ and $f_{\tilde{c}}^2(a)$ are also disjoint. We see that $f_{\tilde{c}}(a)$ is disjoint from both a and $f_{\tilde{c}}^2(a)$. It follows that $(a, f_{\tilde{c}}^2(a))$ does not fill S, which shows that $K = 3$ for some $f \in \mathscr{F}$.

Our next result states that the above example is the only incidence for the pair $(a, f^2(a))$ not to fill S. More precisely, we will also prove the following result.

THEOREM 1.2. *Let $f \in \mathscr{F}$ and let a be a simple closed geodesic. Assume that $(a, f^2(a))$ does not fill S. Then there is a unique simple closed geodesic b that is disjoint from* *both a and $f^2(a)$. Furthermore, $b = f(a)$ and $\{a, b\}$ forms the boundary of an x-punctured cylinder on S.*

An immediate consequence of Theorem 1.2 is the following corollary.

COROLLARY 1.1. *Let $f \in \mathscr{F}$ and let a be a simple closed geodesic. Assume that a and $f(a)$ are not disjoint. Then $(a, f^m(a))$ for any $m \geq 2$ fills S.*

This paper is organized as follows. In Section 2, we discuss properties of Dehn twists and their lifts to the universal covering space via the Bers isomorphism [1]. In Section 3, we study the actions of pseudo-Anosov maps in $\mathscr{F}$ on the set of simple closed geodesics on S. As an outcome, we obtain a family of pseudo-Anosov maps of S determined by simple closed geodesics. In Section 4, we prove Theorem 1.1. In Section 5, we prove Theorem 1.2 and Corollary 1.1. Finally, we include some remarks and questions in Section 6.

2. Preliminaries

Let $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ be the unit disk equipped with the Poincaré metric $\rho(z)dz = 2dz/\left(1 - |z|^2\right)$. Let $\varrho : \mathbf{D} \to \tilde{S}$ be the universal covering map with the covering group G. Let $Q(G)$ be the group of quasiconformal automorphisms w of $\mathbf{D}$ such that $wGw^{-1} = G$. Two maps $w, w' \in Q(G)$ are said equivalent if they share the common boundary values on $\partial\mathbf{D} = \mathbf{S}^1$. Denote by $[w]$ the equivalence class of an element $w \in Q(G)$ and by $Q(G)/\sim$ the quotient group under this equivalence relation. An important theorem of Bers [1] states that there is an isomorphism φ^* of the quotient group $Q(G)/\sim$ onto the x-pointed mapping class group Mod_S^x of S which consists of all mapping classes fixing x. In what follows, the isomorphism $\varphi^* : Q(G)/\sim \to \mathrm{Mod}_S^x$ is called the Bers isomorphism. Under the Bers isomorphism, the image $\varphi^*(G)$ is a subgroup of Mod_S^x consisting of elements that project to a trivial mapping class of $\tilde{S} = \mathbf{D}/G$. Thus $\varphi^*(G) = \mathscr{F}$. By abuse of language, we use $[w]^*$ to denote the mapping class $\varphi^*([w])$ as well as a representative of $\varphi^*([w])$ for an element $w \in Q(G)$. In particular, for an element $h \in G$, we use h^* to denote the mapping class $\varphi^*(h)$ as well as a representative of $\varphi^*(h)$.

Let $\tilde{c} \subset \tilde{S}$ be a closed geodesic. Note that any geodesic $\hat{c} \subset \mathbf{D}$ with $\varrho(\hat{c}) = \tilde{c}$ is an invariant geodesic under a hyperbolic element $g_{\hat{c}} \in G$. Following Kra [6], $g_{\hat{c}}$ is called an essential hyperbolic element if $g_{\hat{c}}$ corresponds to an element $g_{\hat{c}}^*$ of $\mathscr{F}$. In this case, $\tilde{c} = \varrho(\hat{c})$ is a filling closed geodesic on $\tilde{S}$.

Consider now some special elements in $Q(G)/\sim$. Let $a \subset S$ be a simple closed geodesic that is non-trivial on $\tilde{S}$ as x is filled in. Let $\tilde{a}$ denote the (non-trivial) simple closed geodesic homotopic to a on $\tilde{S}$. We can construct a Dehn twist along $\tilde{a}$ as follows. We first cut $\tilde{S}$ along $\tilde{a}$, rotate one of the copies of $\tilde{a}$ by 360 degrees in the counterclockwise direction and then glue the two copies back together.

Let $\hat{a} \subset \mathbf{D}$ be a geodesic such that $\varrho(\hat{a}) = \tilde{a}$. Denote by $\{U, U'\}$ the components of $\mathbf{D}\backslash\{\hat{a}\}$. It is readily seen that $\hat{a}$, U and U' are invariant under the action of a primitive simple hyperbolic element of G. The Dehn twist $t_{\tilde{a}}$ can be lifted to a map $\tau_a : \mathbf{D} \to \mathbf{D}$ with respect to U, which satisfies the conditions:

$$\text{(i)}\ \ \tau_a G \tau_a^{-1} = G \quad \text{and} \quad \text{(ii)}\ \ \varrho \circ \tau_a = t_{\tilde{a}} \circ \varrho\,.$$

In addition to (i) and (ii) above, τ_a defines a collection $\mathscr{U}_a$ of half planes of $\mathbf{D}$ in a partial order defined by inclusion. An element of the first order is usually called a maximal element. There are infinitely many maximal elements of $\mathscr{U}_a$, and all maximal elements U_i (U is one of them) of $\mathscr{U}_a$ are mutually disjoint, and the complement

$$\Omega_a = \mathbf{D} \backslash \bigcup_i U_i \subset U' \tag{2.1}$$

is not empty and is a convex region bounded by a collection of disjoint geodesics $\hat{a}$ with $\varrho(\hat{a}) = \tilde{a}$. Clearly, U' contains infinitely many maximal elements of $\mathscr{U}_a$. The map τ_a keeps each maximal element invariant, and restricts to the identity on Ω_a.

We remark that τ_a so obtained depends on the choice of a geodesic $\hat{a}$ with $\varrho(\hat{a}) = \tilde{a}$, but does not depend on the choice of a boundary component of Ω_a. Note also that all lifts of $t_{\tilde{a}}$ are in the forms $h \circ \tau_a$ for $h \in G$. Furthermore, τ_a determines an element $[\tau_a]$ of $Q(G)/\sim$. By Lemma 3.2 of [13], we can choose $\hat{a}$ (and hence U) properly so that $[\tau_a]^* \in \mathrm{Mod}_S^x$ is represented by the Dehn twist t_a along the geodesic a. See [13, 14] for more details.

Throughout the paper we call the triple $(\tau_a, \Omega_a, \mathscr{U}_a)$ the configuration corresponding to the geodesic a. The following lemma, deduced from the definition of τ_a, plays a crucial role in this paper.

LEMMA 2.1. *Let $U \in \mathscr{U}_a$ be a maximal element. Let $h \in G$ be a hyperbolic element whose axis c_h crosses the boundary ∂U of U. Assume that U covers the repelling fixed point of h. Then $h(\mathbf{D}\backslash\overline{U})$ is contained in another maximal element $U_0 \in \mathscr{U}_a$. Moreover, $h(\mathbf{D}\backslash\overline{U}) = U_0$ if and only if the geodesic $\varrho(c_h)$ intersects $\tilde{a}$ exactly once.*

PROOF. Under the universal covering map $\varrho : \mathbf{D} \to \tilde{S}$, ∂U projects to $\tilde{a}$. Parametrize $c_h = c_h(t)$, $-\infty < t < +\infty$, so that $c_h(-\infty)$ is the repelling fixed point of h while $c_h(+\infty)$ is the attracting fixed point of h. Let $c_h(t_0) = c_h \cap \partial U$. Let $\tilde{\gamma}$ denote the projection of c_h under the map ϱ. Then $\tilde{\gamma}$ is a non-trivial closed geodesic.

Now $\varrho(c_h(t_0))$ is one of the intersection points between $\tilde{a}$ and $\tilde{\gamma}$. Since both $\tilde{a}$ and $\tilde{\gamma}$ are closed, there exists a number $T > 0$ such that $\varrho(c_h(t_0 + T))$ is also an intersection point of $\tilde{a}$ and $\tilde{\gamma}$ which may or may not be the intersection point mentioned earlier (if the parametrization $c_h = c_h(t)$ is selected to be the "arc length" parametrization, then the number T satisfies the condition that the length of the geodesic segment $c_h(t_0 + T)\backslash c_h(t_0)$ is the translation length of h which is the hyperbolic length of $\varrho(c_h)$. See Section 3 for more information).

It follows that there is an element $U_1 \in \mathscr{U}_a$, disjoint from U, such that $c_h(t_0 + T) = \partial U_1 \cap c_h$. But U_1 must be included in a maximal element U_0 of $\mathscr{U}_a$. Clearly, $U_0 \subset U_1$ and U_0 is disjoint from U, and $U_0 = U_1$ if and only if $\varrho(c_h)$ intersects $\tilde{a}$ exactly once. □

3. Simple closed geodesics under the actions of elements of $\mathscr{F}$

In this section, we study the actions of essential hyperbolic elements g of G on various configurations. The results lead to new constructions of pseudo-Anosov maps on S.

Let $f \in \mathscr{F}$. Write $f = g^*$ for some $g \in G$. Then g is an essential hyperbolic element whose axis $\hat{c} = \hat{c}_g$ projects to a filling closed geodesic $\tilde{c}$ on $\tilde{S}$ under the universal covering map $\varrho : \mathbf{D} \to \tilde{S}$.

Let $a \subset S$ be a simple closed geodesic. If a projects to a trivial curve on $\tilde{S}$, then by Theorem 1.1 of [15], $(a, f^m(a))$ fills S for all $m \geq 1$. Thus we may assume throughout the paper that a projects to a non-trivial curve that is homotopic to $\tilde{a}$ on $\tilde{S}$.

Let $(\tau_a, \Omega_a, \mathscr{U}_a)$ be the configuration corresponding to a (Lemma 3.2 of [13]). Assume first that $\hat{c} \cap \Omega_a \neq \emptyset$. Since $\tilde{c}$ intersects $\tilde{a}$, $\hat{c}$ intersects a geodesic $\hat{a}$ for which $\varrho(\hat{a}) = \tilde{a}$. Let $U \in \mathscr{U}_a$ be a maximal element such that $\partial U = \hat{a}$. Assume also that U covers the attracting fixed point of g (by reversing the orientation of $\hat{c}$ if necessary). By Lemma 2.1, there is another maximal element $U_0 \in \mathscr{U}_a$ that covers the repelling fixed point of g. Let $\hat{b} = \partial U_0$. See Figure 1 (a). In the figure both $\hat{a}$ and $\hat{b}$ project to $\tilde{a}$ under $\varrho : \mathbf{D} \to \tilde{S}$. Observe also that $g(\hat{b})$ does not intersect $\hat{a}$ transversely, but $g(\hat{b})$ could be equal to $\hat{a}$.

By the construction of τ_a, $g(\hat{b}) \subseteq \overline{U}$. Thus $g(\mathbf{D} \backslash \overline{U}_0) \subseteq U$. We conclude that for $k \geq 2$,

$$g^k(U_0) \cap U \neq \emptyset, \; g^k(U_0) \cup U = \mathbf{D}, \; \text{and } \partial\big(g^k(U_0)\big) \cap \{\hat{a}\} = \emptyset. \tag{3.1}$$

The case of $k = 1$ is interesting. If this occurs, then either (i) $g(U_0) \cap U \neq \emptyset$ and thus $g(U_0) \cup U = \mathbf{D}$; or (ii) $\{g(U_0), U\}$ tessellates the hyperbolic disk $\mathbf{D}$. The later occurs if and only if $\tilde{c}$ and $\tilde{a}$ intersect exactly once. See Section 5 for more details.

For simplicity, we write $U_k = g^k(U_0)$, $U_k' = g^k(\mathbf{D} \backslash \overline{U}_0)$, and $\hat{b}_k = g^k(\hat{b})$. Figure 1 (b) depicts the situation that after the action of g^k for $k \geq 2$ is performed, the boundary geodesics

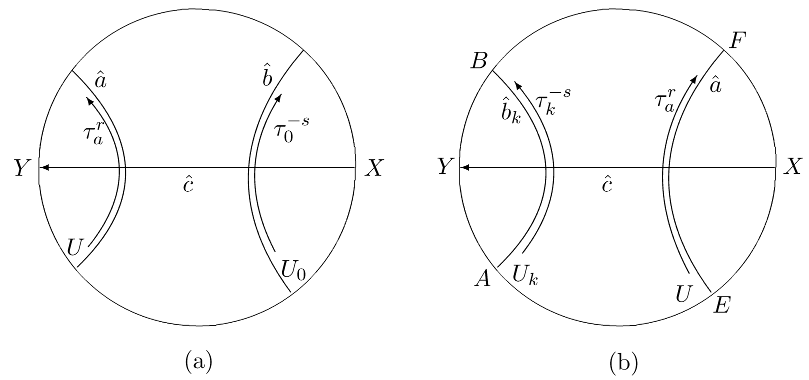

FIGURE 1

$\hat{a}$ and $\hat{b}_k$ are disjoint and the union $U \cup U_k$ covers $\mathbf{D}$.

Write $\mathscr{D}_k = U \cap U_k$ and $\mathbf{D}\backslash\mathscr{D}_k = \{\mathscr{R}_k, \mathscr{L}_k\}$, where $\mathscr{L}_k$ and $\mathscr{R}_k$ be the half planes covering the attracting and repelling fixed point of g, respectively. Clearly, the axis $\hat{c}$ of g traverses the region $\mathscr{D}_k$. In what follows the hyperbolic length of the geodesic segment

$$\varepsilon_k = \hat{c} \cap \mathscr{D}_k \tag{3.2}$$

is called the width of $\mathscr{D}_k$ with respect to $\hat{c}$. Let T_g denotes the translation length of g which is defined by

$$T_g = \inf\{\rho\,(z, g(z)) : z \in \mathbf{D}\}\ .$$

It is evident that $0 \leq \varepsilon_1 < T_g$; and $\varepsilon_1 = 0$ if and only if $g(U_0) = \mathbf{D}\backslash\overline{U}$.

Let $\gamma \subset S$ be another simple closed geodesic so that $\tilde{\gamma} \subset \tilde{S}$ is non-trivial. The positive Dehn twist $t_{\tilde{\gamma}}$ is well defined. By Lemma 3.2 of [13], we can choose a lift $\tau_\gamma : \mathbf{D} \to \mathbf{D}$ of $t_{\tilde{\gamma}}$ so that $[\tau_\gamma]^* = t_\gamma$. As usual, let $(\tau_\gamma, \Omega_\gamma, \mathscr{U}_\gamma)$ be the configuration corresponding to γ. Let $W \in \mathscr{U}_\gamma$ be a maximal element. Then $\varrho(\partial W) = \tilde{\gamma}$. Assume that ∂W intersects $\hat{c}$ and that $\tilde{\gamma}$ is disjoint from or equal to $\tilde{a}$. Denote by

$$W' = \mathbf{D}\backslash\overline{W}\ .$$

LEMMA 3.1. *With the above notation and assumption, assume also that* $\varepsilon_k > T_g$ *for some integer* $k > 1$. *Then there is a maximal element* $W_0 \in \mathscr{U}_\gamma$ *such that* $W_0' \subset U$ *or* $W_0' \subset U_k$.

PROOF. If $\partial W \subset \mathscr{D}_k$, then obviously, either $W' \subset U$ or $W' \subset U_k$. We are done with the choice $W_0 = W$. If $\partial W \subset \mathscr{R}_k$ and $W' \subset \mathscr{R}_k$, then $W' \subset U_k$. If $\partial W \subset \mathscr{R}_k$ and $W \subset \mathscr{R}_k$, then by Lemma 2.1, $g(\mathbf{D}\backslash\overline{W}')$ is contained in a maximal element W_0 of $\mathscr{U}_\gamma$. Since $\varepsilon_k > T_g$, we have $W_0' \subset U_k$.

It remains to consider the case in which $\partial W \subset \mathscr{L}_k$. If $W' \subset \mathscr{L}_k$, then $W' \subset U$. If $W \subset \mathscr{L}_k$, then by Lemma 2.1 again, $g^{-1}(\mathbf{D}\backslash\overline{W}')$ is contained in a maximal element W_0 of $\mathscr{U}_\gamma$. Since $\varepsilon_k > T_g$, we have $W_0' \subset U$. The lemma is proved. □

REMARK 1. If we know that $\partial W \neq \partial U$ or ∂U_0, then the conclusion of the lemma remains valid even if $\varepsilon_k = T_g$. This remark is useful in the proof of Theorem 1.2.

Define

$$\tau_k = g^k \tau_a g^{-k}\ . \tag{3.3}$$

Again, let $(\tau_k, \Omega_k, \mathscr{U}_k)$ be the configuration determined by τ_k; that is, $\mathscr{U}_k = g^k(\mathscr{U}_a)$ and $\Omega_k = g^k(\Omega_a)$. It is clear that for $k \geq 1$, $\Omega_k \cap \Omega_a = \emptyset$. As mentioned earlier, the equivalence classes $[\tau_k]$ and $[\tau_a]$ are elements of $Q(G)/\sim$, and if a geodesic $\hat{a}$ is chosen properly, $[\tau_a]^*$ is represented by t_a. By Lemma 3.2 of [13] again, $[\tau_k]^*$ is represented by the Dehn twist t_{b_k} for some simple closed geodesic $b_k \subset S$. For simplicity, we write $t_k = t_{b_k}$. It follows that

$$[\tau_a^r \tau_k^{-s}]^* = [\tau_a^r]^* \circ [\tau_k^{-s}]^* = t_a^r \circ t_k^{-s} . \tag{3.4}$$

As usual, we let $\tilde{b}_k$ denote the geodesic on $\tilde{S}$ homotopic to b_k on $\tilde{S}$ (if b_k is also viewed as a curve on $\tilde{S}$). Then it is easy to show that $\tilde{b}_k = \tilde{a}$.

LEMMA 3.2. *Assume that $\varepsilon_k > T_g$ for some integer $k > 1$ and that there is a maximal element $W \in \mathscr{U}_\gamma$ such that ∂W intersects $\hat{c}$. Then for any integer N and any positive integers m, n with $r \neq s$, the mapping class $[(\tau_a^r \tau_k^{-s})^N]^*$ has the property that it fixes the puncture x and for any simple closed geodesic γ on S, $[(\tau_a^r \tau_k^{-s})^N]^*(\gamma)$ is not homotopic to γ.*

PROOF. We prove the lemma in the case where $N = 1$. The argument works equally well for the general cases (by iterating the map ζ or ζ^{-1} N times). Suppose that

$$[\tau_a^r \tau_k^{-s}]^*(\gamma) = t_a^r \circ t_k^{-s}(\gamma) \cong \gamma . \tag{3.5}$$

By Lemma 4.1 of [14], $\tau_a^r \tau_k^{-s}$ sends every maximal element of $\mathscr{U}_\gamma$ to a maximal element of $\mathscr{U}_\gamma$. This tells us that for any maximal element $W_1 \in \mathscr{U}_\gamma$,

$$\tau_a^r \tau_k^{-s}(W_1) = W_2, \quad \text{where } W_2 \text{ is a maximal element of } \mathscr{U}_\gamma . \tag{3.6}$$

From (3.5), $t_{\tilde{a}}^r \circ t_{\tilde{a}}^{-s}(\tilde{\gamma}) = \tilde{\gamma}$. We have $t_{\tilde{a}}^{r-s}(\tilde{\gamma}) = \tilde{\gamma}$ for $r \neq s$. So $\tilde{\gamma}$ must be disjoint from $\tilde{a}$. It follows that the sets $\{\varrho^{-1}(\tilde{\gamma})\}$ and $\{\varrho^{-1}(\tilde{a})\}$ are disjoint. We conclude that all boundary geodesics of maximal elements of $\mathscr{U}_\gamma$ must be disjoint from $\hat{a}$ and $\hat{b}_k$.

By hypothesis, there is a maximal element $W \in \mathscr{U}_\gamma$ such that ∂W intersects $\hat{c}$. Thus Lemma 3.1 states that there is a maximal element $W_0 \in \mathscr{U}_\gamma$ such that $W_0' \subset U$ or $W_0' \subset U_k$.

We refer to Figure 1 (b). If $W_0' \subset U_k$, then since $\tilde{\gamma}$ is simple, the region $\tau_k^{-1}(W_0') \subset U_k$ is disjoint from W_0'. So $\tau_k^{-s}(W_0')$ is near to the point B for large s (s does not depend on W_0' since W_0' is away from the point A). Thus the region $\tau_a^r \tau_k^{-s}(W_0') \subset U$ is near to the point F. Hence one of the following conditions holds (with $\zeta = \tau_a^r \tau_k^{-s}$):

(i) $\zeta(W_0') \subset W_0'$ and $W_0' \neq \zeta(W_0')$, or
(ii) The two half planes $\zeta(W_0')$ and W_0' are disjoint.

In both cases, we have $W_0 \cap \tau_a^r \tau_k^{-s}(W_0) \neq \emptyset$ and $W_0 \neq \tau_a^r \tau_k^{-s}(W_0)$. So $\tau_a^r \tau_k^{-s}(W_0)$ is not a maximal element of $\mathscr{U}_\gamma$. This contradicts (3.6).

If $W_0' \subset U$, we consider the inverse map $\tau_k^s \tau_a^{-r}$ of $\tau_a^r \tau_k^{-s}$. Observe that $\tau_a^{-r}(W_0') \subset U$ is near to the point E, and thus $\tau_k^s \tau_a^{-r}(W_0') \subset U_k$ is near to the point A. This implies that (i) and (ii) above remain valid with $\zeta = \tau_k^s \tau_a^{-r}$. It follows that $\tau_k^s \tau_a^{-r}(W_0)$ is not a maximal element of $\mathscr{U}_\gamma$. This also contradicts (3.6). □

LEMMA 3.3. *Assume that $\varepsilon_k > T_g$ for some positive integer k. Then for all sufficiently large positive integers r, s with $r \neq s$, $[\tau_a^r \tau_k^{-s}]^*$ are pseudo-Anosov mapping classes.*

REMARK 2. The condition $k \geq 3$ implies that $\varepsilon_k > T_g$. In fact, we have $0 \leq \varepsilon_1 < T_g$ and $T_g \leq \varepsilon_2 < 2T_g$. So $\varepsilon_k > T_g$ for all $k \geq 3$.

PROOF. Suppose that $[\tau_a^r \tau_k^{-s}]^*$ is not pseudo-Anosov for some k with $\varepsilon_k > T_g$ and some large positive integers r and s with $r \neq s$. By the Nielsen–Thurston classification of mapping classes [9], $[\tau_a^r \tau_k^{-s}]^*$ is either periodic or reducible. If $[\tau_a^r \tau_k^{-s}]^*$ is periodic, then from (3.4), $t_a^r \circ t_k^{-s}$ is periodic, which means that $t_{\tilde{a}}^r \circ t_{\tilde{a}}^{-s} \in \mathrm{Mod}_{\tilde{S}}$ is also periodic. Note that $t_{\tilde{a}}^r \circ t_{\tilde{a}}^{-s} = t_{\tilde{a}}^{r-s}$. We deduce that $t_{\tilde{a}}^{r-s}$ is periodic. But $t_{\tilde{a}}^{r-s}$ is a power of a non-trivial Dehn twist. This is a contradiction.

We conclude that $[\tau_a^r \tau_k^{-s}]^*$ cannot be periodic. Hence $[\tau_a^r \tau_k^{-s}]^*$ must be a reducible mapping class. Let $\mathscr{C} = \{\gamma_1, \dots, \gamma_q\}$, $q \geq 1$, be the corresponding curve simplex reduced by a representative of $[\tau_a^r \tau_k^{-s}]^*$ (the curve simplex depends on r, s and k). If $\mathscr{C}$ contains a curve γ that bounds a twice punctured disk enclosing the puncture x, then such a γ is unique in $\mathscr{C}$ (since any two twice punctured disks Δ, Δ' must intersect if both Δ and Δ' enclose x). From Lemma 5.1 and Lemma 5.2 of [11], the restriction $\tau_a^r \tau_k^{-s}|_{\partial \mathbf{D}}$ fixes a parabolic fixed point of G. But Lemma 3.3 of [13] (see also Theorem 3.5.1 of [18]) asserts that for sufficiently large integers r and s, this does not occur.

We consider the case in which $\mathscr{C}$ does not contain any curve that is the boundary of a twice punctured disk enclosing x. This means that $\mathscr{C}$ does not contain any curve γ with $\tilde{\gamma}$ being trivial (where as usual $\tilde{\gamma}$ denotes the geodesic homotopic to γ on $\tilde{S}$).

Suppose that $\hat{c}$ intersects two maximal elements $U, U_0 \in \mathscr{U}_\gamma$ (Figure 1 (a)) and that (by taking a suitable power if necessary) $t_a^r \circ t_k^{-s}(\gamma) = \gamma$ for a $\gamma \in \mathscr{C}$ with $\tilde{\gamma}$ being non-trivial. As we saw, the minimum width $\varepsilon_2 \geq T_g$ with the equality holds only when $\{g(U_0), U\}$ tessellates the hyperbolic plane $\mathbf{D}$. By Lemma 3.2, $\hat{c}$ does not cross any maximal element of $\mathscr{U}_\gamma$. So either $\hat{c} \subset \Omega_a$ or $\hat{c} \subset W$ for a maximal element W of $\mathscr{U}_\gamma$.

If $\hat{c} \subset \Omega_a$, then g commutes with τ_γ. So f commutes with t_γ. This is impossible. Alternatively, the condition $\hat{c} \subset \Omega_a$ leads to that $\varrho(\hat{c})$ is disjoint from $\tilde{a}$, contradicting that $\varrho(\hat{c})$ is a filling closed curve.

It remains to consider the case that $\hat{c} \subset W$ for some maximal element $W \in \mathscr{U}_\gamma$. In the following discussion we denote by (XY) the minor arc in $\mathbf{S}$ connecting two points X and Y on $\mathbf{S}$. Notice that the boundary geodesic ∂W of W is disjoint from $\hat{a}$ and $\hat{b}_k$. We see that ∂W is disjoint from $\hat{c}$, $\hat{a}$ and $\hat{b}_k$. Hence the arc $W' \cap \mathbf{S}^1$, where $W' = \mathbf{D} \backslash \overline{W}$, is a subarc of one of the six arc components of $\mathbf{S}^1 \backslash \{A, B, E, F, X, Y\}$. See Figure 1 (b) for these labeling points.

Consider the case that $W' \cap \mathbf{S}^1 \subset (AE)$. Since $\hat{c} \subset W$, W' is disjoint from U_k', U' and $\hat{c}$. Note also that the Euclidean diameter of $\tau_k^s \tau_a^{-r}(W')$ is strictly smaller than that of W' for large s, r with $s \neq r$. Here r and s depend on U, U_k, $\hat{c}$, and does not depend on a particular half-plane H with $H \cap \mathbf{S}^1 \subset (AE)$ (since $H \cap \mathbf{S}^1$ is away from the points B and F).

We see that $\tau_k^s \tau_a^{-r}(W') \neq W'$. From Lemma 4.3 of [14], $\tau_k^s \tau_a^{-r}(W) \cap W \neq \emptyset$. Clearly, $W \neq \tau_k^s \tau_a^{-r}(W)$. It follows that $\tau_k^s \tau_a^{-r}$ and hence $\tau_a^r \tau_k^{-s}$ does not send W to any maximal element of $\mathscr{U}_\gamma$. If $W' \cap \mathbf{S}^1 \subset (BF)$, then the same argument as above (by considering the inverse map $\tau_k^s \tau_a^{-r}$ once again) shows that $\tau_a^r \tau_k^{-s}(W)$ is not an element of $\mathscr{U}_\gamma$.

If $W' \cap \mathbf{S}^1 \subset (BY) \cup (YA)$, then since ∂W projects to a simple closed geodesic on $\tilde{S}$, $\tau_k^s \tau_a^{-r}(W')$ is disjoint from W'. Hence $\tau_k^s \tau_a^{-r}(W)$ is not an element of $\mathscr{U}_\gamma$. Similar discussion also applies to the case when $W' \cap \mathbf{S}^1 \subset (EX) \cup (XF)$. Details are omitted. □

4. Proof of Theorem 1.1

Let $(\tau_a, \Omega_a, \mathscr{U}_a)$ be the configuration corresponding to a. Write $f = g^*$ for an essential hyperbolic element $g \in G$ with axis $\hat{c}$. We first assume that $\hat{c} \cap \Omega_a \neq \emptyset$ and continue to use the same notation and terminology as in Section 3. In this case, by Lemma 3.3 and Remark 2, the mapping classes $[\tau_a^r \tau_k^{-s}]^* \in \mathrm{Mod}_S^x$ are pseudo-Anosov when $k \geq 3$ and r, s are sufficiently large positive integers with $r \neq s$ (r, s depend on k).

Now we consider the case in which $\hat{c} \cap \Omega_a = \emptyset$. This means that $\hat{c}$ lies in a maximal element U of $\mathscr{U}_a$. We can assume that $\hat{a} = \partial U$. Let Y and X be the attracting and repelling fixed points of g, and denote by $\{E, F\}$ the two endpoints $\hat{a} \cap \mathbf{S}^1$. See Figure 2. Since $\tilde{a}$ is a simple closed geodesic on $\tilde{S}$, for $k \geq 1$, $g^k(U') \cap U' = \emptyset$. More precisely, $g^k(U') \cap \mathbf{S}^1$ is a subarc contained in (YE).

In Figure 2, $g^k(U')$ is shown as the region U_k'. Let $U_k = \mathbf{D} \backslash U_k'$. Then $g^k(U) = U_k$. We see that $U_k \cap U \neq \emptyset$, $U_k \cup U = \mathbf{D}$ and $\hat{b}_k \cap \hat{a} = \emptyset$, where $\hat{b}_k = \partial U_k$. Following the notation introduced in Lemma 3.3, we can similarly define τ_k as in (3.3) and claim that for all $k \geq 1$, all mapping classes $[\tau_a^r \tau_k^{-s}]^*$ are pseudo-Anosov when r and s are sufficiently large. For otherwise, suppose that $[\tau_a^r \tau_k^{-s}]^*$ is reducible. By taking a suitable power if necessary we assume that there is a geodesic $\gamma \subset S$ such that $[\tau_a^r \tau_k^{-s}]^*(\gamma) = \gamma$. Let $(\tau_\gamma, \Omega_\gamma, \mathscr{U}_\gamma)$ be the configuration corresponding to γ. If $\hat{c}$ is contained in an element W of $\mathscr{U}_\gamma$, we can use the same argument as in the proof of Lemma 3.3 to conclude that this does not occur.

We now assume that $\hat{c}$ crosses a maximal element W of $\mathscr{U}_\gamma$, as shown in Figure 2. Notice that $\tilde{\gamma} = \varrho(\hat{\gamma})$ is disjoint from $\tilde{a}$ and that ∂U_k is a geodesic in $\{\varrho^{-1}(\tilde{a})\}$. We see that ∂W must be disjoint from $\hat{a}$ and ∂U_k. By Lemma 2.1, $g(\mathbf{D} \backslash \overline{W}) \subset W_0$, where $W_0 \in \mathscr{U}_\gamma$ is another maximal element disjoint from W.

Write $\partial W \cap \mathbf{S}^1 = \{P, Q\}$. Since $g^k(U') = U_k'$, we have $g(E) = B$ and $g(F) = A$. This implies the point $g(P)$ stays in the arc connecting X and Y; while $g(Q)$ either lies in the arc connecting Q and F, or the arc connecting E and A (the point $g(Q)$ cannot lie in the arc connecting F and E. For otherwise, $g(\partial W)$ would intersect $\hat{a}$. This contradicts that $g(\partial W)$ projects to $\tilde{\gamma}$ which is disjoint from $\tilde{a}$).

If $g(Q)$ lies in the arc connecting Q and F, then for $W_0' = \mathbf{D} \backslash \overline{W}_0$, $\tau_a^r \tau_k^{-s}(W_0') \cap W_0' = \emptyset$, which tells us that $\tau_a^r \tau_k^{-s}(W_0')$ is not a maximal element of $\mathscr{U}_\gamma$. If $g(Q)$ lies in the arc connecting A and E, then W_0 contains the geodesic connecting $g(P)$ and $g(Q)$. Let $\partial W_0 \cap \mathbf{S}^1 = \{M, L\}$, where $M \in (XY)$ and $L \in (AE)$. For $W_0' = \mathbf{D} \backslash \overline{W}_0$, we have $\tau_k^{-s}(W_0') \cap \mathbf{S}^1 \subset$

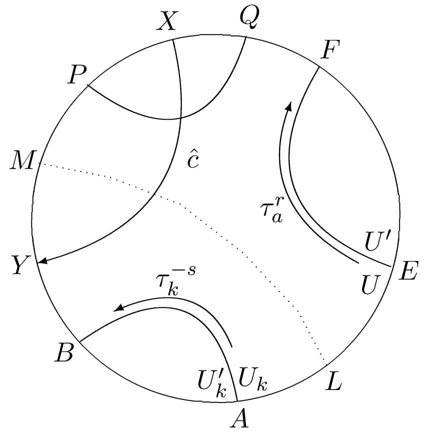

FIGURE 2

(BM), and so $\tau_a^r \tau_k^{-s}(W_0') \subset W_0'$. It follows that $\tau_a^r \tau_k^{-s}(W_0)$ is not a maximal element of $\mathscr{U}_\gamma$.

We conclude that for $k \geq 1$, the mapping classes $[\tau_a^r \tau_k^{-s}]^* \in \mathrm{Mod}_S^x$ are pseudo-Anosov whenever $\hat{c} \cap \Omega_a = \emptyset$ and r, s are sufficiently large positive integers with $r \neq s$.

In summary, we see that no matter whether or not $\hat{c} \cap \Omega_a \neq \emptyset$, for any integer $k \geq 3$, the mapping class $[\tau_a^r \tau_k^{-s}]^* \in \mathrm{Mod}_S^x$ are pseudo-Anosov when r and s are sufficiently large positive integers with $r \neq s$. From (3.4), this is equivalent to that $t_a^r \circ t_k^{-s}$ represents a pseudo-Anosov mapping class on S. We claim that all (a, b_k) for $k \geq 3$ fill S. For otherwise, there is a non-trivial curve $\delta \subset S$ disjoint from a and b_k. This implies that $t_a^r \circ t_k^{-s}(\delta) = \delta$ for any integers r and s, which says that $t_a^r \circ t_k^{-s}$ are reducible. This is a contradiction.

But from (3.3) we have

$$t_k = [\tau_k]^* = (g^*)^k \circ [\tau_a]^* \circ (g^*)^{-k} = f^k \circ t_a \circ f^{-k} = t_{f^k(a)}. \tag{4.1}$$

We conclude that $b_k = f^k(a)$. Hence $(a, f^k(a))$ fills S for $k \geq 3$. Thus $K \leq 3$. We have already seen that for some $f \in \mathscr{F}$, $K = 3$. This proves the first statement of Theorem 1.1.

To prove the second statement of the result, we notice that if $\tilde{c}$ intersects $\tilde{a}$ more than once, then the translation length T_g is larger than the hyperbolic length of the segment of $\hat{c}$ in $\mathbf{D} \backslash (U \cup U_0)$. See Figure 1 (a). This means that $\varepsilon_2 > T_g$. By Lemma 3.3, the mapping classes $[\tau_a^r \tau_2^{-s}]^*$ are pseudo-Anosov for all large integers r and s with $r \neq s$, i.e., $t_a^r \circ t_2^{-s}$ are pseudo-Anosov. It follows that (a, b_2) fills S. But a simple calculation similar to (4.1) reveals that $b_2 = f^2(a)$. As a consequence, $(a, f^2(a))$ fills S. Now by a similar argument, $(a, f^m(a))$ also fills S for all $m \geq 2$.

The last statement of the theorem is the restatement of Theorem 1.1 of [15]. □

5. Proof of Theorem 1.2 and Corollary 1.1

We continue to write $f = g^*$, where we recall that $g \in G$ is an essential hyperbolic element with axis $\hat{c}$. Let ε_k be as defined in (3.2). We refer to Figure 1 (a).

LEMMA 5.1. *$\varepsilon_2 \geq T_g$, and $\varepsilon_2 = T_g$ if and only if $\varepsilon_1 = 0$.*

PROOF. By the definition, ε_2 is the width of $\mathscr{D}_2 = U \cup g^2(U_0)$. We see that $\varepsilon_2 = \varepsilon_1 + T_g \geq T_g$, and $\varepsilon_2 = T_g$ if and only if $\varepsilon_1 = 0$. □

LEMMA 5.2. *$0 \leq \varepsilon_1 < T_g$, and $\varepsilon_1 = 0$ if and only if a and $f(a)$ are disjoint and furthermore $\{a, f(a)\}$ forms the boundary of an x-punctured cylinder on S.*

PROOF. It is trivial that $\varepsilon_1 \geq 0$. Since Ω_a has a non-empty interior, $\varepsilon_1 < T_g$. Suppose that $\varepsilon_1 = 0$. Then $\{g(U_0), U\}$ tessellates the hyperbolic plane $\mathbf{D}$, which says that $\partial U = \partial\,(g(U_0))$. Note that U is a maximal element of $\mathscr{U}_a$ and that $\hat{a} = \partial U$. By (3.3), $\tau_1 = g\tau_a g^{-1}$. By Lemma 3.2 of [13], there is a geodesic $\alpha \subset S$ such that

$$[\tau_1]^* = t_\alpha\,. \tag{5.1}$$

Let $(\tau_1, \Omega_\alpha, \mathscr{U}_\alpha)$ be the configuration corresponding to α. Let $h \in G$ be the primitive hyperbolic element that keeps $\hat{a} = \partial U = \partial\,(g(U_0))$ invariant and takes the same orientation as that of τ_a. We need to inspect the actions of τ_a and τ_1 on $\mathbf{D}$.

Let Λ be the disjoint union of all small “crescent” neighborhoods of geodesics $\hat{a}$ with $\varrho(\hat{a}) = \tilde{a}$. Thus $\Lambda \cap \mathbf{S}^1$ consists of infinitely (but countably) many points and the projection $\varrho(N)$ for every $N \in \Lambda$ is the prescribed cylinder with the central geodesic $\tilde{a}$. It is clear that $\Lambda = g(\Lambda)$ and that the Beltrami coefficients $\partial_{\bar{z}}(\tau_a(z))/\partial_z(\tau_a(z))$ and $\partial_{\bar{z}}(\tau_1(z))/\partial_z(\tau_1(z))$ of τ_a and τ_1, respectively, are both supported in Λ. It turns out that $\tau_a\tau_1^{-1}$ is conformal off Λ. Let $\Lambda(U)$ be the subset of Λ contained in U.

Observe (by the construction of τ_a) that the restriction of τ_a to the region

$$\Theta = U \backslash \{(\text{elements of } \mathscr{U}_a \text{ of the second order}) \cup (\Lambda(U))\}$$

is the same as the restriction of h to Θ, where an element $U(2) \in \mathscr{U}_a$ is called to be of second order if $U(2)$ is contained in a maximal element but is not contained in any other element of $\mathscr{U}_a$. By construction, we know that the restriction of τ_1 to Θ is the identity. We conclude that $\tau_a\tau_1^{-1}|_\Theta = h|_\Theta$.

Now for each second-order element $U(2) \in \mathscr{U}_a$ contained in U, let $N \in \Lambda$ be the neighborhood of $\partial\,(U(2))$. Write $\tilde{U}(2) = U(2) \cup N$. We have $\tau_a(\tilde{U}(2)) = h(\tilde{U}(2))$. Notice that $U(2)$ is a maximal element of $\mathscr{U}_\alpha$, which is an invariant half-plane of τ_1. Hence $\tau_a\tau_1^{-1}|_{\tilde{U}(2)} = h|_{\tilde{U}(2)}$. Since $U(2)$ is an arbitrary element of the second order, the action of $\tau_a\tau_1^{-1}$ on $U\backslash\Lambda$ coincides with the action of h on $U\backslash\Lambda$. Likewise, we can show that the action of $\tau_a\tau_1^{-1}$ coincides with the action of h on $g(U_0)\backslash\Lambda$. As it turns out, the restriction

$\tau_a\tau_1^{-1}|_{\mathbf{D}\setminus\Lambda} = h|_{\mathbf{D}\setminus\Lambda}$. Recall that $\Lambda \cap \mathbf{S}^1$ consists of infinitely (but countably) many points. We conclude that $\tau_a\tau_1^{-1}|_{\partial\mathbf{D}} = h|_{\partial\mathbf{D}}$, which tells us that

$$[\tau_a\tau_1^{-1}]^* = h^* . \tag{5.2}$$

From Theorem 2 of [6] and Theorem 2 of [8], $h^* = t_a \circ t_{a_0}^{-1}$, where $\{a, a_0\}$ is the boundary of an x-punctured cylinder P on S. It follows from (5.1) and (5.2) that

$$t_a \circ t_\alpha^{-1} = [\tau_a\tau_1^{-1}]^* = h^* = t_a \circ t_{a_0}^{-1} .$$

So $t_\alpha = t_{a_0}$ and hence $\alpha = a_0$. A calculation similar to (4.1) yields $a_0 = \alpha = g^*(a) = f(a)$. This proves that $(a, f(a))$ forms the boundary of P.

Conversely, if $\varepsilon_1 > 0$, then $\mathscr{D}_1 \neq \emptyset$. By Lemma 4 of [12], τ_a and τ_1 do not commute. Thus t_a and t_α do not commute, which is equivalent to that a and $f(a)$ intersect. In particular, $\{a, f(a)\}$ does not form a boundary of any x-punctured cylinder on S. □

PROOF OF THEOREM 1.2. By assumption, $(a, f^2(a))$ does not fill S. By Theorem 1.1 of [15], a cannot be the boundary component of a twice punctured disk enclosing x. That is, $\tilde{a}$ is non-trivial. Let τ_a be the lift of $t_{\tilde{a}}$ so that $[\tau_a]^* = t_a$. From the argument of Theorem 1.1, we also claim that $\hat{c} \cap \Omega_a \neq \emptyset$. We are now in the situation depicted in Figure 1 (a). By Lemma 5.1, $\varepsilon_2 \geq T_g$.

If $\varepsilon_2 > T_g$, then by Lemma 3.3, for large r, s with $r \neq s$, the mapping classes $[\tau_a^r\tau_2^{-s}]^*$ are pseudo-Anosov, i.e., $t_a^r \circ t_2^{-s}$ are pseudo-Anosov. This implies that (a, b_2) fills S. But a calculation similar to (4.1) demonstrates that $b_2 = f^2(a)$. As a result, $(a, f^2(a))$ fills S. This is a contradiction.

We thus conclude that $\varepsilon_2 = T_g$. Hence by Lemma 5.1, $\varepsilon_1 = 0$. So by Lemma 5.2, we see that a is disjoint from $f(a)$ and $\{a, f(a)\}$ forms the boundary of an x-punctured cylinder P on S. Note that $f : S \to S$ is a homeomorphism, $f(a)$ and $f^2(a)$ are also disjoint. We see that $f(a)$ is disjoint from both a and $f^2(a)$.

We need to show that if $\gamma \subset S$ is a geodesic disjoint from both a and $f^2(a)$, then $\gamma = f(a)$. For this purpose, we notice that t_γ commutes with $t_a^r \circ t_{f^2(a)}^{-s}$. But $t_a^r \circ t_{f^2(a)}^{-s} = t_a^r \circ \left(f^2 \circ t_a^{-s} \circ f^{-2}\right)$. By the Bers isomorphism, τ_γ commutes with $\tau_a^r\left(g^2\tau_a^{-s}g^{-2}\right)$. By Lemma 4.1 of [14], we know that $\tau_a^r\left(g^2\tau_a^{-s}g^{-2}\right)$ sends every maximal element W of $\mathscr{U}_\gamma$ to a maximal element.

First we claim that $g(U_0) \cap U = \emptyset$. See Figure 1 (a). Suppose $g(U_0) \cap U \neq \emptyset$. Then $g(U_0) \cup U = \mathbf{D}$ and $\varepsilon_2 > T_g$. By the same argument of Lemma 3.3, $\left[\tau_a^r\left(g^2\tau_a^{-s}g^{-2}\right)\right]^*$ are pseudo-Anosov. This contradicts that $\tau_a^r\left(g^2\tau_a^{-s}g^{-2}\right)$ sends every maximal element $W \in \mathscr{U}_\gamma$ to a maximal element.

Next we claim that there exists a maximal element $W \in \mathscr{U}_\gamma$ such that

$$W = g(U_0) . \tag{5.3}$$

Since $\tilde{c}$ is a filling geodesic, if there is no maximal element W with $\partial W \cap \hat{c} \neq \emptyset$, there is a maximal element $W_0 \in \mathscr{U}_\gamma$ such that $\hat{c} \subset W_0$. Since $\partial W_0 = \partial W_0'$ does not intersect $\hat{a}$, $\hat{b}_k$ and $\hat{c}$, we see that $W_0' \cap \mathbf{S}^1$ lies in one of the six arcs of $\mathbf{S}^1 \backslash \{A, B, E, F, X, Y\}$ (see Figure 1 (b)). The same argument of Lemma 3.3 shows that this is impossible.

Consider the case in which there is a maximal element $W \in \mathscr{U}_\gamma$ with $\partial W \cap \hat{c} \neq \emptyset$. Since $g(U_0) \cap U = \emptyset$, we see that $\varepsilon_1 = 0$ and thus $\varepsilon_2 = T_g$. If $W \neq g(U_0)$, the same argument of Lemma 3.1 and Remark 1 can be used to assert that there is a maximal element $W_0 \in \mathscr{U}_\gamma$ such that either $W_0' \subset U$ or $W_0' \subset U_2$, where we recall that $U_2 = g^2(U_0)$. Then we can use the argument of Lemma 3.2 to deduce that $\tau_a^r \tau_2^{-s}(W_0)$ is not a maximal element of $\mathscr{U}_\gamma$. This contradicts the fact that $\tau_a^r \left(g^2 \tau_a^{-s} g^{-2}\right)$ sends every maximal element $W \in \mathscr{U}_\gamma$ to a maximal element.

We conclude that (5.3) holds. Since U_0 is a maximal element of $\mathscr{U}_a$, (5.3) tells us that $\tau_\gamma = g\tau_a g^{-1}$. With the help of the Bers isomorphism, we obtain $t_\gamma = f \circ t_a \circ f^{-1} = t_{f(a)}$. That is, $\gamma = f(a)$. This completes the proof of Theorem 1.2. □

PROOF OF COROLLARY 1.1. Suppose that $(a, f^{k_0}(a))$ does not fill S for some integer $k_0 \geq 2$. Then $(a, f^2(a))$ does not fill S. This means that there is a simple closed geodesic $b \subset S$ such that b is disjoint from both a and $f^2(a)$. By Theorem 1.2, $b = f(a)$. This implies that a and $f(a)$ are disjoint, which leads to a contradiction. □

6. Remarks

For any simple closed geodesic $a \subset S$, if $\tilde{a} \subset \tilde{S}$ is trivial, then Theorem 1.1 of [15] states that $(a, f(a))$ fills S for all elements $f \in \mathscr{F}$. We know that the set $\mathscr{F}$ is a disjoint union of $\mathscr{F}_{\tilde{c}}$, where every $\mathscr{F}_{\tilde{c}}$ consists of pseudo-Anosov maps conjugate (in the fundamental group $\pi_1(\tilde{S}, x)$) to a map $f_{\tilde{c}}$ that can be obtained from an oriented filling closed geodesic $\tilde{c} \subset \tilde{S}$. From the argument of Theorem 1.1, for every filling closed geodesic $\tilde{c}$, most elements f in $\mathscr{F}_{\tilde{c}}$ have the property that $(a, f(a))$ fill S. However, there are some exceptions.

From Theorem 1.2, some oriented filling closed geodesics $\tilde{c} \subset \tilde{S}$ (and thus elements $f \in \mathscr{F}_{\tilde{c}}$) can be identified so that $(a, f^2(a))$ does not fill S (while Theorem 1.1 guarantees that $(a, f^3(a))$ always fills S). We can ask whether some oriented filling geodesic $\tilde{c}$ (and thus elements $f \in \mathscr{F}_{\tilde{c}}$) can be identified so that $(a, f(a))$ fills S; or $(a, f^2(a))$ fills S if $(a, f(a))$ does not fill S. These problems are related to the problem of estimating the minimum value among the logarithms of the dilatations $\lambda(f)$ for $f \in \mathscr{F}$. See [16, 17] for more discussions.

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