TOKYO J. MATH.
VOL. 35, NO. 2, 2012

Sasaki-Einstein Metrics on $S^2 \times S^3$

Mitsuhiro IMADA

Keio University

(Communicated by K. Ota)

Abstract. In [9], Boyer and Galicki introduced a contact reduction method in the context of Sasakian manifolds, which produces 5-dimentional Sasaki-Einstein manifolds from a 7-sphere. In this paper, we compute very explicitly the metric obtained from the above mentioned reduction via a projection, $S^3 \times S^3 \to S^2 \times S^3$, and show that this metric is the homogeneous Kobayashi-Tanno metric.

1. Introduction

Reduction techniques in symplectic geometry, such as Marsden and Ratiu [1], have natural analogues in the context of contact geometry. Depending on the geometric situation, various specializations have been considered in the literature, such as the Sasakian case by Geiges [2], Grantcharov and Ornea [3]. Later on the Sasaki-Einstein case, by Boyer and Galicki in [9]. In the latter approach (on which this paper is based) the authors constructed a 5-dimentional Sasaki-Einstein manifold by means of a S^1 reduction of the zero set of a moment map defined on S^7.

In this paper, we explicitized the above construction, and compute explicitly the reduced metric on the reduced space by means of a projection from the zero set to the reduced space, which is diffeomorphic to $S^2 \times S^3$. More precisely, we consider the following moment map on $\mathbf{C}^4$,

$$\mu(z_1, z_2, z_3, z_4) := |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 ,$$

with the associated $U(1)$ action,

$$(z_1, z_2, z_3, z_4) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2, e^{-i\theta} z_3, e^{-i\theta} z_4) \quad (\theta \in \mathbf{R}) ,$$

and we show that $\mu^{-1}(0)|_{S^7}$ is diffeomorphic to $S^3 \times S^3$. Using this identification, we define a smooth projection π from $S^3 \times S^3$ to $\left(\mu^{-1}(0)|_{S^7}\right)/S^1$ (see §4):

$$\pi(z_1, z_2, z_3, z_4) := (2z_1\bar{z}_2,\ |z_1|^2 - |z_2|^2,\ z_1 z_3 + \bar{z}_2\bar{z}_4,\ z_2 z_3 - \bar{z}_1\bar{z}_4) .$$

Received January 18, 2011; revised March 22, 2011; revised April 12, 2011

Later in §4, we show that this image is diffeomorphic to $S^2 \times S^3$. We notice that a $SU(2) \times SU(2)$ acts on $S^3 \times S^3$ naturally from the left, which gives π is an equivariant map, that is, $S^2 \times S^3$ becomes a homogeneous space by this action. We then define an inner product $\langle \cdot, \cdot \rangle_o$ on $T_o(S^2 \times S^3)$ $(o = (0, 0, -1, 1, 0, 0, 0))$ and extend it to any point x as follows

$$\langle u, v \rangle_x := \langle dk^{-1}(u), dk^{-1}(v) \rangle_o \quad (u, v \in T_x(S^2 \times S^3))$$

where k is a $(SU(2) \times SU(2))/U(1)$ free action such that $x = k \cdot o$. This is a representation of the metric named the homogeneous Kobayashi-Tanno metric [10], [11]. Our main result (Theorem 4.1) is an explicit calculation of the metric.

2. Sasaki-Einstein manifolds

In this section, we recall the definition of a Sasaki-Einstein manifold [9].

DEFINITION 2.1. A Sasakian manifold is a $(2n - 1)$-dimentional Riemannian manifold (M, g) whose metric cone $(C(M), r^2 g + dr^2, J)$ is a Kähler manifold, where $C(M) := M \times \mathbf{R}_+ = \{(x, r) \mid x \in M, r \in \mathbf{R}_+\}$.

Now we check if there exists a complex structure on $C(M)$. There is a contact metric structure (Φ, ξ, η, g) on Sasakian M where Φ is a field of endomorphisms of TM, ξ is a Killing vector field and η is a 1-form satisfying

$$\eta(\xi) = 1\,,$$
$$\Phi^2 = -I + \eta \otimes \xi\,.$$

We denote a vector field on $C(M)$ by $\left(X, f\frac{\partial}{\partial r}\right)$ where X is tangent to M and f is a C^∞ function on $C(M)$. Then we define a field of endomorphisms of $TC(M)$ by

$$J\left(X, f\frac{\partial}{\partial r}\right) := \left(\Phi X - f\xi,\ \eta(X)\frac{\partial}{\partial r}\right)\,.$$

It is easy to check that $J^2 = -I$. Since J is integrable [8], it follows that J is a complex structure on $C(M)$.

DEFINITION 2.2. A Sasaki-Einstein manifold is a $(2n - 1)$-dimentional Riemannian manifold (M, g) whose metric cone $(C(M), r^2 g + dr^2, J)$ is a Ricci-flat (i.e. Ricci curvature $= 0$) Kähler manifold.

We recall the definition of Ricci curvature;

$$Ric(X, Y) := Tr(Z \to R(Z, X)Y)$$

where R is the curvature tensor of the metric $r^2 g + dr^2$,

$$R(Z, X)Y := \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y\,.$$

Note that R is a tri-linear map, and $'Tr'$ is the trace of the linear map $Z \to R(Z, X)Y$ for any given X and Y.

EXAMPLE 2.3. An odd-dimentional sphere S^{2n-1} with induced metric g_0 from $\mathbf{C}^n$ is Sasaki-Einstein, as its cone $(C(S^{2n-1}), r^2 g_0 + dr^2)$ is isometric to $(\mathbf{C}^n, g_{std.})$, where g_{std} is the standard Ricci-flat Kähler metric on $\mathbf{C}^n$.

3. Sasakian reduction by Boyer and Galicki

In this section, we recall the special Sasakian reduction constructed by Boyer and Galicki in [9]. In particular, they focus on $n = 4$ case.

DEFINITION 3.1. Let $p, q \in \mathbf{Z}_{\geq 0}$ be coprime and $p > q$, or $p = 1, q = 0$. We define a moment map $\mu_{p,q} : \mathbf{C}^4 \longrightarrow \mathbf{R}$ as follows

$$\mu_{p,q}(z_1, z_2, z_3, z_4) := p|z_1|^2 + p|z_2|^2 - (p-q)|z_3|^2 - (p+q)|z_4|^2 ,$$

and $S^1_{p,q}$ is the associated S^1 action on $(\mathbf{C}^*)^4$,

$$(z_1, z_2, z_3, z_4) \mapsto (z_1 e^{ip\theta}, z_2 e^{ip\theta}, z_3 e^{-i(p-q)\theta}, z_4 e^{-i(p+q)\theta}) .$$

THEOREM 3.2. *We set an inclusion ι and a projection π as*

$$\iota : \mu^{-1}_{p,q}(0)|_{S^7} \hookrightarrow S^7$$
$$\pi : \mu^{-1}_{p,q}(0)|_{S^7} \to \big(\mu^{-1}_{p,q}(0)|_{S^7}\big)/S^1_{p,q} .$$

Then we have the following:

1. $\mu^{-1}_{p,q}(0)|_{S^7}$ *is diffeomorphic to* $S^3 \times S^3$.
2. $\big(\mu^{-1}_{p,q}(0)|_{S^7}\big)/S^1_{p,q}$ *is diffeomorphic to* $S^2 \times S^3$.
3. *There is a Sasaki-Einstein metric* $g_{p,q}$ *on* $\big(\mu^{-1}_{p,q}(0)|_{S^7}\big)/S^1_{p,q} satisfying$ $\iota^* g_0 = \pi^* g_{p,q}$ *where* g_0 *is the induced metric on* S^7 *from* $\mathbf{C^4}$ (*Example* 2.3).

4. Computing the case of $p = 1, q = 0$

Let us restrict our attention for the case of $p = 1$ and $q = 0$, and consider the zero level set

$$\mu^{-1}_{1,0}(0)|_{S^7} = \left\{(z_1, z_2, z_3, z_4) \in S^7; |z_1|^2 + |z_2|^2 = |z_3|^2 + |z_4|^2 = \frac{1}{2}\right\}$$
$$= S^3\left(\frac{1}{\sqrt{2}}\right) \times S^3\left(\frac{1}{\sqrt{2}}\right) .$$

For any point in $\mu_{1,0}^{-1}(0) \subset S^3 \times S^3$, we identify S^3 and $SU(2)$ as follows:

$$(z_1, z_2) \in S^3 \leftrightarrow \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \in SU(2)\,.$$

The reduced space $S^3 \times S^3/S^1$ is diffeomorphic to $S^2 \times S^3$ with a projection π defined by,

$$\pi(h_1, h_2) := ([h_1], h_1\, {}^t h_2)$$

where $h_1, h_2 \in SU(2)$ and $[\cdot]$ is the equivalence class $\sim$ given by

$$h_1 \sim h_2 \Leftrightarrow h_2 = h_1 \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

This equivalence relation is the same as in the definition of the projective space $\mathbf{C}P^1$. In complex coordinates, π is given explicitly by

$$\pi(z_1, z_2, z_3, z_4) := (2z_1\bar{z}_2,\ |z_1|^2 - |z_2|^2,\ z_1z_3 + \bar{z}_2\bar{z}_4,\ z_2z_3 - \bar{z}_1\bar{z}_4)\,.$$

Then we have a left $SU(2) \times SU(2)$ action $\phi = (\phi_1, \phi_2)$ on $S^3 \times S^3$,

$$\phi(h_1, h_2) := (\phi_1 h_1, \phi_2 h_2) \quad (\phi_1, \phi_2 \in SU(2))\,.$$

Let us define a $(SU(2) \times SU(2))/U(1)$ action $\tilde{\phi} = ([\tilde{\phi}_1], \tilde{\phi}_2)$ on $S^2 \times S^3$ as follows

$$\tilde{\phi}([h_1], h_1\, {}^t h_2) := ([\tilde{\phi}_1 h_1], \tilde{\phi}_1 h_1\, {}^t h_2\, {}^t \tilde{\phi}_2) \quad (\tilde{\phi}_1, \tilde{\phi}_2 \in SU(2))\,,$$

such that ϕ induces $\tilde{\phi}$, and π is $(\phi, \tilde{\phi})$-equivariant:

$$\begin{array}{ccc} S^3 \times S^3 & \xrightarrow{\phi} & S^3 \times S^3 \\ \pi \downarrow & & \downarrow \pi \\ S^2 \times S^3 & \xrightarrow{\tilde{\phi}} & S^2 \times S^3\,. \end{array}$$

Since $S^2 \times S^3$ is a homogeneous space for $(SU(2) \times SU(2))/U(1)$, we can define an inner product $\langle \cdot, \cdot \rangle_o$ on $T_o(S^2 \times S^3)$, where o is written with an unit matrix I_2,

$$o := (0, 0, -1, 1, 0, 0, 0) = ([I_2], I_2) = \pi(I_2, I_2) = \pi(1, 0, 0, 0, 1, 0, 0, 0)\,,$$

for the Sasaki-Einstein metric $g_{1,0}$. By Theorem 3.2, the inner product $\langle \cdot, \cdot \rangle_o$ satisfies a condition:

$$\begin{aligned} &d\pi\big(\{\text{an orthonormal basis of } T_{(I_2,I_2)}(S^3 \times S^3)\}\big) \\ &\quad = \{\text{an orthonormal basis of } T_o(S^2 \times S^3)\}\,. \end{aligned}$$

By this, if we choose $\left\{\frac{\partial}{\partial s_2}, \frac{\partial}{\partial s_3}, \frac{\partial}{\partial s_4}, \frac{\partial}{\partial s_6}, \frac{\partial}{\partial s_7}, \frac{\partial}{\partial s_8}\right\}$ an orthonormal basis of $T_{(I_2,I_2)}(S^2 \times S^3)$, thus

$$\left\{d\pi\left(\frac{\partial}{\partial s_2}\right) = d\pi\left(\frac{\partial}{\partial s_6}\right) = \left(\frac{\partial}{\partial x_5}\right)_o, d\pi\left(\frac{\partial}{\partial s_3}\right) = 2\left(\frac{\partial}{\partial x_1}\right)_o + \left(\frac{\partial}{\partial x_6}\right)_o,\right.$$
$$\left. d\pi\left(\frac{\partial}{\partial s_4}\right) = 2\left(\frac{\partial}{\partial x_2}\right)_o + \left(\frac{\partial}{\partial x_7}\right)_o, d\pi\left(\frac{\partial}{\partial s_7}\right) = -\left(\frac{\partial}{\partial x_6}\right)_o, d\pi\left(\frac{\partial}{\partial s_8}\right) = \left(\frac{\partial}{\partial x_7}\right)_o\right\}$$

is an orthonormal basis of $T_o(S^2 \times S^3)$. Then the metric $g_o(\cdot,\cdot) = \langle\cdot,\cdot\rangle_o$ defined by

$$\left(\left\langle\left(\frac{\partial}{\partial x_i}\right)_o, \left(\frac{\partial}{\partial x_j}\right)_o\right\rangle_o\right)_{ij} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 1 \end{pmatrix}, \quad (i, j = 1, 2, 5, 6, 7).$$

Choosing the local chart (U_0, ψ_0) such that

$$U_0 = \{(x_1, \dots, x_7) \in S^2 \times S^3; x_3 < 0, x_4 > 0\},$$
$$\psi_0 : (x_1, \dots, x_7) \mapsto (x_1, x_2, x_5, x_6, x_7),$$

we extend this metric to any point $x := ([k_1], k_2)$ by another $(SU(2) \times SU(2))/U(1)$ action on $S^2 \times S^3$: for $k = (k_1, k_2)$,

$$k([h_1], h_2) := ([k_1 h_1], k_1 h_2 k_1^{-1} k_2),$$

noting that $x = k \cdot o$. We define the metric g at x by

$$g_x(u, v) := g_0(dk^{-1}(u), dk^{-1}(v)) \quad (u, v \in T_x(S^2 \times S^3)).$$

For $y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7) \in U_o$, we can write k^{-1} as

$$k^{-1}(y) = \left(k_1^{-1}(y), k_2^{-1}(y), k_3^{-1}(y), k_4^{-1}(y), k_5^{-1}(y), k_6^{-1}(y), k_7^{-1}(y)\right)$$
$$= \Big(\tfrac{(1-x_3-x_1^2)y_1 - x_1x_2y_1 + x_1(1-x_3)y_3}{1-x_3}, \tfrac{-x_1x_2y_1 + (1-x_3-x_2^2)y_2 + x_2(1-x_3)y_3}{1-x_3},$$
$$-x_1y_1 - x_2y_2 - x_3y_3, \tfrac{X_1Y_1 + X_2Y_2 + X_3Y_3 + X_4Y_4}{2(1-x_3)}, \tfrac{-X_2Y_1 - X_1Y_2 + X_4Y_3 - X_3Y_4}{2(1-x_3)},$$
$$\tfrac{-X_3Y_1 - X_4Y_2 - X_1Y_3 + X_2Y_4}{2(1-x_3)}, \tfrac{-X_4Y_1 + X_3Y_2 - X_2Y_3 - X_1Y_4}{2(1-x_3)}\Big), \text{ where}$$
$$X_1 = (1-x_3)x_4 + x_1x_6 + x_2x_7,\ X_2 = x_2x_6 - x_1x_7 - (1-x_3)x_5,$$
$$X_3 = x_1x_4 - x_2x_5 - (1-x_3)x_6,\ X_4 = x_1x_5 + x_2x_4 - (1-x_3)x_7,$$
$$Y_1 = (1-x_3)y_4 + x_1y_6 + x_2y_7,\ Y_2 = x_2y_6 - x_1y_7 - (1-x_3)y_5,$$
$$Y_3 = x_1y_4 - x_2y_5 - (1-x_3)y_6,\ Y_4 = x_1y_5 + x_2y_4 - (1-x_3)y_7.$$

Next we calculate $g_x\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_5}\right)$. Let us first consider the derivation dk^{-1},

$$
\begin{aligned}
dk^{-1}\left(\tfrac{\partial}{\partial x_1}\right) &= \tfrac{\partial k_1^{-1}}{\partial y_1}(x)\left(\tfrac{\partial}{\partial x_1}\right)_o + \tfrac{\partial k_2^{-1}}{\partial y_1}(x)\left(\tfrac{\partial}{\partial x_2}\right)_o + \tfrac{\partial k_5^{-1}}{\partial y_1}(x)\left(\tfrac{\partial}{\partial x_5}\right)_o + \tfrac{\partial k_6^{-1}}{\partial y_1}(x)\left(\tfrac{\partial}{\partial x_6}\right)_o + \tfrac{\partial k_7^{-1}}{\partial y_1}(x)\left(\tfrac{\partial}{\partial x_7}\right)_o \\
&= \frac{x_2^2 + x_3 - 1}{x_3(1-x_3)}\left(\frac{\partial}{\partial x_1}\right)_o + \frac{-x_1x_2}{x_3(1-x_3)}\left(\frac{\partial}{\partial x_2}\right)_o \qquad \text{and} \\
dk^{-1}\left(\tfrac{\partial}{\partial x_5}\right) &= \tfrac{\partial k_1^{-1}}{\partial y_5}(x)\left(\tfrac{\partial}{\partial x_1}\right)_o + \tfrac{\partial k_2^{-1}}{\partial y_5}(x)\left(\tfrac{\partial}{\partial x_2}\right)_o + \tfrac{\partial k_5^{-1}}{\partial y_5}(x)\left(\tfrac{\partial}{\partial x_5}\right)_o + \tfrac{\partial k_6^{-1}}{\partial y_5}(x)\left(\tfrac{\partial}{\partial x_6}\right)_o + \tfrac{\partial k_7^{-1}}{\partial y_5}(x)\left(\tfrac{\partial}{\partial x_7}\right)_o \\
&= \tfrac{-x_3(x_4^2+x_5^2)+x_1(x_4x_6+x_5x_7)+x_2(x_4x_7-x_5x_6)}{x_4}\left(\tfrac{\partial}{\partial x_5}\right)_o \\
&\quad + \tfrac{(1-x_3-x_2^2)(x_5x_6-x_4x_7)+(1-x_3)x_2(x_4^2+x_5^2)+x_1x_2(x_4x_6+x_5x_7)}{(1-x_3)x_4}\left(\tfrac{\partial}{\partial x_6}\right)_o \\
&\quad + \tfrac{(1-x_3-x_1^2)(x_4x_6+x_5x_7)-(1-x_3)x_1(x_4^2+x_5^2)+x_1x_2(x_5x_6-x_4x_7)}{(1-x_3)x_4}\left(\tfrac{\partial}{\partial x_7}\right)_o .
\end{aligned}
$$

Then the coeffcient of dx_1dx_5 and dx_5dx_1 is given by

$$
\begin{aligned}
g_x\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_5}\right) &= g_0\left(dk^{-1}\left(\frac{\partial}{\partial x_1}\right), dk^{-1}\left(\frac{\partial}{\partial x_5}\right)\right) \\
&= \tfrac{-(x_1^2+x_3^2)(x_4x_7-x_5x_6)-x_2x_3(x_4^2+x_5^2)+x_1x_2(x_4x_6+x_5x_7)}{2x_3x_4} .
\end{aligned}
$$

Also we can find the coefficient of dx_idx_j and dx_jdx_i by calculating $g_x\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$. As the result, with the local coordinates $x = (x_1, x_2, x_5, x_6, x_7)$ on U_0, we have the formula:

$$
\begin{aligned}
g_x = &\sum_{i=1}^{2} \frac{x_i^2 + x_3^2}{2x_3^2} dx_i^2 + \frac{x_1x_2}{x_3^2} dx_1dx_2 + \sum_{i=5}^{7} \frac{x_4^2 + x_i^2}{x_4^2} dx_i^2 \\
&+ \frac{2x_5x_6}{x_4^2} dx_5dx_6 + \frac{2x_5x_7}{x_4^2} dx_5dx_7 + \frac{2x_6x_7}{x_4^2} dx_6dx_7 \\
&+ \frac{-(x_1^2+x_3^2)(x_4x_7 - x_5x_6) - x_2x_3(x_4^2+x_5^2) + x_1x_2(x_4x_6 + x_5x_7)}{x_3x_4} dx_1dx_5 \\
&+ \frac{(x_1^2+x_3^2)(x_4^2+x_6^2) - x_1x_2(x_4x_5 - x_6x_7) - x_2x_3(x_4x_7 + x_5x_6)}{x_3x_4} dx_1dx_6 \\
&+ \frac{x_1x_2(x_4^2+x_7^2) + (x_1^2+x_3^2)(x_4x_5 + x_6x_7) + x_2x_3(x_4x_6 - x_5x_7)}{x_3x_4} dx_1dx_7 \qquad (1) \\
&+ \frac{x_1x_3(x_4^2+x_5^2) - x_1x_2(x_4x_7 - x_5x_6) + (x_2^2+x_3^2)(x_4x_6 + x_5x_7)}{x_3x_4} dx_2dx_5 \\
&+ \frac{x_1x_2(x_4^2+x_6^2) + x_1x_3(x_4x_7 + x_5x_6) - (x_2^2+x_3^2)(x_4x_5 - x_6x_7)}{x_3x_4} dx_2dx_6 \\
&+ \frac{(x_2^2+x_3^2)(x_4^2+x_7^2) + x_1x_2(x_4x_5 + x_6x_7) - x_1x_3(x_4x_6 - x_5x_7)}{x_3x_4} dx_2dx_7 .
\end{aligned}
$$

On other open sets $U_{i^\pm j^\pm}$ of $S^2 \times S^3$ defined by for $i \in \{1, 2, 3\}$, $j \in \{4, 5, 6, 7\}$, i.e.

$$U_{i^+j^+} = \{x_i > 0, x_j > 0\},\ U_{i^-j^+} = \{x_i < 0, x_j > 0\}\,,$$
$$U_{i^+j^-} = \{x_i > 0, x_j < 0\} \text{ and } U_{i^-j^-} = \{x_i < 0, x_j < 0\}\,,$$

we can calculate the metric the same way as the previous case. This is an explicit representation at x of the Sasaki-Einstein metric $g_{1,0}$ called the homogeneous Kobayashi-Tanno metric by Boyer and Galicki in [9].

THEOREM 4.1. *The Sasaki-Einstein metric $g_{1,0}$ on $S^2 \times S^3$ at any point x is given by the fomula* (1).

References

[1] J. E. MARSDEN and T. S. RATIU, *Introduction to Mechanics and Symmetry*, Springer (1999).

[2] H. GEIGES, Constructions of contact manifolds, Math. Proc. Camb. Phil. Soc. **121** (1997), 455–564.

[3] L. ORNEA and P. PICCINI, Induced Hopf bundles and Einstein metrics, in *New developments in differential geometry*, Budapest, 1996, Kluwer Publ. (1998), 295–306.

[4] G. GRANTCHAROV and L. ORNEA, Reduction of Sasakian manifolds, arXiv:math/9909127v1.

[5] O. DRAGULETE and L. ORNEA, Non-zero contact and Sasakian reduction, arXiv:math/0407434v1.

[6] M. Y. WANG and W. ZILLER, Einstein metrics on principal torus bundles, J. Diff. Geom. **31** (1990), 215–248.

[7] A. BESSE, *Einstein manifolds*, Springer Verlag (1987).

[8] D. E. BLAIR, *Contact Manifolds in Riemannian Geometry*, Springer Verlag (1976).

[9] C. BOYER and K. GALICKI, *Sasakian Geometry*, Oxford Science Publ. (2008).

[10] S. KOBAYASHI, Topology of positively pinched Kaehler manifolds, Tōhoku Math. J. **15** (2) (1963), 121–139. MR 27 # 4185

[11] S. TANNO, Geodesic flows on C_L-manifolds and Einstein metrics on $S^2 \times S^3$, in *Minimal submanifolds and geodesics*, Kaigai Publ., Tokyo (1978), 283–292.

Present Address:
DEPARTMENT OF MATHEMATICS,
KEIO UNIVERSITY,
HIYOSHI, KOHOKU-KU, YOKOHAMA, 223–8532 JAPAN.
e-mail: imaddd@a6.keio.jp