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Surfaces with Constant Chebyshev Angle

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Abstract. In this paper we consider surfaces with negative Gaussian curvature parametrized by a generalized Chebyshev net with constant Chebyshev angle in the Euclidean 3-space. We characterize these surfaces in terms of a meromorphic function which satisfies a certain differential equation. Moreover, we show that these surfaces have the geometric property that the asymptotic lines have the same sign of geodesic curvature. As an application we obtain for each constant Chebyshev angle a four-parameter family of complete surfaces.

1. Introduction

Bianchi in [1], [2] studies a class of surfaces with negative Gaussian curvature obtained by generalizing Bäcklund transformation for surfaces with constant negative Gaussian curvature. Fujioka in [7], introduces the notion of generalized Chebyshev nets (a natural generalization of Chebyshev nets for surfaces with constant negative Gaussian curvature) and shows that a Bianchi surface with constant Chebyshev angle parametrized by a generalized Chebyshev net is a piece of a right helicoid; in this case the Chebyshev angle is $\pi/2$.

In this work we obtain a characterization of a class of surfaces with a generalized Chebyshev net and constant Chebyshev angle different from $\pi/2$. The characterization is obtained by showing that the coefficients of the first and second fundamental form of these surfaces depend on a meromorphic function which satisfies a differential equation. The characterization is based on the results obtained in [3], [4] and [5]. We show that these surfaces have the geometric property that the asymptotic lines have the same sign of geodesic curvature. As an application we obtain solutions of this differential equation which allows us to obtain a four-parameter family of complete surfaces.

2. Preliminaries

In the following we consider only surfaces with negative Gaussian curvature in the Euclidean 3-space $\mathbf{R}^3$. Since such a surface has two directions, called the asymptotic directions,

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in which the normal curvature vanishes, we can parametrize the surface locally by asymptotic line coordinates (x, y):

$$\chi : \Omega \subset \mathbf{R}^2 \to \mathbf{R}^3$$

If the Gaussian curvature is $-\frac{1}{\rho^2}$ for a positive function ρ on Ω then the fundamental forms become as follows:

$$I = A^2 dx^2 + 2AB\cos\varphi dxdy + B^2 dy^2\,, \quad II = \frac{2AB\sin\varphi}{\rho} dxdy\,, \tag{1}$$

where $A = |\chi_x|$, $B = |\chi_y|$ and φ is the angle between the asymptotic lines, called the *Chebyshev angle*. Changing the coordinates if necessary, we may assume that $0 < \varphi < \pi$.

For a regular surface $\chi(x, y)$ in $\mathbf{R}^3$, the Gauss-Codazzi equations are given by

$$-EK = (\Gamma_{12}^2)_x - (\Gamma_{11}^2)_y + \Gamma_{12}^1\Gamma_{11}^2 - \Gamma_{11}^1\Gamma_{12}^2 + (\Gamma_{12}^2)^2 - \Gamma_{11}^2\Gamma_{22}^2 \tag{2}$$

$$\begin{aligned} e_y - f_x &= e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2\,, \\ f_y - g_x &= e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\Gamma_{12}^2\,, \end{aligned} \tag{3}$$

where E, F, G, e, f, g are the coefficients of the first and second fundamental form, respectively and Γ_{jk}^i are the Christoffel symbols of the second kind given by

$$\begin{aligned} \Gamma_{11}^1 &= \frac{GE_x - 2FF_x + FE_y}{2(EG - F^2)}\,, \quad \Gamma_{11}^2 = \frac{2EF_x - EE_y - FE_x}{2(EG - F^2)}\,, \\ \Gamma_{12}^1 &= \frac{GE_y - FG_x}{2(EG - F^2)}\,, \quad \Gamma_{12}^2 = \frac{EG_x - FE_y}{2(EG - F^2)}\,, \\ \Gamma_{22}^1 &= \frac{2GF_y - GG_x - FG_y}{2(EG - F^2)}\,, \quad \Gamma_{22}^2 = \frac{EG_y - 2FF_y + FG_x}{2(EG - F^2)}\,. \end{aligned} \tag{4}$$

Given a curve $C : x = x(s), y = y(s)$ on a surface $\chi(x, y)$ where s is arc length. Beltrami's formula for the geodesic curvature at point P of the curve is:

$$\begin{aligned} k_g \quad &= \left[\Gamma_{11}^2\left(\frac{dx}{ds}\right)^3 + (2\Gamma_{12}^2 - \Gamma_{11}^1)\left(\frac{dx}{ds}\right)^2\frac{dy}{ds} + (\Gamma_{22}^2 - 2\Gamma_{12}^1)\frac{dx}{ds}\left(\frac{dy}{ds}\right)^2\right. \\ &\quad \left. -\Gamma_{22}^1\left(\frac{dy}{ds}\right)^3 + \frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{d^2x}{ds^2}\frac{dy}{ds}\right]\sqrt{EG - F^2}\,. \end{aligned} \tag{5}$$

For a surface parametrized by asymptotic lines, if we put $a = \frac{A}{\rho}$, $b = \frac{B}{\rho}$ then the

Gauss-Codazzi equations (3) have the following form [6]:

$$\varphi_{xy} + \left(\frac{\rho_x b \sin\varphi}{2a\rho}\right)_x + \left(\frac{\rho_y a \sin\varphi}{2b\rho}\right)_y - ab\sin\varphi = 0\,. \tag{6}$$

$$\begin{aligned} a_y + \frac{\rho_y}{2\rho}a - \frac{\rho_x}{2\rho}b\cos\varphi = 0\,,\\ b_x + \frac{\rho_x}{2\rho}b - \frac{\rho_y}{2\rho}a\cos\varphi = 0\,. \end{aligned} \tag{7}$$

DEFINITION 1. A parametrization of a surface is called a generalized Chebyshev net if $A = B$, i.e $a = b$.

Now we consider the Liouville equation in two real variables

$$\Delta u = Re^{-2u}\,,\ (x, y) \in \Omega\,, \tag{8}$$

where R is a real constant, u is a real C^2 function and $\Omega \subset \mathbf{R}^2 \simeq \mathbf{C}$ is a planar domain, that is, a connected open subset of the plane.

One hundred and fifty years ago Liouville [8] obtained a local representation formula for the general solution of

$$\frac{\partial^2 \log\lambda}{\partial z\partial\bar{z}} \pm \frac{\lambda}{2a^2} = 0 \tag{9}$$

in terms of a holomorphic function. The changes $\lambda = e^{-2u}$ and $\pm 1/a^2 = R \neq 0$ transform (8) into (9), which is solved by Liouville's remarkable formula

$$u(x, y) = \log\left(\frac{1 + R|h(z)|^2}{2|h'(z)|}\right), \quad z = x + iy\,, \tag{10}$$

where h is holomorphic with $h'(z) \neq 0$ in a neighborhood of a given point.

PROPOSITION 1. *A solution of the equation*

$$\Delta u = Re^{-Su}\,, \quad R, S \in \mathbf{R} \tag{11}$$

is given by

$$u(x, y) = \log\left(\frac{2 + RS|h(z)|^2}{4|h'(z)|}\right)^{\frac{2}{S}}, \quad z = x + iy \in \mathbf{C}\,, \tag{12}$$

where h is a global meromorphic function such that $h'(z) \neq 0$ at all regular points and it admits only simple poles.

PROOF. From (11) we obtain $\Delta\frac{Su}{2} = \frac{RS}{2}e^{-2(\frac{Su}{2})}$. The result follows from (8) and (10). □

PROPOSITION 2. *A surface with a generalized Chebyshev net and negative Gaussian curvature* $K = -\dfrac{1}{e^{2u}}$ *, can be parametrized such that the fundamental forms are given by*

$$I = e^{v+2u}[dx^2 + 2\cos\varphi dxdy + dy^2]\,, \tag{13}$$

$$II = 2e^{v+u}\sin\varphi dxdy\,, \tag{14}$$

where $0 < \varphi < \pi$ *is the Chebyshev angle*, u *and* v *are functions of* x *and* y. *The Gauss-Codazzi equations are given by*

$$2\varphi_{xy} + (u_x \sin\varphi)_x + (u_y \sin\varphi)_y = 2e^v \sin\varphi \tag{15}$$

$$v_x + u_x - u_y \cos\varphi = 0\,,$$

$$v_y + u_y - u_x \cos\varphi = 0\,. \tag{16}$$

PROOF. Putting $A = B = a\rho,\ v = \log(a^2)\,,\quad u = \log(\rho)$ into (1), (6) and (7), the result follows. □

3. Main results

The following Theorem characterizes the surfaces in $\mathbf{R}^3$ with a generalized Chebyshev net, negative Gaussian curvature and constant Chebyshev angle $\varphi \neq \pi/2$.

THEOREM 1. *Let* $M \subset \mathbf{R}^3$ *be a connected orientable Riemann surface and* φ *a constant different of* $\pi/2$. *There exists an immersion* $X : M \to \mathbf{R}^3$ *with a generalized Chebyshev net, negative Gaussian curvature* $K = -\dfrac{1}{e^{2u}}$ *and Chebyshev angle* φ, *if and only if, there exists a global meromorphic function* $h : M \to \mathbf{C}$ *such that* $h'(z) \neq 0$ *at all regular points and it admits only simple poles, satisfying the following*

$$\begin{aligned} &2e^c(1+\varepsilon\cos\varphi)\langle h, (1+\varepsilon i)h'\rangle\langle h', h'\rangle \\ &\quad -[1 + e^c(1+\varepsilon\cos\varphi) \mid h \mid^2]\langle h', (1+\varepsilon i)h''\rangle = 0\,. \end{aligned} \tag{17}$$

Moreover, locally the fundamental forms of X are given by

$$I = e^{(1-\varepsilon\cos\varphi)u+c}[dx^2 + 2\cos\varphi dxdy + dy^2]\,, \tag{18}$$

$$II = 2e^{-\varepsilon\cos\varphi u + c}\sin\varphi dxdy\,, \tag{19}$$

where

$$u(x,y) = \log\left(\frac{1 + e^c(1+\varepsilon\cos\varphi) \mid h(z) \mid^2}{2 \mid h'(z) \mid}\right)^{\frac{2}{1+\varepsilon\cos\varphi}}\,, \tag{20}$$

$$c \in \mathbf{R}\,,\quad z = x + iy \in \mathbf{C}, \varepsilon = \pm 1\,.$$

PROOF. From (15) the Gauss equation is given by

$$\triangle u = 2e^v \tag{21}$$

and the Codazzi equations are given by (16).
Differentiating (16), we have

$$v_{xy} + u_{xy} - u_{yy}\cos\varphi = 0\,,$$
$$v_{yx} + u_{yx} - u_{xx}\cos\varphi = 0\,.$$

From these equations, we obtain

$$(u_{xx} - u_{yy})\cos\varphi = 0\,.$$

Since $\varphi \neq \pi/2$, we get

$$u_{xx} - u_{yy} = 0\,. \tag{22}$$

Similarly, differentiating (16) we have

$$v_{xx} + u_{xx} - u_{yx}\cos\varphi = 0\,,$$
$$v_{yy} + u_{yy} - u_{xy}\cos\varphi = 0\,.$$

We get from these equations and (22) that

$$v_{xx} - v_{yy} = 0\,. \tag{23}$$

Substituting (22) into (21), we obtain

$$u_{xx} = e^{v}\,.$$

Differentiating twice with respect to y , we get

$$u_{xxyy} = e^{v}[(v_y)^2 + v_{yy}]\,. \tag{24}$$

In the same way, substituting (22) into (21), we have

$$u_{yy} = e^{v}\,.$$

Differentiating twice with respect to x, we have

$$u_{yyxx} = e^{v}[(v_x)^2 + v_{xx}]\,. \tag{25}$$

It follows from (23), (24) and (25) that

$$(v_x)^2 - (v_y)^2 = 0\,, \tag{26}$$

hence,

$$v_x + \varepsilon v_y = 0,\ \ \varepsilon = \pm 1\,. \tag{27}$$

Using (16), we get

$$(u_x + \varepsilon u_y)(1 - \varepsilon\cos\varphi) = 0\,,$$

Since $(1 - \varepsilon\cos\varphi) \neq 0$, we obtain

$$u_x + \varepsilon u_y = 0\,, \tag{28}$$

Substituting (28) into (16) we have

$$v_x = -(1+\varepsilon\cos\varphi)u_x \quad \text{and} \quad v_y = -(1+\varepsilon\cos\varphi)u_y\,,$$

and thus

$$v = -(1+\varepsilon\cos\varphi)u + c,\ \ c \in \mathbf{R}\,. \tag{29}$$

By (21) and (29), we obtain

$$\triangle u = 2e^{-(1+\varepsilon\cos\varphi)u+c}\,.$$

By Proposition 1, we get (20). From (20) and (28), we obtain (17). Finally, substituting (29) into (13) and (14) we obtain (18) and (19).

The converse follows from the fundamental theorem of surfaces in $\mathbf{R}^3$. □

COROLLARY 1. *In a surface with a generalized Chebyshev net of negative Gaussian curvature* $K = -\dfrac{1}{e^{2u}}$ *and constant Chebyshev angle* $\varphi \neq \pi/2$, *the asymptotic lines has the same geodesic curvature at least of sign.*

PROOF. Let $X(x, y)$ a parametrization of the surface. From Theorem 1, the coefficients of the first fundamental E, F and G is given by

$$E = G = e^{(1-\varepsilon\cos\varphi)u+c}\ ,\ \ F = \cos\varphi e^{(1-\varepsilon\cos\varphi)u+c}\,. \tag{30}$$

From Beltrami's formula (5), we get that the geodesic curvature of the asymptotic lines is given by

$$(k_g)_{x=constant} = -\Gamma^1_{22}\frac{\sqrt{EG-F^2}}{G\sqrt{G}} \tag{31}$$

$$(k_g)_{y=constant} = \Gamma^2_{11}\frac{\sqrt{EG-F^2}}{E\sqrt{E}} \tag{32}$$

From (4), (28) and (30) we get $\Gamma^2_{11} = -\dfrac{u_y}{2}$ and $\Gamma^1_{22} = \varepsilon\dfrac{u_y}{2}$.
The substitution of this expressions into (31) and (32) gives the result. □

COROLLARY 2. *There exists a four-parameter family of surfaces with a generalized Chebyshev net and constant Chebyshev angle* $\varphi \neq \pi/2$ *whose first and second fundamental forms are given by*

$$I = e^c\left(\frac{1+e^c(1+\varepsilon\cos\varphi)e^{2k(\varepsilon x-y)+2a}}{2\sqrt{2}|k|e^{k(\varepsilon x-y)+a}}\right)^{\frac{2(1-\varepsilon\cos\varphi)}{1+\varepsilon\cos\varphi}}[dx^2+2\cos\varphi dxdy+dy^2]\,, \tag{33}$$

$$II = 2e^c\left(\frac{1+e^c(1+\varepsilon\cos\varphi)e^{2k(\varepsilon x-y)+2a}}{2\sqrt{2}|k|e^{k(\varepsilon x-y)+a}}\right)^{\frac{-2\varepsilon\cos\varphi}{1+\varepsilon\cos\varphi}}\sin\varphi dxdy\,. \tag{34}$$

Moreover, if $c \geq -\log(1+\varepsilon\cos\varphi)$ *then the surfaces defined by* (33) *and* (34) *are complete.*

PROOF. It is easy to verify that the function $h : \mathbf{C} \to \mathbf{C}$ defined by $h = e^{(\varepsilon+i)kz+z_1}$, $z_1 = a+bi$, $a, b, k \in \mathbf{R}$, $k \neq 0$, is a solution to the equation (17). In fact $(1+\varepsilon i)h' = i2kh$ and $(1+\varepsilon i)h'' = i2kh'$ hence $\langle h, (1+\varepsilon i)h'\rangle = 0$ and $\langle h', (1+\varepsilon i)h''\rangle = 0$. Substituting the function h into (20), we obtain

$$u(x, y) = \log\left(\frac{1+e^c(1+\varepsilon\cos\varphi)e^{2k(\varepsilon x-y)+2a}}{2\sqrt{2}|k|e^{k(\varepsilon x-y)+a}}\right)^{\frac{2}{1+\varepsilon\cos\varphi}},$$

hence, substituting $u(x, y)$ above in (18) and (19) we get the expressions (33) and (34). Therefore, from Theorem 1, there exists a four-parameter family $X : \mathbf{C} \to \mathbf{R}^3$ of surfaces with a generalized Chebyshev net and constant Chebyshev angle. On the other hand, if $c \geq -\log(1+\varepsilon\cos\varphi)$, we can show that

$$\left(\frac{1+e^c(1+\varepsilon\cos\varphi)e^{2k(\varepsilon x-y)+2a}}{2\sqrt{2}|k|e^{k(\varepsilon x-y)+a}}\right)^{\frac{2(1-\varepsilon\cos\varphi)}{1+\varepsilon\cos\varphi}} > \left(\frac{1}{\sqrt{2}|k|}\right)^{\frac{2(1-\varepsilon\cos\varphi)}{1+\varepsilon\cos\varphi}} \tag{35}$$

As a consequence of (35) we get that the metric ds^2 of X given by

$$ds^2 = e^c\left(\frac{1+e^c(1+\varepsilon\cos\varphi)e^{2k(\varepsilon x-y)+2a}}{2\sqrt{2}|k|e^{k(\varepsilon x-y)+a}}\right)^{\frac{2(1-\varepsilon\cos\varphi)}{1+\varepsilon\cos\varphi}}[dx^2+2\cos\varphi dxdy+dy^2],$$

satisfies

$$ds^2 > Cd\bar{s}^2, \tag{36}$$

where

$$C = e^c\left(\frac{1}{\sqrt{2}|k|}\right)^{\frac{2(1-\varepsilon\cos\varphi)}{1+\varepsilon\cos\varphi}},$$
$$d\bar{s}^2 = dx^2+2\cos\varphi dxdy+dy^2.$$

Since, the metric $Cd\bar{s}^2$ is complete, it follow that the metric ds^2 also is complete. This concludes the proof of Corollary 2. □

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