

Deformations of a Holomorphic Map and Its Degeneracy Locus

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Abstract. Let $f : X \rightarrow Y$ be a surjective holomorphic map of compact complex manifolds and Δ the degeneracy locus of f . In this paper we shall discuss relationship between infinitesimal deformations of f and the corresponding infinitesimal displacements of Δ in Y . We shall prove that two kinds of Kodaira-Spencer maps are compatible under certain assumptions. As an application of our main theorem, deformations of quadric bundles shall be discussed.

1. Introduction

Let $f : X \rightarrow Y$ be a surjective holomorphic map between compact complex manifolds X and Y of dimension n and m respectively ($n \geq m$). We define the ramification locus R and the degeneracy locus Δ of f as follows:

$$R = \{x \in X \mid f \text{ is not smooth at } x\}, \quad \Delta = f(R).$$

In this paper we shall discuss relationship between infinitesimal deformations of f and the corresponding infinitesimal deformations of Δ .

Let $\{f_t : X_t \rightarrow Y\}_{t \in M}$ be a deformation family of the map f (see §2 for precise definition). We denote by R_t and Δ_t the ramification locus and the degeneracy locus of f_t respectively.

E. Horikawa [6] studied deformations of holomorphic maps. Due to [6] we have the map

$$\tau : T_o(M) \rightarrow D_{X/Y} = \mathbf{H}^1(F : \Theta_X \rightarrow f^*\Theta_Y) \quad (1)$$

of Kodaira-Spencer type, where $T_o(M)$ denotes the tangent space of the base space M at $o \in M$ and $F : \Theta_X \rightarrow f^*\Theta_Y$ is the natural homomorphism induced by $f : X \rightarrow Y$ (see §2 for precise construction).

On the other hand, K. Kodaira [7] studied displacements of submanifolds of a complex manifold. Applying arguments of [7] to the family $\{\Delta_t\}_{t \in M}$, we have the map

$$\rho : T_o(M) \rightarrow H^0(\Delta, N_{\Delta/Y}), \quad (2)$$

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which is another type of Kodaira-Spencer map, where $N_{\Delta/Y}$ denotes the normal sheaf of Δ in Y .

Now we ask the following question.

Question. *How are the maps τ and ρ related to each other?*

In the papers [2], [3], [4] and [5] we discussed this question in case where $f : X \rightarrow Y$ is a conic bundle and $\dim X = 3$. In this paper we shall generalize [2, Theorem 2.12] as follows (see Theorem 1 for precise statement).

- (i) Let $f : X \rightarrow Y$ be a surjective holomorphic map between compact complex manifolds. We assume that Δ is a *smooth* submanifold of Y of codimension one. We furthermore assume that $f|_R : R \rightarrow \Delta$ is *isomorphic*.
- (ii) Due to [6], we have the natural homomorphism

$$P : D_{X/Y} \rightarrow H^0(X, \mathcal{S}_{X/Y}), \quad (3)$$

where $\mathcal{S}_{X/Y} = \text{Coker}(F : \Theta_X \rightarrow f^*\Theta_Y)$.

- (iii) Let R' be the scheme-theoretic ramification locus of f and $j : R \rightarrow R'$ the natural inclusion map (cf. Definition 1). Then $\mathcal{S}_{X/Y}$ is an $\mathcal{O}_{R'}$ -module (cf. Proposition 1). The map j induces the map

$$j^* : H^0(X, \mathcal{S}_{X/Y}) \rightarrow H^0(R, j^*\mathcal{S}_{X/Y}). \quad (4)$$

- (iv) We shall prove that there exists a natural isomorphism

$$\psi : H^0(R, j^*\mathcal{S}_{X/Y}) \rightarrow H^0(\Delta, N_{\Delta/Y}) \quad (5)$$

(cf. Proposition 3).

- (v) Moreover we shall prove that

$$\rho = \psi \circ j^* \circ P \circ \tau, \quad (6)$$

which shows the compatibility of τ and ρ (cf. Theorem 1).

In §5, §6 and §7 we shall discuss deformations of quadric bundles, especially of conic bundles, applying Theorem 1 and its corollary (Corollary 1). We shall study local structures of a quadric bundle in §5. In §6 we shall give a general formula on the direct image sheaf $f_*\Theta_{X/Y}$ for a conic bundle $f : X \rightarrow Y$, which is a generalization of [3, Theorem 3.3] (see Theorem 2 for precise statement). Using these results, we shall finally discuss a kind of rigidity of a conic bundle; we shall prove that certain conic bundles admit no non-trivial small deformation families fixing the discriminant loci, which is a generalization of [3, Corollary 3.14] to higher-dimensional cases (see Corollary 4 for precise statement). In §7 we shall prove a technical lemma (Lemma 12) which is needed to prove Theorem 2.

2. Preliminaries

First we shall briefly recall the deformation theory of holomorphic maps due to E. Horikawa [6] in order to fix our notation.

Let Y be a compact complex manifold of dimension m . By a family of holomorphic maps into Y , we mean a quadruplet $(\mathcal{X}, \Phi, p, M)$ of complex manifolds \mathcal{X} , M and holomorphic maps $\Phi : \mathcal{X} \rightarrow Y \times M$, $p : \mathcal{X} \rightarrow M$ with the following properties:

- (i) p is smooth and proper.
- (ii) $q \circ \Phi = p$, where $q : Y \times M \rightarrow M$ denotes the second projection.

Putting $X_t = p^{-1}(t)$ and $f_t = \Phi|_{X_t}$ for $t \in M$, we denote the family $(\mathcal{X}, \Phi, p, M)$ by $\{f_t : X_t \rightarrow Y\}_{t \in M}$. Let $o \in M$, $X = X_o$ and $f = f_o$. Then the family $\{f_t : X_t \rightarrow Y\}_{t \in M}$ is called a deformation family of $f : X \rightarrow Y$.

From now on, $\dim X$ shall be supposed to be n unless otherwise mentioned.

Let $F : \Theta_X \rightarrow f^*\Theta_Y$ be the natural homomorphism induced by f . We put $\Theta_{X/Y} = \text{Ker}(f)$, $\mathcal{S}_{X/Y} = \text{Coker}(F)$ and $D_{X/Y} = \mathbf{H}^1(F : \Theta_X \rightarrow f^*\Theta_Y)$. Then we have an exact sequence

$$0 \rightarrow H^1(X, \Theta_{X/Y}) \rightarrow D_{X/Y} \rightarrow H^0(X, \mathcal{S}_{X/Y}) \rightarrow H^2(X, \Theta_{X/Y}). \quad (7)$$

Horikawa showed that the infinitesimal deformations of f are classified by $D_{X/Y}$. He also defined a kind of Kodaira-Spencer map $\tau : T_o(M) \rightarrow D_{X/Y}$.

Let $\mathcal{U} = \{U_i\}$ be a finite Stein open covering of X . For a sheaf \mathcal{F} on X , we denote the group of the q -cochains and the q -cocycles with coefficients in \mathcal{F} by $C^q(\mathcal{U}, \mathcal{F})$ and $Z^q(\mathcal{U}, \mathcal{F})$ respectively. We denote the q -th coboundary map by $\delta : C^q(\mathcal{U}, \mathcal{F}) \rightarrow C^{q+1}(\mathcal{U}, \mathcal{F})$. Then we have

$$D_{X/Y} = \frac{\{(\tau, \sigma) \in C^0(\mathcal{U}, f^*\Theta_Y) \times Z^1(\mathcal{U}, \Theta_X) \mid \delta(\tau) = F(\sigma)\}}{\{(F(g), \delta(g)) \mid g \in C^0(\mathcal{U}, \Theta_X)\}} \quad (8)$$

The map τ is determined as follows. Shrinking M , if necessary, we assume the following.

- (i) M is an open subset of \mathbf{C}^k with coordinates $t = (t_1, \dots, t_k)$ and $o = (0, \dots, 0)$.
- (ii) \mathcal{X} is covered by a finite number of Stein coordinate open sets $\{\mathcal{U}_i\}$. Each \mathcal{U}_i is covered by a system of coordinates (z_i, t) such that $p(z_i, t) = t$, where $(z_i, t) = (z_i^1, \dots, z_i^n, t_1, \dots, t_k)$.
- (iii) $\Phi(\mathcal{U}_i) \subset V_i \times M$, where V_i is an Stein open subset of Y covered by a system of coordinates $w_i = (w_i^1, \dots, w_i^m)$.
- (iv) Φ is given by $w_i^l = \Phi_i^l(z_i, t)$ for $l = 1, \dots, m$.
- (v) $(z_i, t) \in \mathcal{U}_i$ coincides with $(z_j, t) \in \mathcal{U}_j$ if and only if $z_i^l = \phi_{ij}^l(z_j^1, \dots, z_j^n, t)$ for $l = 1, \dots, n$.
- (vi) $w_i \in V_i$ coincides with $w_j \in V_j$ if and only if $w_i^l = \psi_{ij}^l(w_j^1, \dots, w_j^m)$ for $l = 1, \dots, m$.

Let $U_i = \mathcal{U}_i \cap X$ and \mathcal{U} denote the covering $\{U_i\}$ of X . Let $T_o(M)$ denote the tangent space of M at $o \in M$, that is to say, $T_o(M)$ is the complex vector space of dimension k whose elements are tangent vectors of M at $o \in M$ of the following form:

$$\frac{\partial}{\partial t} = \sum_{l=1}^k c_l \frac{\partial}{\partial t_l},$$

where $c_l \in \mathbb{C}$ for $1 \leq l \leq k$. For any element $\partial/\partial t \in T_o(M)$, we put

$$\tau_i = \sum_{l=1}^m \frac{\partial \Phi_i^l}{\partial t} \Big|_{t=0} \cdot \frac{\partial}{\partial w_i^l} \in \Gamma(U_i, f^* \Theta_Y), \quad (9)$$

$$\sigma_{ij} = \sum_{l=1}^n \frac{\partial \phi_{ij}^l}{\partial t} \Big|_{t=0} \cdot \frac{\partial}{\partial z_i^l} \in \Gamma(U_{ij}, \Theta_X). \quad (10)$$

Then $\tau = \{\tau_i\} \in C^0(\mathcal{U}, f^* \Theta_Y)$ and $\sigma = \{\sigma_{ij}\} \in Z^1(\mathcal{U}, \Theta_X)$ represents an element of $D_{X/Y}$, which we define to be $\tau(\partial/\partial t)$. Thus we can define the map

$$\tau : T_o(M) \rightarrow D_{X/Y} \quad (11)$$

Let $P : f^* \Theta_Y \rightarrow \mathcal{S}_{X/Y}$ be the natural homomorphism. For an element of $D_{X/Y}$, we take a representative $(\tau, \sigma) \in C^0(\mathcal{U}, f^* \Theta_Y) \times Z^1(\mathcal{U}, \Theta_X)$ with $\tau = \{\tau_i\}$ and $\sigma = \{\sigma_{ij}\}$. Then the collection $\{P(\tau_i)\}$ patches together to an element of $H^0(X, \mathcal{S}_{X/Y})$. In this way, we can define the map

$$P : D_{X/Y} \rightarrow H^0(X, \mathcal{S}_{X/Y}), \quad (12)$$

which is nothing but the homomorphism appearing in (7), where we use the same symbol P as above.

Next we shall briefly recall infinitesimal displacements of a divisor on a complex manifold.

Let Y be a compact complex manifold of dimension m and $\{V_i\}$ a Stein open covering of Y . Assume that each V_i is a sufficiently small open set with a system of coordinates $w_i = (w_i^1, \dots, w_i^m)$. Let Δ be a smooth divisor on Y defined locally by $w_i^m = 0$ and \mathcal{I} the defining ideal sheaf of Δ in Y . Then we have $\Gamma(V_i, \mathcal{I}) = w_i^m \Gamma(V_i, \mathcal{O}_Y)$.

Now we define the homomorphism $\zeta_i : \Gamma(\Delta \cap V_i, \mathcal{I}/\mathcal{I}^2) \rightarrow \Gamma(\Delta \cap V_i, \mathcal{O}_\Delta)$ by $\zeta_i(w_i^m \bmod \mathcal{I}^2) = 1$. Then we have

$$\Gamma(\Delta \cap V_i, N_{\Delta/Y}) = \Gamma(\Delta \cap V_i, \mathcal{O}_\Delta) \cdot \zeta_i. \quad (13)$$

Let $\{\Delta_t\}_{t \in M}$ be a family of displacements of $\Delta = \Delta_o$ defined by

$$w_i^m = \varepsilon_i(w_i^1, \dots, w_i^{m-1}, t) \quad (14)$$

on V_i , where $\varepsilon_i(w_i^1, \dots, w_i^{m-1}, 0) = 0$. For any element $\partial/\partial t \in T_o(M)$, we put

$$\rho_i \left(\frac{\partial}{\partial t} \right) = \left(\frac{\partial \varepsilon_i}{\partial t} \Big|_{t=0} \bmod \mathcal{I} \right) \cdot \zeta_i. \quad (15)$$

Then the collection $\{\rho_i(\partial/\partial t)\}$ patches together to an element $\rho(\partial/\partial t)$ of $H^0(\Delta, N_{\Delta/Y})$. In this way we can define a map

$$\rho : T_o(M) \rightarrow H^0(\Delta, N_{\Delta/Y}), \quad (16)$$

which is the Kodaira-Spencer map with respect to infinitesimal displacements of a smooth divisor.

3. Basic lemmas and propositions

Let X and Y be compact complex manifolds of dimension n and m respectively ($n \geq m$). Let $f : X \rightarrow Y$ be a surjective holomorphic map and $\{f_t : X_t \rightarrow Y\}_{t \in M}$ a deformation family of $f = f_o$.

We denote by R and Δ the ramification locus and the degeneracy locus of f respectively:

$$R = \{x \in X \mid f \text{ is not smooth at } x\}, \quad \Delta = f(R).$$

Let us furthermore denote by $J_i(z_i, t) = J_i(z_i^1, \dots, z_i^n, t_1, \dots, t_k)$ the Jacobian matrix of $f_t|_{U_i}$:

$$J_i(z_i, t) = \left(\frac{\partial \Phi_i^l}{\partial z_i^q} \right)_{1 \leq l \leq m, 1 \leq q \leq n}. \quad (17)$$

We denote by $D_1^{(i)}, \dots, D_s^{(i)}$ the minor determinants of $J_i(z_i, 0)$ of degree m , where $s = \binom{n}{m} = n!/(m!(n-m)!)$.

DEFINITION 1. We denote by R' the complex subspace of X defined locally by the ideal $(D_1^{(i)}, \dots, D_s^{(i)})$ on U_i . We call R' the *scheme-theoretic ramification locus* of f .

It is easy to see the definition of R' above does not depend on the choice of local coordinates. Let us note that R' is not necessarily reduced and that $R'_{\text{red}} = R$, since $R \cap U_i$ is the locus at which the rank of $J_i(z_i, 0)$ is less than m . We have the natural inclusion map $j : R \rightarrow R'$. If R' is reduced itself, then j is an isomorphism.

We also use the following notation. We denote by $\iota : \Delta \rightarrow Y$ and $h : R' \rightarrow X$ the natural inclusion maps. We put $g = f \circ h : R' \rightarrow Y$. Then the map g factors through $g' : R' \rightarrow \Delta$, that is to say, $g = \iota \circ g'$. We put $g'' = g' \circ j : R \rightarrow \Delta$.

From now on, we assume that the degeneracy locus Δ of f is a smooth submanifold of Y of codimension one and that $g'' : R \rightarrow \Delta$ is isomorphic. Then, after suitable change of coordinates, we may assume the following:

- (i) $\Delta \cap V_i$ is defined by $w_i^m = 0$ on each V_i ;
- (ii) $R \cap U_i$ is defined by $z_i^m = z_i^{m+1} = \dots = z_i^n = 0$ on each U_i .

We put $z'_i = (z_i^1, \dots, z_i^{m-1})$, $z''_i = (z_i^m, z_i^{m+1}, \dots, z_i^n)$, $w'_i = (w_i^1, \dots, w_i^{m-1})$ and $\Phi'_i = (\Phi_i^1, \dots, \Phi_i^{m-1})$. We shall write $z_i = (z_i^1, \dots, z_i^n) = (z'_i, z''_i)$, $w_i = (w_i^1, \dots, w_i^m) = (w'_i, w_i^m)$, $\Phi_i^l(z_i^1, \dots, z_i^n, t_1, \dots, t_k) = \Phi_i^l(z'_i, z''_i, t)$ ($1 \leq l \leq m$), $\Phi_i = (\Phi_i^1, \dots, \Phi_i^m) = (\Phi'_i, \Phi_i^m)$, $J_i(z_i, t) = J_i(z'_i, z''_i, t)$, and so on. Since g'' is isomorphic, we may furthermore assume the following:

- (iii) $\Phi_i^l(z'_i, 0, 0) = z_i^l$ for $l = 1, \dots, m-1$.

Under the assumptions and the notation above, we have the following lemmas.

LEMMA 1. We have $\frac{\partial \Phi_i^m}{\partial z_i^q}(z'_i, 0, 0) = 0$ for $1 \leq q \leq m-1$.

PROOF. Since $f(R) = \Delta$, we have $\Phi_i^m(z'_i, 0, 0) = 0$. Then we have

$$\frac{\partial \Phi_i^m}{\partial z_i^q}(z'_i, 0, 0) = \frac{\partial (\Phi_i^m(z'_i, 0, 0))}{\partial z_i^q} = 0$$

for $1 \leq q \leq m-1$. □

LEMMA 2. We have $\frac{\partial \Phi_i^l}{\partial z_i^q}(z'_i, 0, 0) = \delta_{lq}$ for $1 \leq l \leq m-1$ and $1 \leq q \leq m-1$, where δ_{lq} denotes Kronecker's symbol.

PROOF. It is straightforward from the assumption (iii) above. □

LEMMA 3. We have $\frac{\partial \Phi_i^m}{\partial z_i^q}(z'_i, 0, 0) = 0$ for $m \leq q \leq n$.

PROOF. From Lemma 1 and Lemma 2, we can write $J_i(z'_i, 0, 0)$ in the following form:

$$J_i(z'_i, 0, 0) = \begin{pmatrix} E_{m-1} & * \\ {}^t\mathbf{0} & {}^t\mathbf{a} \end{pmatrix},$$

where E_{m-1} denotes the unit matrix. Since the rank of $J_i(z'_i, 0, 0)$ is less than m , we have $\mathbf{a} = \mathbf{0}$, that is to say, $(\partial \Phi_i^m / \partial z_i^q)(z'_i, 0, 0) = 0$ for $m \leq q \leq n$. □

Now we have the following propositions on $\mathcal{S}_{X/Y}$.

PROPOSITION 1. The sheaf $\mathcal{S}_{X/Y}$ is an $\mathcal{O}_{R'}$ -module.

PROOF. Let p be a point of R . We discuss locally around p . Let U_i be a small neighbourhood of p with a system z_i of coordinates satisfying the conditions (i) to (iii) above. We may assume that p is defined by $z_i^1 = \dots = z_i^n = 0$.

Let us put $A = \mathcal{O}_{X,p}$. Then we have $A \cong \mathbf{C}\{z_i^1, \dots, z_i^n\}$, where $\mathbf{C}\{z_i^1, \dots, z_i^n\}$ denotes the convergent power series ring. We also have $\Theta_{X,p} = \bigoplus_{q=1}^n A \cdot (\partial/\partial z_i^q) \cong A^n$ and $f^*\Theta_{Y,p} = \bigoplus_{l=1}^m A \cdot (\partial/\partial w_i^l) \cong A^m$. Via these isomorphisms, the homomorphism $F : \Theta_{X,p} \rightarrow f^*\Theta_{Y,p}$ is determined by multiplying the matrix $J_i(z_i', z_i'', 0) \in M(m, n; A)$:

$$F : \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} \mapsto J_i(z_i', z_i'', 0) \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix},$$

where $\varphi_1, \dots, \varphi_n \in A$. Let us write $J_i(z_i', z_i'', 0)$ as follows:

$$J_i(z_i', z_i'', 0) = \begin{pmatrix} J' & J'' \\ {}^t\mathbf{c} & {}^t\mathbf{d} \end{pmatrix},$$

where $J' = J'(z_i', z_i'', 0) \in M(m-1, m-1; A)$, $J'' \in M(m-1, n-m+1; A)$, $\mathbf{c} \in A^{m-1}$ and $\mathbf{d} \in A^{n-m+1}$. Since $J'(z_i', 0, 0) = E_{m-1}$ by Lemma 2, we have $J'(z_i', z_i'', 0) \in GL(m-1, A)$.

Then Proposition 1 follows from Lemma 4 below. \square

LEMMA 4. *Let $B = (b_{ij}) = \begin{pmatrix} B' & B'' \\ {}^t\mathbf{c} & {}^t\mathbf{d} \end{pmatrix} \in M(m, n; A)$, where $B' \in M(m-1, m-1; A)$, $B'' \in M(m-1, n-m+1; A)$, $\mathbf{c} \in A^{m-1}$ and $\mathbf{d} \in A^{n-m+1}$. Let $T_B : A^n \rightarrow A^m$ be the map defined by multiplying B . Let D_1, \dots, D_s be the minor determinants of B of degree m , where $s = \binom{n}{m}$, and I the ideal of A generated by $\{D_1, \dots, D_s\}$. Assume that $B' \in GL(m-1, A)$. Then $\text{Coker}(T_B)$ is an A/I -module.*

PROOF. The module $\text{Coker}(T_B)$ and the ideal I do not vary after replacing B by BP , where $P \in GL(n, A)$. Taking $\begin{pmatrix} E_{m-1} & -B'^{-1}B'' \\ O & E_{n-m+1} \end{pmatrix}$ as the matrix P , we may assume that $B'' = O$. Then the ideal I is generated by $\{b_{ml} \mid m \leq l \leq n\}$. In fact, non-zero minors of B are the form

$$\begin{vmatrix} B' & \mathbf{0} \\ {}^t\mathbf{c} & b_{ml} \end{vmatrix} = \det B' \cdot b_{ml}$$

with $m \leq l \leq n$. Note that $\det B'$ is a unit of A .

Now let $\psi = (\psi_i)$ be any element of A^m . Then, for each l with $m \leq l \leq n$, there exists an element $\varphi \in A^n$ satisfying $B\varphi = b_{ml}\psi$. In fact, if we put

$$\begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{m-1} \end{pmatrix} = B'^{-1} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{m-1} \end{pmatrix}$$

and

$$\varphi_i = \begin{cases} b_{ml}\eta_i, & \text{if } 1 \leq i \leq m-1; \\ \psi_m - \sum_{j=1}^{m-1} b_{mj}\eta_j, & \text{if } i = l; \\ 0, & \text{otherwise,} \end{cases}$$

then $\varphi = (\varphi_i)$ satisfies $B\varphi = b_{ml}\psi$. Therefore we have $b_{ml}\psi \in \text{Im}(T_B)$, whence each element of I is an annihilator of $\text{Coker}(T_B)$. Thus Lemma 4 is proved. \square

PROPOSITION 2. (1) *The sheaf $j^*\mathcal{S}_{X/Y}$ is an invertible \mathcal{O}_R -module.*

(2) *We have a natural surjective homomorphism $\bar{P} : g''^*(\Theta_Y \otimes \mathcal{O}_\Delta) \rightarrow j^*\mathcal{S}_{X/Y}$. Let $p \in R$. We take suitably as before a small open neighbourhood U_i of p and coordinates (z_i) and (w_i) on U_i and $f(U_i)$ respectively. Then, taking a suitable local generator v_i of $j^*\mathcal{S}_{X/Y}$ on $R \cap U_i$, we have*

$$\Gamma(R \cap U_i, j^*\mathcal{S}_{X/Y}) = \Gamma(R \cap U_i, \mathcal{O}_R) \cdot v_i$$

and

$$\bar{P} : \sum_{l=1}^m \varphi_l \frac{\partial}{\partial w_i^l} \mapsto \varphi_m v_i,$$

where $\varphi_l \in \Gamma(R \cap U_i, \mathcal{O}_R)$, $l = 1, \dots, m$.

PROOF. Pulling back the exact sequence $\Theta_X \rightarrow f^*\Theta_Y \rightarrow \mathcal{S}_{X/Y} \rightarrow 0$ by $h \circ j$ and noting that $j^*h^*f^*\Theta_Y = g''^*(\Theta_Y \otimes \mathcal{O}_\Delta)$, we have the exact sequence:

$$j^*h^*\Theta_X \xrightarrow{\bar{F}} g''^*(\Theta_Y \otimes \mathcal{O}_\Delta) \xrightarrow{\bar{P}} j^*\mathcal{S}_{X/Y} \rightarrow 0. \quad (18)$$

Let p be a point of R . We furthermore put $\bar{A} = \mathcal{O}_{R,p}$. Then we have $j^*h^*\Theta_{X,p} = \bigoplus_{q=1}^n \bar{A} \cdot (\partial/\partial z_i^q) \cong \bar{A}^n$ and $g''^*(\Theta_Y \otimes \mathcal{O}_\Delta) = \bigoplus_{l=1}^m \bar{A} \cdot (\partial/\partial w_i^l) \cong \bar{A}^m$. Via these isomorphisms, the homomorphism \bar{F} above is determined by multiplying the matrix $J_i(z'_i, 0, 0)$. By Lemma 1, Lemma 2 and Lemma 3, we have

$$J_i(z'_i, 0, 0) = \begin{pmatrix} E_{m-1} & J'' \\ {}_t\mathbf{0} & {}_t\mathbf{0} \end{pmatrix},$$

where $J'' \in M(m-1, n-m+1; \bar{A})$. Then we have

$$\text{Im}(\bar{F}) = \left\{ \sum_{l=1}^{m-1} \varphi_l \frac{\partial}{\partial w_i^l} \mid \varphi_l \in \bar{A}, l = 1, \dots, m-1 \right\}$$

and $\text{Coker}(\bar{F}) \cong \bar{A}$, which implies the assertion (1). Putting $v_i = \bar{P}(\partial/\partial w_i^m)$, we can check the assertion (2).

Thus Proposition 2 is proved. \square

4. Compatibility of Kodaira-Spencer maps

In this section we shall discuss compatibility of the maps τ and ρ .

First we shall construct a natural isomorphism $\psi : g''_* j^* \mathcal{S}_{X/Y} \rightarrow N_{\Delta/Y}$. In [2] we proved that there exists such an isomorphism if $f : X \rightarrow Y$ is a conic bundle with $\dim X = 3$ and Δ is smooth. Here we shall generalize arguments in [2].

Under the same assumptions as before, we shall first construct two homomorphisms $\lambda : \Theta_Y \otimes \mathcal{O}_\Delta \rightarrow g''_* j^* \mathcal{S}_{X/Y}$ and $\mu : \Theta_Y \otimes \mathcal{O}_\Delta \rightarrow N_{\Delta/Y}$ as follows.

Applying g''_* to $\bar{P} : g''^*(\Theta_Y \otimes \mathcal{O}_\Delta) \rightarrow j^* \mathcal{S}_{X/Y}$ and noting that g'' is isomorphic by assumption, we have the homomorphism

$$\lambda : \Theta_Y \otimes \mathcal{O}_\Delta \rightarrow g''_* j^* \mathcal{S}_{X/Y}. \quad (19)$$

Let $p \in R$ and $p' = f(p) \in \Delta$. Under the same notation as before, we can describe λ locally around p' as follows:

$$\lambda : \sum_{l=1}^m \varphi_l \frac{\partial}{\partial w_i^l} \mapsto \varphi_l v_i, \quad (20)$$

where $\varphi_l \in \Gamma(\Delta \cap V_i, \mathcal{O}_\Delta)$, $l = 1, \dots, m$.

On the other hand, we have the standard homomorphism $\mu : \Theta_Y \otimes \mathcal{O}_\Delta \rightarrow N_{\Delta/Y}$, which we can describe locally around p' as follows. Since $\Delta \cap V_i$ is defined by $w_i^m = 0$, we have $\mathcal{I}|_{V_i} = (w_i^m)$, where \mathcal{I} denotes the defining ideal of Δ in Y . We define a homomorphism $\zeta_i : \Gamma(\Delta \cap V_i, \mathcal{I}/\mathcal{I}^2) \rightarrow \Gamma(\Delta \cap V_i, \mathcal{O}_\Delta)$ in the same way as in §2, that is to say, we define ζ_i by $\zeta_i(w_i^m \bmod \mathcal{I}^2) = 1$. Then we have

$$\Gamma(\Delta \cap V_i, N_{\Delta/Y}) = \Gamma(\Delta \cap V_i, \mathcal{O}_\Delta) \cdot \zeta_i \quad (21)$$

(cf. (13)). The homomorphism μ is described as follows:

$$\mu : \sum_{l=1}^m \varphi_l \frac{\partial}{\partial w_i^l} \mapsto \varphi_l \zeta_i, \quad (22)$$

where $\varphi_l \in \Gamma(\Delta \cap V_i, \mathcal{O}_\Delta)$, $l = 1, \dots, m$.

PROPOSITION 3. (1) *The homomorphisms λ and μ are both surjective.*

(2) *We have $\text{Ker}(\lambda) = \text{Ker}(\mu)$.*

(3) *There exists an isomorphism $\psi : g''_* j^* \mathcal{S}_{X/Y} \rightarrow N_{\Delta/Y}$ that satisfies $\mu = \psi \circ \lambda$. Moreover, ψ is locally determined by $\psi(v_i) = \zeta_i$.*

PROOF. Straightforward from the local descriptions of λ and μ . □

The isomorphism ψ above induces the isomorphism

$$\psi : H^0(R, j^* \mathcal{S}_{X/Y}) \rightarrow H^0(\Delta, N_{\Delta/Y}), \quad (23)$$

where we use the same symbol ψ as above.

Then we have the following theorem.

THEOREM 1. *Let $f : X \rightarrow Y$ be a surjective holomorphic map between compact complex manifolds. Let R , R' and Δ denote the ramification locus, the scheme-theoretic ramification locus and the degeneracy locus of f , respectively. Assume that Δ is a smooth submanifold of Y of codimension one and that $g'' = f|_R : R \rightarrow \Delta$ is an isomorphism. Let $\{f_t : X_t \rightarrow Y\}_{t \in M}$ be a deformation family of $f : X \rightarrow Y$ with $X_o = X$ and $f_o = f$. Let $\tau : T_o(M) \rightarrow D_{X/Y}$ denote the Kodaira-Spencer map due to E. Horikawa. Let Δ_t denote the degeneracy locus of f_t and $\rho : T_o(M) \rightarrow H^0(\Delta, N_{\Delta/Y})$ the Kodaira-Spencer map of the family $\{\Delta_t\}_{t \in M}$ of displacements of Δ . Then we have*

$$\rho = \psi \circ j^* \circ P \circ \tau,$$

where ψ is the isomorphism above, $j : R \rightarrow R'$ the natural inclusion map, and $P : D_{X/Y} \rightarrow H^0(X, S_{X/Y})$ the map of (12).

PROOF. Let $p \in R$ and $p' = f(p) \in \Delta$. We discuss locally around p' . We take open neighbourhoods U_i and V_i of p and p' , respectively, and coordinates (z_i) and (w_i) as before.

Let R_t be the ramification locus of f_t . Suppose that the family $\{R_t \cap U_i\}_{t \in M}$ is determined by the equations

$$z_i^q = \eta_i^q(z'_i, t) \quad \text{for } m \leq q \leq n, \quad (24)$$

where $\eta_i^q(z'_i, 0) = 0$. Let us put

$$\eta_i''(z'_i, t) = (\eta_i^m(z'_i, t), \dots, \eta_i^n(z'_i, t)). \quad (25)$$

Suppose that the family $\{\Delta_t \cap V_i\}_{t \in M}$ is determined by the equation

$$w_i^m = \varepsilon_i(w'_i, t) \quad (26)$$

with $\varepsilon_i(w'_i, 0) = 0$.

Since we have $f_t(R_t) = \Delta_t$, we obtain

$$\Phi_i^m(z'_i, \eta_i''(z'_i, t), t) = \varepsilon_i(\Phi_i'(z'_i, \eta_i''(z'_i, t), t), t). \quad (27)$$

Let $\partial/\partial t$ be any element of $T_o(M)$. Putting $t = 0$ after applying $\partial/\partial t$ to the equality (27) above, we have

$$\begin{aligned} & \sum_{q=m}^n \frac{\partial \Phi_i^m}{\partial z_i^q}(z'_i, 0, 0) \cdot \frac{\partial \eta_i^q}{\partial t}(z'_i, 0) + \frac{\partial \Phi_i^m}{\partial t}(z'_i, 0, 0) \\ &= \sum_{l=1}^{m-1} \frac{\partial \varepsilon_i}{\partial w_i^l}(\Phi_i'(z'_i, 0, 0), 0) \cdot \frac{\partial \Phi_i^l(z'_i, \eta_i''(z'_i, t), t)}{\partial t} \Big|_{t=0} + \frac{\partial \varepsilon_i}{\partial t}(\Phi_i'(z'_i, 0, 0), 0). \end{aligned}$$

By Lemma 3 we have

$$\frac{\partial \Phi_i^m}{\partial z_i^q}(z'_i, 0, 0) = 0 \quad \text{for } m \leq q \leq n. \quad (29)$$

Since $\varepsilon_i(w'_i, 0) = 0$, we also have

$$\frac{\partial \varepsilon_i}{\partial w_i^l}(\Phi'_i(z'_i, 0, 0), 0) = 0 \quad \text{for } 1 \leq l \leq m-1, \quad (30)$$

whence we obtain:

$$\frac{\partial \Phi_i^m}{\partial t}(z'_i, 0, 0) = \frac{\partial \varepsilon_i}{\partial t}(\Phi'_i(z'_i, 0, 0), 0). \quad (31)$$

On the other hand, $\psi \circ j^* \circ P(\partial/\partial t)$ and $\rho(\partial/\partial t)$ are calculated locally as follows. Let us put

$$\tau_i = \sum_{l=1}^m \frac{\partial \Phi_i^l}{\partial t} \Big|_{t=0} \cdot \frac{\partial}{\partial w_i^l} \in \Gamma(U_i, f^* \Theta_Y). \quad (32)$$

Then, by Proposition 2, we have

$$j^* \circ P(\tau_i) = \frac{\partial \Phi_i^m}{\partial t}(z'_i, 0, 0) \cdot v_i, \quad (33)$$

whence, by Proposition 3, we obtain

$$\psi \circ j^* \circ P \left(\frac{\partial}{\partial t} \right) \Big|_{V_i} = \frac{\partial \Phi_i^m}{\partial t}(w'_i, 0, 0) \cdot \zeta_i, \quad (34)$$

while we have

$$\rho \left(\frac{\partial}{\partial t} \right) \Big|_{V_i} = \frac{\partial \varepsilon_i}{\partial t}(w'_i, 0) \cdot \zeta_i. \quad (35)$$

Noting that $\Phi'_i(z'_i, 0, 0) = z'_i$, we have

$$\psi \circ j^* \circ P \left(\frac{\partial}{\partial t} \right) \Big|_{V_i} = \rho \left(\frac{\partial}{\partial t} \right) \Big|_{V_i} \quad (36)$$

by (31) above.

Thus Theorem 1 is proved. \square

COROLLARY 1. *Assume furthermore that R' is reduced, that is to say, $j : R \rightarrow R'$ is an isomorphism. Then we have an exact sequence*

$$0 \rightarrow H^1(X, \Theta_{X/Y}) \rightarrow D_{X/Y} \xrightarrow{\psi \circ j^* \circ P} H^0(\Delta, N_{\Delta/Y})$$

with $\rho = \psi \circ j^* \circ P \circ \tau$. In particular, if furthermore $H^1(X, \Theta_{X/Y}) = 0$, then there does not exist non-trivial small deformation of $f : X \rightarrow Y$ with the same degeneracy locus Δ .

PROOF. Straightforward from the assumption and the exact sequence (7). \square

EXAMPLE 1. Let $f : X \rightarrow Y$ be a double cover branching along a smooth divisor Δ of Y . Then $g'' : R \rightarrow \Delta$ is an isomorphism and R' is reduced. In fact, choosing local coordinates (z_i) and (w_i) around $p \in R$ and $f(p) \in \Delta$ suitably, we may assume that f is locally defined by $(w_i^1, \dots, w_i^m) = (z_i^1, \dots, z_i^{m-1}, (z_i^m)^2)$. Then both R and R' are defined by the ideal (z_i^m) locally around p .

EXAMPLE 2. Let $f : X \rightarrow Y$ be a quadric bundle (see §5 for the definition). Assume that $\dim Y = 2$ and that f is ordinary, that is to say, every singular fibre has only one singular point. Then $g'' : R \rightarrow \Delta$ is an isomorphism. Moreover, we can prove that R' is reduced (cf. Corollary 2 in §5).

EXAMPLE 3. Let $\dim X = 3$ and $\dim Y = 2$. If f is locally defined by

$$\begin{cases} w_i^1 = z_i^1, \\ w_i^2 = (z_i^2)^2 - (z_i^3)^3, \end{cases}$$

then R' is locally defined by the ideal $(z_i^2, (z_i^3)^2)$, while R is defined by (z_i^2, z_i^3) , whence j is not isomorphic.

5. Local descriptions of quadric bundles

In the rest of this paper, we shall discuss deformations of quadric bundles. In [2], [3], [4] and [5] we discussed deformations of a conic bundle $f : X \rightarrow Y$ with $\dim X = 3$. Here we shall generalize some of the results in [2] and [3] to higher-dimensional cases.

Let us explain more precisely. In §4 we have proved Corollary 1, which claims a kind of rigidity of a holomorphic map. We recall Corollary 1 here, since it shall play a central role in our later discussions.

Let $f : X \rightarrow Y$ be a surjective holomorphic map of compact complex manifolds. We use the same notation as before. Let Δ , R , R' denote the degeneracy locus, the ramification locus and the scheme-theoretic ramification locus respectively. Assume that the following four conditions (a), (b), (c) and (d) are satisfied:

- (a) The degeneracy locus Δ of f is a smooth submanifold of Y of codimension one;
- (b) The scheme-theoretic ramification locus R' of f is reduced and the natural inclusion map $j : R \rightarrow R'$ is an isomorphism;
- (c) The map $g'' = f|_R : R \rightarrow \Delta$ is an isomorphism;
- (d) The cohomology group $H^1(X, \Theta_{X/Y})$ vanishes: $H^1(X, \Theta_{X/Y}) = 0$.

Then Corollary 1 claims that there does not exist non-trivial small deformation of $f : X \rightarrow Y$ with the same degeneracy locus Δ .

From now on, we shall restrict ourselves to discussing deformations of quadric bundles (see Definition 2 below for precise definition).

In this section we shall discuss local structure of an ordinary quadric bundle (cf. Proposition 5) and prove that the conditions (a), (b) and (c) above are satisfied for an ordinary quadric bundle (cf. Corollary 2). (See Definition 3 for the definition of ordinary quadric bundles.)

In §6 and §7 we shall discuss the condition (d) in case where $f : X \rightarrow Y$ is a conic bundle, that is to say, in case where $f : X \rightarrow Y$ is a quadric bundle of relative dimension one.

Let us begin with the definition of a quadric bundle.

DEFINITION 2. A surjective holomorphic map $f : X \rightarrow Y$ between compact complex manifolds is called a *quadric bundle* if every fibre of f is isomorphic to a quadric hypersurface of constant dimension d . In case where $d = 1$, $f : X \rightarrow Y$ is called a *conic bundle*.

We have the following proposition due to A. Beauville.

PROPOSITION 4 ([1, Proposition 1.2, Lemma 1.5.2]). Assume that $\dim Y = 2$.

- (1) The map f is flat.
- (2) There exist a locally free sheaf \mathcal{E} on Y of rank $d + 2$, an invertible sheaf \mathcal{M} on Y and a section $q \in H^0(Y, S^2(\mathcal{E}) \otimes \mathcal{M})$ such that X is identified with the zero locus of q in $\mathbf{P}_Y(\mathcal{E})$.
- (3) The degeneracy locus Δ of f is a normal crossing divisor of Y .
- (4) For a smooth point y of Δ , the fibre $f^{-1}(y)$ has exactly one singular point. For a singular point y of Δ , the singular locus of $f^{-1}(y)$ is isomorphic to \mathbf{P}^1 .
- (5) Assume furthermore that f is a conic bundle. Let $y \in \Delta$. We take a small open neighbourhood U of y , local coordinates (u, v) on U , and homogeneous coordinates $(X_0 : X_1 : X_2)$ on the fibre $\pi^{-1}(y) \cong \mathbf{P}^2$ of $\mathbf{P}_Y(\mathcal{E})$ suitably, where $\pi : \mathbf{P}_Y(\mathcal{E}) \rightarrow Y$ denotes the natural projection. If y is a smooth point of Δ , then $X_U = f^{-1}(U)$ is defined by the equation $uX_0^2 + X_1^2 + X_2^2 = 0$ in $\pi^{-1}(U)$. If y is a singular point of Δ , then X_U is defined by $uX_0^2 + vX_1^2 + X_2^2 = 0$.

Although A. Beauville [1] originally proved Proposition 4 in case where $Y = \mathbf{P}^2$, we can apply the same arguments in [1] to the case where $\dim Y = 2$. Moreover, the assertions (1) and (2) of Proposition 4 hold true even if $\dim Y$ is greater than two, which is also proved by the same arguments in [1]. However, some of the assertions (3), (4) and (5) do not hold true if $\dim Y \geq 3$.

EXAMPLE 4. Let $Z = \mathbf{P}^2 \times \mathbf{P}^3$ and $\pi : Z \rightarrow \mathbf{P}^3$ the second projection. Let X be a subvariety of Z defined by the equation

$$T_0X_0^2 + 2T_1X_0X_1 + T_2X_1^2 + T_3X_2^2 = 0,$$

where $(X_0 : X_1 : X_2)$ and $(T_0 : T_1 : T_2 : T_3)$ denote the homogeneous coordinates on \mathbf{P}^2 and \mathbf{P}^3 respectively. Then X is smooth and $f = \pi|_X : X \rightarrow \mathbf{P}^3$ is a conic bundle. Since the degeneracy locus Δ is determined by $T_3(T_0T_2 - T_1^2) = 0$, it is not a normal crossing divisor.

The following definition is also due to Beauville [1].

DEFINITION 3 (cf. [1, Definition 1.3]). A quadric bundle $f : X \rightarrow Y$ is called an *ordinary quadric bundle* if every singular fibre of f has only one singular point.

Proposition 4 (5) implies that, if $\dim Y = 2$, then local structure of a conic bundle $f : X \rightarrow Y$ around a smooth (resp. singular) point $y \in Y$ is unique, which does not hold true in general if $\dim Y \geq 3$. However, the following proposition shows that, as to an ordinary quadric bundle, local structure around $y \in \Delta$ is unique.

Let $f : X \rightarrow Y$ be a quadric bundle. Suppose that $\dim Y = m$ and $\dim X = m + d$. Then, by Proposition 4 (2), there exist a locally free sheaf \mathcal{E} on Y of rank $d + 2$, an invertible sheaf M on Y and a section $q \in H^0(Y, S^2(\mathcal{E}) \otimes M)$ such that X is identified with the zero locus q in $\mathbf{P}_Y(\mathcal{E})$.

Now let us put $r = d + 1$ for later use. Then we have $\dim X = m + r - 1$ and $\mathrm{rk}(\mathcal{E}) = r + 1$, where $\mathrm{rk}(\mathcal{E})$ denotes the rank of \mathcal{E} .

PROPOSITION 5. *Under the assumptions and the notation above, we furthermore assume that $f : X \rightarrow Y$ is an ordinary quadric bundle. Let $y \in \Delta$. Let U be such a small open neighbourhood of y that we have $\pi^{-1}(U) \cong \mathbf{P}^r \times U$, where $\pi : \mathbf{P}_Y(\mathcal{E}) \rightarrow Y$ denotes the natural projection. Let $(w) = (w_1, \dots, w_m)$ be local coordinates on U and $(X_0 : \dots : X_r)$ homogeneous coordinates of fibres of $\pi|_{\pi^{-1}(U)}$. Then, after shrinking the open set U and changing coordinates on U , if necessary, we can write the defining equation of X in $\pi^{-1}(U)$ as follows:*

$$w_1 X_0^2 + X_1^2 + \dots + X_r^2 = 0.$$

In particular, the degeneracy locus Δ is a smooth divisor.

PROOF. Let us put $X_y = f^{-1}(y)$ and $X_U = f^{-1}(U)$. Let us denote by \mathbf{x} the vector ${}^t(X_0, \dots, X_r)$. Let $A = \Gamma(U, \mathcal{O}_U)$. Then the defining equation of X_U is written as follows:

$${}^t \mathbf{x} Q(w) \mathbf{x} = 0, \quad (37)$$

where $Q(w) = (q_{ij}(w))_{0 \leq i \leq r, 0 \leq j \leq r} \in M(r + 1, r + 1, A)$ with $q_{ij}(w) = q_{ji}(w) \in A$.

We may assume that X_y is defined by $X_1^2 + \dots + X_r^2 = 0$. Then we have:

$$q_{ij}(0) = \begin{cases} 1 & \text{if } i = j \geq 1, \\ 0 & \text{if } i = j = 0 \text{ or } i \neq j. \end{cases} \quad (38)$$

From now on, we shall shrink U if necessary. First we have the following claim.

CLAIM 1. *By replacing $Q(w)$ by ${}^t P(w) Q(w) P(w)$ for some $P(w) \in GL(r + 1, A)$,*

we may assume that $Q(w)$ is a diagonal matrix of the following form:

$$\begin{pmatrix} q_{00}(w) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

with $q_{00}(0) = 0$.

PROOF OF CLAIM 1. For each i with $1 \leq i \leq r$, there exists a unit $b_{ii}(w)$ of A that satisfies $(b_{ii}(w))^2 = 1/q_{ii}(w)$ and $b_{ii}(0) = 1$, since $q_{ii}(0) = 1$. Replacing $Q(w)$ by ${}^tP(w)Q(w)P(w)$ with the diagonal matrix

$$P(w) = \begin{pmatrix} 1 & & & \\ & b_{11}(w) & & \\ & & \ddots & \\ & & & b_{rr}(w) \end{pmatrix},$$

we may assume that $q_{ii}(w) = 1$ for $1 \leq i \leq r$. Moreover, replacing $Q(w)$ by ${}^tP(w)Q(w)P(w)$ with

$$P(w) = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ -q_{r0}(w) & -q_{r1}(w) & \cdots & -q_{r,r-1}(w) & 1 \end{pmatrix},$$

we can sweep out the last row and column of $Q(w)$, that is to say, we obtain new $Q(w)$ with $q_{ri}(w) = q_{ir}(w) = 0$ for $0 \leq i \leq r-1$ and $q_{rr}(w) = 1$. After sweeping out rows and columns successively, we finally obtain a diagonal matrix as in Claim 1. Since $Q(0)$ does not change under these procedure, $q_{00}(0)$ remains zero.

Thus Claim 1 is proved. \square

Let us continue the proof of Proposition 5. By the claim above, we may assume that X_U is defined by:

$$H(X_0, \dots, X_r; w) := q_{00}(w)X_0^2 + X_1^2 + \cdots + X_r^2 = 0.$$

Let p be the point of X defined by $(X_0 : \cdots : X_r) = (1 : 0 : \cdots : 0)$ and $w = 0$. Noting that

$$\frac{\partial H}{\partial w_i}(1, 0, \dots, 0; 0) = \frac{\partial q_{00}}{\partial w_i}(0) \quad \text{for } i = 1, \dots, m$$

and that X is smooth at p , we have $(\partial q_{00}/\partial w_j)(0) \neq 0$ for some j with $1 \leq j \leq m$. Therefore we can take a new system of coordinates (w') that satisfies $w'_1 = q_{00}(w)$. Then X_U is defined

by $w'_i X_0^2 + X_1^2 + \cdots + X_r^2 = 0$. The degeneracy locus Δ is locally defined by $w'_1 = 0$, whence Δ is smooth.

COROLLARY 2. *Let $f : X \rightarrow Y$ be an ordinary quadric bundle. Then the scheme-theoretic ramification locus R' of f is reduced and $g'' = f|_R : R \rightarrow \Delta$ is isomorphic.*

PROOF. We can prove it by direct calculation using the local description of $f : X \rightarrow Y$ in Proposition 5. We omit details. \square

COROLLARY 3. *Let $f : X \rightarrow Y$ be an ordinary quadric bundle with the degeneracy locus Δ . Assume that $H^1(X, \Theta_{X/Y}) = 0$. Then there does not exist non-trivial small deformation of $f : X \rightarrow Y$ with the same degeneracy locus Δ .*

PROOF. Straightforward from Corollary 1, Proposition 5 and Corollary 2. \square

6. Rigidity of certain conic bundles

Let us now discuss when $H^1(X, \Theta_{X/Y})$ vanishes. Let $f : X \rightarrow Y$ be a holomorphic map of complex manifolds in general. Suppose that $R^i f_* \Theta_{X/Y} = 0$ for $i > 0$. Then we have $H^j(X, \Theta_{X/Y}) \cong H^j(Y, f_* \Theta_{X/Y})$ for any integer j , whence we can reduce discussions on cohomology groups of $\Theta_{X/Y}$ to those on cohomology groups of $f_* \Theta_{X/Y}$.

From now on, we restrict ourselves to the case where $f : X \rightarrow Y$ is a conic bundle.

Let $f : X \rightarrow Y$ be a conic bundle. By applying Proposition 4 (2) for $d = 1$, we have a locally free sheaf \mathcal{E} on Y of rank 3, an invertible sheaf \mathcal{M} on Y and a section $q \in H^0(Y, S^2(\mathcal{E}) \otimes \mathcal{M})$ such that X is identified with the zero locus q in $\mathbf{P}_Y(\mathcal{E})$.

The central result in this section is the following theorem, which shall be proved later.

THEOREM 2. *Let $f : X \rightarrow Y$ be a conic bundle determined by a locally free sheaf \mathcal{E} on Y of rank three, an invertible sheaf \mathcal{M} on Y , and an element $q \in H^0(Y, S^2(\mathcal{E}) \otimes \mathcal{M})$ as above. Then we have:*

- (1) $R^i f_* \Theta_{X/Y} = 0$ for $i > 0$;
- (2) $f_* \Theta_{X/Y} \cong \mathcal{E} \otimes (\det \mathcal{E})^{-1} \otimes \mathcal{M}^{-1}$.

Calculating $H^1(X, \Theta_{X/Y})$ by Theorem 2, we obtain the following corollary.

COROLLARY 4. *Let $f : X \rightarrow Y$ be a conic bundle determined by a locally free sheaf \mathcal{E} on Y of rank three, an invertible sheaf \mathcal{M} on Y , and an element $q \in H^0(Y, S^2(\mathcal{E}) \otimes \mathcal{M})$. Assume that the following three conditions are satisfied:*

- (1) $Y = \mathbf{P}^m$ ($m \geq 2$);
- (2) \mathcal{E} is a direct sum of invertible sheaves;
- (3) $f : X \rightarrow Y$ is an ordinary conic bundle.

Then there does not exist non-trivial small deformation of $f : X \rightarrow Y$ with the same degeneracy locus Δ .

PROOF OF COROLLARY 4. By the assumption and Theorem 2, we have $H^1(X, \mathcal{O}_{X/Y}) = 0$. Then Corollary 4 follows immediately from Corollary 3. \square

REMARK 1. (1) In the previous paper [3] we proved Theorem 2 and Corollary 4 in case where $\dim X = 3$ (cf. [3, Lemma 3.1], [3, Theorem 3.3] and [3, Corollary 3.14]). Here we generalize them to higher-dimensional cases.

- (2) The conclusion of Theorem 2 holds true for *any* conic bundle $f : X \rightarrow Y$, even if it is not ordinary.
- (3) On the other hand, we need the assumption that f is ordinary in Corollary 4, since the assumption that the degeneracy locus Δ is smooth is indispensable in Theorem 1.
- (4) If Δ has singularity, there does not exist such an isomorphism ψ as in (23) in general (cf. [4]).
- (5) In the paper [5] we generalize Corollary 4; if $Y = \mathbf{P}^2$ and if \mathcal{E} is a direct sum of invertible sheaves, then we have the same conclusion as in Corollary 4 without assuming that $f : X \rightarrow Y$ is ordinary. The proof is done by using subtle arguments on deformations that admit no smoothing of degeneracy loci. In these arguments, it is essential that Theorem 2 holds true for *any* conic bundle.
- (6) We do not know yet whether the conclusion of Corollary 4 holds true for general m in case where $f : X \rightarrow Y (= \mathbf{P}^m)$ is not ordinary, since arguments of [5] essentially need the fact that Δ is normal crossing, which does not hold true if $\dim Y \geq 3$ (cf. Proposition 4 and Example 4).

PROOF OF THEOREM 2. The rest of this paper is devoted to proving Theorem 2. The proof is done by almost the same arguments as in [3], except for Lemma 12 below, which shall be proved later in §7. Although the other arguments are the same as those of [3], we shall sketch them here for readers' convenience.

Let $f : X \rightarrow Y$ be as in Theorem 2. Let us put $Z = \mathbf{P}_Y(\mathcal{E})$. Let $\pi : Z \rightarrow Y$ denote the natural projection. First we have: \square

LEMMA 5 ([3, Lemma 3.1]). $R^i f_* \mathcal{O}_{X/Y} = 0$ for $i > 0$.

PROOF. We can easily check that $H^i(X_y, \mathcal{O}_{X/Y} \otimes \mathbf{C}(y)) = 0$ for $i > 0$ on each fibre X_y of f over $y \in Y$. Since f is flat and $\mathcal{O}_{X/Y}$ is invertible, we are done. \square

We have an exact sequence

$$0 \rightarrow \mathcal{O}_{X/Y} \rightarrow \mathcal{O}_{Z/Y} \otimes \mathcal{O}_X \rightarrow N_{X/Z} = \mathcal{O}_X(X).$$

Taking the direct images by f , we have an exact sequence

$$0 \rightarrow f_* \mathcal{O}_{X/Y} \rightarrow f_*(\mathcal{O}_{Z/Y} \otimes \mathcal{O}_X) \xrightarrow{G} f_* \mathcal{O}_X(X).$$

Denoting the last homomorphism in this sequence by G , we have an isomorphism

$$f_* \mathcal{O}_{X/Y} \cong \operatorname{Ker}(G : f_*(\mathcal{O}_{Z/Y} \otimes \mathcal{O}_X) \rightarrow f_* \mathcal{O}_X(X)).$$

LEMMA 6 ([3, Lemma 3.4]). (1) $R^i \pi_* \mathcal{O}_Z = 0$ for $i > 0$.
 (2) $R^i \pi_*(\mathcal{O}_{Z/Y}(-X)) = 0$ for all i .

PROOF. We have $H^i(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}) = 0$ for $i > 0$ and $H^i(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-2)) = 0$ for all i , from which Lemma 6 follows. \square

LEMMA 7 ([3, Lemma 3.5]). (1) The natural homomorphism $\pi_* \mathcal{O}_{Z/Y} \rightarrow f_*(\mathcal{O}_{Z/Y} \otimes \mathcal{O}_X)$ is isomorphic.
 (2) The natural homomorphism $\pi_* \mathcal{O}_Z(X) \rightarrow f_* \mathcal{O}_X(X)$ is surjective and its kernel is \mathcal{O}_Y .

PROOF. We have the natural exact sequences $0 \rightarrow \mathcal{O}_{Z/Y}(-X) \rightarrow \mathcal{O}_{Z/Y} \rightarrow \mathcal{O}_{Z/Y} \otimes \mathcal{O}_X \rightarrow 0$ and $0 \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_Z(X) \rightarrow \mathcal{O}_X(X) \rightarrow 0$. We apply Lemma 6 after taking the direct images by π . \square

Via isomorphisms $\pi_* \mathcal{O}_{Z/Y} \cong f_*(\mathcal{O}_{Z/Y} \otimes \mathcal{O}_X)$ and $f_* \mathcal{O}_X(X) \cong (\pi_* \mathcal{O}_Z(X))/\mathcal{O}_Y$, we have a homomorphism

$$\bar{G} : \pi_* \mathcal{O}_{Z/Y} \rightarrow (\pi_* \mathcal{O}_Z(X))/\mathcal{O}_Y.$$

Then the sheaf $f_* \mathcal{O}_{X/Y}$ is isomorphic to $\text{Ker}(\bar{G})$.

Now we discuss \bar{G} locally on Y . Let $y \in Y$ and $A = \mathcal{O}_{Y,y}$. Then the ring A is isomorphic to the convergent power series ring $\mathbf{C}\{w_1, \dots, w_m\}$, where w_1, \dots, w_m denote local coordinates around y . We localize $f : X \rightarrow Y$ over $\text{Spec } A$. Let $Z_A = \mathbf{P}_A^2 = \mathbf{P}^2 \times \text{Spec } A$ and $X_A = X \times_Y \text{Spec } A$. Let $(X_0 : X_1 : X_2)$ be a system of homogeneous coordinates on \mathbf{P}^2 and $U_i = \{X_i \neq 0\}$. Let us put $x_i = X_{i+1}/X_i$ and $y_i = X_{i+2}/X_i$ ($X_3 = X_0$). Let $U_{A,i} = U_i \times \text{Spec } A$. Localizing the homomorphism \bar{G} , we discuss

$$\bar{G}_A : H^0(\mathbf{P}_A^2, \mathcal{O}_{\mathbf{P}_A^2}) \rightarrow H^0(\mathbf{P}_A^2, \mathcal{O}(X_A))/H^0(\mathbf{P}_A^2, \mathcal{O}_{\mathbf{P}_A^2}).$$

LEMMA 8 ([3, Lemma 3.6]). We have a standard exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbf{P}_A^2} \rightarrow \mathcal{O}_{\mathbf{P}_A^2}(1)^{\oplus 3} \rightarrow \mathcal{O}_{\mathbf{P}_A^2} \rightarrow 0.$$

The homomorphisms appearing in this sequence are described as follows. We denote the natural basis of $\Gamma(U_{A,i}, \mathcal{O}(1)^{\oplus 3})$ by $\{p_i, q_i, r_i\}$ and the basis of $\Gamma(U_{A,i}, \mathcal{O}_{\mathbf{P}_A^2})$ by $\{\partial/\partial x_i, \partial/\partial y_i\}$ for $i = 0, 1, 2$.

(1) The homomorphism $\Gamma(U_{A,i}, \mathcal{O}) \rightarrow \Gamma(U_{A,i}, \mathcal{O}(1)^{\oplus 3})$ is given by

$$1 \mapsto p_0 + x_0 q_0 + y_0 r_0 \quad (i = 0),$$

$$1 \mapsto y_1 p_1 + q_1 + x_1 r_1 \quad (i = 1),$$

$$\text{and } 1 \mapsto x_2 p_2 + y_2 q_2 + r_2 \quad (i = 2), \text{ respectively.}$$

(2) The homomorphism $\Gamma(U_{A,i}, \mathcal{O}(1)^{\oplus 3}) \rightarrow \Gamma(U_{A,i}, \Theta_{\mathbf{P}_A^2})$ is given by

$$\begin{aligned} p_0 &\mapsto -x_0 \frac{\partial}{\partial x_0} - y_0 \frac{\partial}{\partial y_0}, & q_0 &\mapsto \frac{\partial}{\partial x_0}, & r_0 &\mapsto \frac{\partial}{\partial y_0} & (i=0); \\ p_1 &\mapsto \frac{\partial}{\partial y_1}, & q_1 &\mapsto -x_1 \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial y_1}, & r_1 &\mapsto \frac{\partial}{\partial x_1} & (i=1); \\ p_2 &\mapsto \frac{\partial}{\partial x_2}, & q_2 &\mapsto \frac{\partial}{\partial y_2}, & r_2 &\mapsto -x_2 \frac{\partial}{\partial x_2} - y_2 \frac{\partial}{\partial y_2} & (i=2). \end{aligned}$$

PROOF. We omit it, since it is well-known. \square

LEMMA 9 ([3, Lemma 3.7]). We have natural isomorphisms

$$H^0(\mathbf{P}_A^2, \Theta_{\mathbf{P}_A^2}) \cong H^0(\mathbf{P}_A^2, \mathcal{O}(1))^{\oplus 3} / H^0(\mathbf{P}_A^2, \mathcal{O}) \cong M_3(A) / \langle E \rangle,$$

where $M_3(A)$ denotes the A -module of the 3×3 matrices with coefficients in A and $\langle E \rangle$ the A -submodule of $M_3(A)$ generated by the unit matrix E .

PROOF. The first isomorphism follows from Lemma 8. Let $S_1(A)$ denote the A -module of the homogeneous polynomials of degree one of X_0, X_1 and X_2 with coefficients in A . We have natural isomorphisms $S_1(A) \cong H^0(\mathbf{P}_A^2, \mathcal{O}(1))$ and $H^0(\mathbf{P}_A^2, \mathcal{O}(1))^{\oplus 3} \cong S_1(A)^{\oplus 3} \cong M_3(A)$. We fix the last isomorphism as follows. For $(f_0, f_1, f_2) \in S_1(A)^{\oplus 3}$ with $f_j = \sum_{i=0}^2 b_{ij} X_i$ ($b_{ij} \in A$), we attach the matrix $B = (b_{ij}) \in M_3(A)$. Since the image of $1 \in H^0(\mathbf{P}_A^2, \mathcal{O}_{\mathbf{P}_A^2})$ in $S_1(A)^{\oplus 3}$ is (X_0, X_1, X_2) (cf. Lemma 8 (1)), its image in $M_3(A)$ is the unit matrix E . Thus Lemma 9 is proved. \square

Suppose that X_A is defined in Z_A by $q \in H^0(\mathbf{P}_A^2, \mathcal{O}(2))$ and that $Q = (q_{ij}) \in M_3(A)$ ($q_{ij} = q_{ji}$) is the symmetric matrix corresponding to q . Let $\mathbf{x} = {}^t(X_0, X_1, X_2)$. Then we have $q = {}^t \mathbf{x} Q \mathbf{x}$.

Next we shall discuss $H^0(\mathbf{P}_A^2, \mathcal{O}(X_A)) / H^0(\mathbf{P}_A^2, \mathcal{O})$. Let $\mathcal{G}(A)$ denote the A -module of the symmetric matrices in $M_3(A)$.

LEMMA 10 ([3, Lemma 3.8]). We have an isomorphism

$$H^0(\mathbf{P}_A^2, \mathcal{O}(X_A)) / H^0(\mathbf{P}_A^2, \mathcal{O}) \cong \mathcal{G}(A) / \langle Q \rangle.$$

PROOF. The space $H^0(\mathbf{P}_A^2, \mathcal{O}(X_A)) = H^0(\mathbf{P}_A^2, \mathcal{O}(2))$ can be identified with the set of the homogeneous polynomials of degree two of X_0, X_1 and X_2 with coefficients in A , which can also be identified with $\mathcal{G}(A)$. The image of $1 \in H^0(\mathbf{P}_A^2, \mathcal{O})$ in $H^0(\mathbf{P}_A^2, \mathcal{O}(X_A))$ is nothing but q , whose image in $\mathcal{G}(A)$ is Q . Thus Lemma 10 is proved. \square

Via isomorphisms in Lemma 9 and Lemma 10, the homomorphism \bar{G}_A is equivalent to the homomorphism

$$\hat{G}_A : M_3(A)/\langle E \rangle \rightarrow \mathcal{G}(A)/\langle Q \rangle,$$

which can be explicitly described as follows.

LEMMA 11 ([3, Lemma 3.9]). *For $B \in M_3(A)$, we have*

$$\hat{G}_A(B \bmod \langle E \rangle) = BQ + Q^t B \bmod \langle Q \rangle.$$

PROOF. The proof is done by pursuing homomorphisms appearing in discussions above. We refer it to [3, Lemma 3.9], since it still holds true in our case. (Note that the notation used in [3] is a little bit different.) \square

Next we shall discuss $\text{Ker}(\hat{G}_A)$.

In general, for $S \in \mathcal{G}(A)$, we define $\varphi_S : M_3(A) \rightarrow \mathcal{G}(A)$ as follows:

$$\begin{array}{ccc} \varphi_S : & M_3(A) & \rightarrow & \mathcal{G}(A) \\ & \downarrow & & \downarrow \\ & B & \mapsto & BS + S^t B \end{array} \quad (39)$$

Note that, if S is the unit matrix E , then $\text{Ker}(\varphi_E)$ is the set of the skew-symmetric matrices in $M_3(A)$ and it is generated by the following three matrices:

$$H_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (40)$$

We have the following key lemma, which is a generalization of [3, Lemma 3.10].

LEMMA 12 ([3, Lemma 3.10]). *If $Q \in \mathcal{G}(A)$ corresponds to the defining equation of a nonsingular conic bundle X_A , we have*

$$\text{Ker}(\varphi_Q) = Q \cdot \text{Ker}(\varphi_E).$$

In particular, $\text{Ker}(\varphi_Q)$ is a free A -module generated by $\{QH_i \mid i = 0, 1, 2\}$.

In the proof of [3, Lemma 3.10] we used Proposition 4 (5), which does not hold true in higher-dimensional cases. So we have to prove Lemma 12 in another way that holds true in general. The proof of Lemma 12 shall be given later in §7.

Let us continue the proof of Theorem 2. Let $\{V_i\}$ be a sufficiently fine open covering of Y . Let $\{X_0^{(i)}, X_1^{(i)}, X_2^{(i)}\}$ be a local basis of \mathcal{E} on V_i . Let us put $\mathbf{x}^{(i)} = {}^t(X_0^{(i)}, X_1^{(i)}, X_2^{(i)})$. Suppose that the relation

$$\mathbf{x}^{(i)} = T_{ij} \mathbf{x}^{(j)} \quad (41)$$

is satisfied on V_{ij} with the transition matrix $T_{ij} = (t_{ij;kl})_{0 \leq k \leq 2, 0 \leq l \leq 2}$.

Let μ_i be a local basis of \mathcal{M} on V_i . Suppose that the transition relation

$$\mu_i = \lambda_{ij} \mu_j \quad (42)$$

is satisfied on V_{ij} . We can take $(X_0^{(i)} : X_1^{(i)} : X_2^{(i)})$ as the homogeneous coordinates of the fibre of $Z = \mathbf{P}_Y(\mathcal{E})$ on V_i . Suppose that the defining element $q \in H^0(Y, S^2(\mathcal{E}) \otimes \mathcal{M})$ of X is written as

$$q = {}^t \mathbf{x}^{(i)} Q_i \mathbf{x}^{(i)} \mu_i$$

on V_i with a symmetric matrix $Q_i = (q_{kl}^{(i)})$.

Then we can take a local basis $\{e_0^{(i)}, e_1^{(i)}, e_2^{(i)}\}$ of $f_* \mathcal{O}_{X/Y}$ on V_i as follows. Let H_v be the skew-symmetric matrix as in (40) and $A_v^{(i)} = Q_i H_v$ ($v = 0, 1, 2$). Then $\{A_0^{(i)}, A_1^{(i)}, A_2^{(i)}\}$ is a basis of

$$\text{Ker}(M_3(\Gamma(V_i, \mathcal{O}_Y)) / \langle E \rangle \rightarrow \mathcal{G}(\Gamma(V_i, \mathcal{O}_Y)) / \langle Q_i \rangle),$$

where $M_3(\Gamma(V_i, \mathcal{O}_Y))$ and $\mathcal{G}(\Gamma(V_i, \mathcal{O}_Y))$ denote the set of the 3×3 matrices and the symmetric 3×3 matrices with coefficients in the ring $\Gamma(V_i, \mathcal{O}_Y)$, respectively. We choose as $e_v^{(i)}$ the element of $\Gamma(V_i, \pi_* \mathcal{O}_{Z/Y})$ which corresponds to $A_v^{(i)}$. Then $\{e_0^{(i)}, e_1^{(i)}, e_2^{(i)}\}$ is the local basis of $f_* \mathcal{O}_{X/Y}$ on V_i .

Let us now put $\mathbf{e}^{(i)} = (e_0^{(i)}, e_1^{(i)}, e_2^{(i)})$. Then, after rather complicated calculation, we have the following transition relation.

LEMMA 13. *We have $\mathbf{e}^{(i)} = \lambda_{ij}^{-1} D_{ij}^{-1} T_{ij} \mathbf{e}^{(j)}$, where T_{ij} and λ_{ij} are as in (41) and (42), and $D_{ij} = \det T_{ij}$.*

PROOF. We refer the proof to arguments from page 30 to page 33 in [3], which still hold true in our case, since they are independent of $\dim Y$.

Lemma 13 implies that

$$f_* \mathcal{O}_{X/Y} \cong \mathcal{E} \otimes (\det \mathcal{E})^{-1} \otimes \mathcal{M}^{-1}.$$

Thus Theorem 2 is proved. \square

7. Proof of Lemma 12

In this section we shall prove Lemma 12. We use the same notation as before.

Suppose that $Q \in \mathcal{G}(A)$ corresponds to the defining equation of a nonsingular conic bundle X_A . We shall prove that $\text{Ker}(\varphi_Q) = Q \cdot \text{Ker}(\varphi_E)$, where φ_S denote the map in (39) for $S \in \mathcal{G}(A)$.

For $C \in \text{Ker}(\varphi_E)$, we have

$$\varphi_Q(QC) = (QC)Q + Q^t(QC) = Q(C + {}^t C)Q = O,$$

whence we have $Q \cdot \text{Ker}(\varphi_E) \subset \text{Ker}(\varphi_Q)$.

Next we prove that, for any $B \in \text{Ker}(\varphi_Q)$, there exists $C \in \text{Ker}(\varphi_E)$ that satisfies $B = QC$.

If we change the homogeneous coordinates of \mathbf{P}_A^2 by $T \in GL(3, A)$, the corresponding symmetric matrix Q is transformed to $Q' = {}^t T Q T$. It is easy to see that there is a one-to-one correspondence between $\text{Ker}(\varphi_Q)$ and $\text{Ker}(\varphi_{Q'})$ with $B \in \text{Ker}(\varphi_Q)$ corresponding to ${}^t T B {}^t T^{-1} \in \text{Ker}(\varphi_{Q'})$. It is also easy to check that $T^{-1} \text{Ker}(\varphi_E) {}^t T^{-1} = \text{Ker}(\varphi_E)$. If we have $\text{Ker}(\varphi_Q) = Q \cdot \text{Ker}(\varphi_E)$ for some Q , then we have

$$\text{Ker}(\varphi_{Q'}) = {}^t T \text{Ker}(\varphi_Q) {}^t T^{-1} = {}^t T Q \text{Ker}(\varphi_E) {}^t T^{-1} = Q' \text{Ker}(\varphi_E).$$

Therefore we can change coordinates, if necessary, to prove Lemma 12.

Now let $Q = Q(w) = (q_{ij}(w))_{0 \leq i \leq 2, 0 \leq j \leq 2} \in \mathcal{G}(A)$ be a symmetric matrix corresponding to the defining equation of X_A . Let y be the point of Y defined by $w = 0$. We consider the following three cases.

CASE 1: X_y is smooth.

In this case, we may assume that $Q(w)$ is the unit matrix, after change of coordinates. Then the assertion is trivial.

CASE 2: X_y has two irreducible components intersecting at one point.

In this case, we may assume that

$$Q(w) = \begin{pmatrix} w_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

by Proposition 5. Then we can check that $\text{Ker}(\varphi_Q) = Q \cdot \text{Ker}(\varphi_E)$ by direct calculation.

CASE 3: X_y is a double line.

In this case we may assume that

$$Q(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $q_{22}(w)$ is a unit of A , we can sweep out the last row and column of $Q(w)$ and we may assume that $Q(w)$ is of the following form:

$$Q(w) = \begin{pmatrix} p(w) & r(w) & 0 \\ r(w) & q(w) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $p(w)$, $q(w)$ and $r(w)$ are elements of A satisfying $p(0) = q(0) = r(0) = 0$.

Let us put

$$H(X_0, X_1, X_2; w) = {}^t \mathbf{x} Q(w) \mathbf{x} = p(w)X_0^2 + q(w)X_1^2 + 2r(w)X_0X_1 + X_2^2. \quad (43)$$

We now consider the following two cases.

CASE 3 (A): $r(w) = 0$.

In this case we have the following claim.

CLAIM 2. *The elements $p(w)$ and $q(w)$ are coprime to each other.*

PROOF. Suppose the contrary. Then there exists an element $\gamma(w) \in A$ satisfying $\gamma(0) = 0$, $p(w) = \gamma(w)p_1(w)$ and $q(w) = \gamma(w)q_1(w)$ for some elements $p_1(w)$ and $q_1(w)$ of A which are coprime to each other.

Let a and b be complex numbers satisfying $(a, b) \neq (0, 0)$ and $p_1(0)a^2 + q_1(0)b^2 = 0$. (Note that such numbers exist.) Let \tilde{P}_1 be the point of X defined by $w = 0$ and $(X_0 : X_1 : X_2) = (a : b : 0)$. Then we have $(\partial H / \partial X_i)(a, b, 0; 0) = 0$ for $0 \leq i \leq 2$ and

$$\frac{\partial H}{\partial w_i}(a, b, 0; 0) = \frac{\partial \gamma}{\partial w_i}(0) \cdot (p_1(0)a^2 + q_1(0)b^2) = 0$$

for $1 \leq i \leq m$. Therefore \tilde{P}_1 is a singular point of X , which contradicts the assumption that X is nonsingular.

Claim 2 is thus proved. \square

Now let $B = (b_{ij}(w))_{0 \leq i \leq 2, 0 \leq j \leq 2} \in \text{Ker}(\varphi_Q)$. Then we have:

$$b_{ii}(w) = 0 \quad \text{for } i = 0, 1, 2; \quad (44)$$

$$p(w)b_{10}(w) + q(w)b_{01}(w) = 0; \quad (45)$$

$$p(w)b_{20}(w) + b_{02}(w) = 0; \quad (46)$$

$$q(w)b_{21}(w) + b_{12}(w) = 0. \quad (47)$$

Since $p(w)$ and $q(w)$ are coprime to each other, the equality (45) implies that there exists an element $a(w) \in A$ satisfying $b_{10}(w) = q(w)a(w)$ and $b_{01}(w) = -p(w)a(w)$. Let us put $b(w) = b_{20}(w)$ and $c(w) = b_{21}(w)$. Then we have

$$B = \begin{pmatrix} 0 & -p(w)a(w) & -p(w)b(w) \\ q(w)a(w) & 0 & -q(w)c(w) \\ b(w) & c(w) & 0 \end{pmatrix} = QC,$$

where $C = \begin{pmatrix} 0 & -a(w) & -b(w) \\ a(w) & 0 & -c(w) \\ b(w) & c(w) & 0 \end{pmatrix} \in \text{Ker}(\varphi_E)$, whence we have $B \in Q \cdot \text{Ker}(\varphi_E)$.

CASE 3 (B): $r(w) \neq 0$.

If $p(w)$ divides $r(w)$, then we can reduce to Case 3 (a) after sweeping out the first row and column of $Q(w)$. We can also reduce to Case 3 (a) if $q(w)$ divides $r(w)$. Thus we may assume that neither $p(w)$ nor $q(w)$ divides $r(w)$.

CLAIM 3. (1) *The elements $p(w)$ and $r(w)$ are coprime to each other.*

(2) *The elements $q(w)$ and $r(w)$ are coprime to each other.*

PROOF. We prove only (1), since (2) can be similarly proved.

Suppose the contrary. Then there exists an element $\gamma(w) \in A$ satisfying $\gamma(0) = 0$, $p(w) = \gamma(w)p_1(w)$ and $r(w) = \gamma(w)r_1(w)$ for some elements $p_1(w)$ and $r_1(w)$ of A which are coprime to each other. If $p_1(0) \neq 0$, then $p_1(w)$ is a unit of A , whence $p(w)$ divides $r(w)$, which contradicts the assumption. Hence we have $p_1(0) = 0$.

Let \tilde{P}_2 be the point of X defined by $w = 0$ and $(X_0 : X_1 : X_2) = (1 : 0 : 0)$. Then we have $(\partial H / \partial X_i)(1, 0, 0; 0) = 0$ for $0 \leq i \leq 2$ and

$$\frac{\partial H}{\partial w_i}(1, 0, 0; 0) = \frac{\partial p}{\partial w_i}(0) = \gamma(0) \frac{\partial p_1}{\partial w_i}(0) + \frac{\partial \gamma}{\partial w_i}(0) p_1(0) = 0$$

for $1 \leq i \leq m$. Therefore \tilde{P}_2 is a singular point of X , which contradicts the assumption that X is nonsingular.

Thus Claim 3 is proved. \square

Now let $B = (b_{ij}(w))_{0 \leq i \leq 2, 0 \leq j \leq 2} \in \text{Ker}(\varphi_Q)$. Then we have:

$$p(w)b_{00}(w) + r(w)b_{01}(w) = 0; \quad (48)$$

$$r(w)b_{10}(w) + q(w)b_{11}(w) = 0; \quad (49)$$

$$b_{22}(w) = 0; \quad (50)$$

$$p(w)b_{10}(w) + r(w)b_{11}(w) + r(w)b_{00}(w) + q(w)b_{01}(w) = 0; \quad (51)$$

$$p(w)b_{20}(w) + r(w)b_{21}(w) + b_{02}(w) = 0; \quad (52)$$

$$r(w)b_{20}(w) + q(w)b_{21}(w) + b_{12}(w) = 0. \quad (53)$$

Since $p(w)$ and $r(w)$ are coprime to each other, the equality (48) implies that there exists an element $a(w) \in A$ satisfying $b_{00}(w) = r(w)a(w)$ and $b_{01}(w) = -p(w)a(w)$. Similarly, the equality (49) implies that there exists an element $\tilde{a}(w) \in A$ satisfying $b_{10}(w) = q(w)\tilde{a}(w)$ and $b_{11}(w) = -r(w)\tilde{a}(w)$. Then the condition (51) implies that

$$(p(w)q(w) - r(w)^2)(\tilde{a}(w) - a(w)) = 0.$$

Noting that the degeneracy locus Δ of f is determined by $p(w)q(w) - r(w)^2 = 0$ and that general fibres of f are smooth, we have $p(w)q(w) - r(w)^2 \in A \setminus \{0\}$. Thus we have $\tilde{a}(w) = a(w)$. Let us put $b(w) = b_{20}(w)$ and $c(w) = b_{21}(w)$. Then we have $b_{02}(w) = -p(w)b(w) - r(w)c(w)$ and $b_{12}(w) = -r(w)b(w) - q(w)c(w)$ by (52) and (53). Thus we have

$$B = \begin{pmatrix} r(w)a(w) & -p(w)a(w) & -p(w)b(w) - r(w)c(w) \\ q(w)a(w) & -r(w)a(w) & -r(w)b(w) - q(w)c(w) \\ b(w) & c(w) & 0 \end{pmatrix} = QC,$$

where $C = \begin{pmatrix} 0 & -a(w) & -b(w) \\ a(w) & 0 & -c(w) \\ b(w) & c(w) & 0 \end{pmatrix} \in \text{Ker}(\varphi_E)$, whence we have $B \in Q \cdot \text{Ker}(\varphi_E)$.

Thus Lemma 12 is proved. \square

References

- [1] A. BEAUVILLE, Variété de Prym et Jacobiennes intermédiaires, Ann. scient. Éc. Norm. Sup., 4^e série. t. **10** (1977), 309–391.
- [2] M. EBIHARA, Some remarks on infinitesimal deformations of a conic bundle I, Saitama Math. J. **18** (2000), 1–21.
- [3] M. EBIHARA, Some remarks on infinitesimal deformations of a conic bundle II, Saitama Math. J. **18** (2000), 23–38.
- [4] M. EBIHARA, Some remarks on infinitesimal deformations of a conic bundle III, Saitama Math. J. **24** (2006 · 2007), 93–104.
- [5] M. EBIHARA, Some remarks on infinitesimal deformations of a conic bundle IV, Saitama Math. J. **28** (2011), 39–54.
- [6] E. HORIKAWA, On deformations of holomorphic maps I, II, III, J. Math. Soc. Japan **25** (1973), 372–396; **26** (1974), 647–667; Math. Ann. **222** (1976), 275–282.
- [7] K. KODAIRA, A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds, Ann. of Math. **75** (1962), 146–162.
- [8] M. Miyanishi, Algebraic methods in the theory of algebraic threefolds, *Algebraic varieties and analytic varieties*, eds. S. Iitaka et al., Advanced Studies in Pure Math. **1**, Kinokuniya, North-Holland (1983), 69–99.

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