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Alexander Ideals of Graphs in the 3-Sphere

Dedicated to Professor Takizo Minagawa for his 70th birthday

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Let $(P\subset S^n)$ be a pair of the oriented n-sphere S^n $(n\geqq 3)$ and a finite subpolyhedron P of S^n with S^n-P connected. Using Fox's free differential calculus ([2], [3], [4]), S. Kinoshita [11] explained that for each non-negative integer d there is the d^{th} elementary ideal E_d of the fundamental group $G(P)\equiv\pi_1(S^n-P)$, associated with each integral $(n-2)$-cycle l on P, so that the collection $\{E_d\}$ forms a topological invariant of the position of P in S^n. He also examined some fundamental properties of it in [11], [12] and [13].

In this paper we discuss the elementary ideals of finite 1-dimensional polyhedra in the 3-sphere S^3 associated with the abelianizer, and give a necessary condition for the exterior of a connected 1-dimensional polyhedron to be retractible and boundary-retractible [9], (Theorems 3.1 and 3.2).

§1. Preliminaries.

Throughout the paper we work in the piecewise linear category.

By P we denote a finite 1-dimensional polyhedron with μ components $P_1, \cdots, P_\mu$, $\mu\geqq 1$. We denote by β_i the 1-dimensional Betti number of P_i for $i=1, \cdots, \mu$, and let $\beta=\beta_1+\cdots+\beta_\mu$. *We always assume that* $\beta_i>0$ for $i=1, \cdots, \mu$, and we will call such a pair $(P\subset S^3)$ of the 3-sphere S^3 and its subpolyhedron P a *graph* in S^3.

For a graph $(P\subset S^3)$, by the *exterior* $M(P)$ of P we mean the closure of $S^3-N(P; S^3)$, where $N(P; S^3)$ is a regular neighborhood of P in S^3, and by $G(P)$ we denote the fundamental group $\pi_1(S^3-P)\cong\pi_1(M(P))$.

We shall consider finitely presentable groups and their finite presentations. For a finite presentation $\langle x_1, \cdots, x_n | r_1, \cdots, r_m\rangle$ of a group G,

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we denote by ϕ the canonical homomorphism of the free group $\langle x_1, \cdots, x_n\rangle$ onto G. The *deficiency* of a presentation is the number of generators minus the number of relators. The *deficiency* $\operatorname{def}(G)$ of a group G is the maximum of the deficiencies of its presentations.

A graph $(P\subset S^3)$ is said to be (geometrically) *splittable* [17], iff there exists a 2-sphere $S^2\subset S^3-P$, such that both components of S^3-S^2 contain points of P. More precisely, we say that a graph $(P\subset S^3)$ has c *factors* $Q_1, \cdots, Q_c$, iff $P=Q_1\cup\cdots\cup Q_c$ and there exist c disjoint 3-cells $D_1^3\cup\cdots\cup D_c^3$ in S^3 such that $Q_j\subset\operatorname{Int}(D_j^3)$ for $j=1, \cdots, c$, and each pair $(Q_j\subset S^3)$ is non-splittable graph. In this case, $G(P)$ is a free product $G(Q_1)*\cdots*G(Q_c)$, where $G(Q_j)=\pi_1(S^3-Q_j)$, $j=1, \cdots, c$.

If a graph $(P\subset S^3)$ is non-splittable, then the space S^3-P is aspherical by Papakyriakopoulos [17], and so as in Trotter [20, p. 478] it follows that $\operatorname{def}(G(P))=\beta-\mu+1$; see also Fox [5, (6.2)], Kinoshita [11, Theorem 7 and Corollary] and Hillman [8, Theorem 12]. Now we have:

1.1. PROPOSITION. *A graph $(P\subset S^3)$ has c factors if and only if* $\operatorname{def}(G(P))=\beta-\mu+c$.

1.2. Definitions and Notation. In order to calculate elementary ideals of $G(P)$, we discuss how to obtain a presentation of $G(P)$ for a given graph $(P\subset S^3)$.

(1.2.1) First we choose a maximal tree T_i of each connected component P_i, and we give an orientation for each 1-simplex $\Delta_{i\lambda}^1$ of P_i-T_i, where $\lambda=1, \cdots, \beta_i$. Then for each $\Delta_{i\lambda}^1\subset P_i-T_i$, there exists a unique simple oriented loop $k_{i\lambda}\subset\Delta_{i\lambda}^1\cup T_i\subset P_i$ such that $k_{i\lambda}\supset\Delta_{i\lambda}^1$ and the orientation of $k_{i\lambda}$ is coherent to that of $\Delta_{i\lambda}^1$. It will be noticed that $k_i=\{k_{i1}, \cdots, k_{i\beta_i}\}$ forms a free abelian basis for the 1st integral homology group $H_1(P_i; \boldsymbol{Z})$ of P_i. We call such a set of 1-cycles k_i a *fundamental cycle* on P_i or an *orientation* on P_i, and $l_0=\{k_1, \cdots, k_\mu\}$ a *fundamental cycle* on P or an *orientation* on P.

Now let p be a regular projection of $P\subset S^3$ in a suitably chosen 2-sphere $S_0^2\subset S^3$, in the sense of the Knot Theory (see Crowell-Fox [2, Chap. I]). Since T_i is contractible, we can deform $P=P_1\cup\cdots\cup P_\mu$ in S^3 isotopically, so that $p(P)$ has no double points in $p(T_1\cup\cdots\cup T_\mu)$. Let m be the number of the crossing points of $p(P)=p((P_1-T_1)\cup\cdots\cup(P_\mu-T_\mu))$. Then these m crossing points divides $\beta=\beta_1+\cdots+\beta_\mu$ arcs $p((P_1-T_1)\cup\cdots\cup(P_\mu-T_\mu))$ into $\beta+m$ overpasses, and using this projection, we obtain a presentation of $G(P)$ as knots and links (Crowell-Fox [2, Chap. VI], see also Suzuki [19, § 4]). In fact, let $m(i, \lambda)$ be the number of overcrossing points of $p(P)$ on the arc $p(\Delta_{i\lambda}^1)$. Then we have the following presentation of $G(P)$:

(1.2.2) $$\left\langle \begin{matrix} x^i_{\lambda 0}, x^i_{\lambda 1}, \cdots, x^i_{\lambda m(i,\lambda)} \\ (i=1, \cdots, \mu;\ \lambda=1, \cdots, \beta_i) \end{matrix} \middle| \begin{matrix} r_1, \cdots, r_m \\ \Omega_1, \cdots, \Omega_\mu \end{matrix} \right\rangle,$$

where the generator $x^i_{\lambda\nu}$ corresponds to the overpass on $p(\Delta^1_{i\lambda})$, and the relation r_k corresponds to a crossing point and the relation Ω_i corresponds to the $p(T_i)$. The relation r_k is the form

$$(x^i_s)(x^j_t)^\varepsilon(x^i_{s+1})^{-1}(x^j_t)^{-\varepsilon}=1 \quad (\varepsilon=1 \text{ or } -1)$$

as shown in Figure 1, and the relation Ω_i is obtained by running clockwise around the boundary of a regular neighborhood $N(p(T_i); S^2_0)$ as shown in Figure 2. If $(P\subset S^3)$ has c factors $Q_1, \cdots, Q_c$, then we may assume

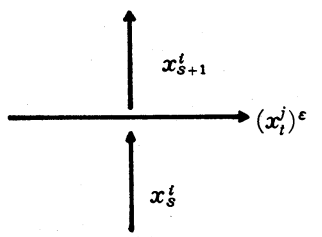
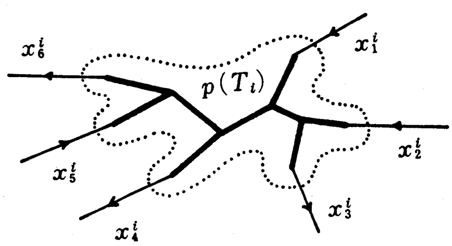


FIGURE 1



$(x^i_1)(x^i_2)(x^i_3)^{-1}(x^i_4)^{-1}(x^i_5)(x^i_6)^{-1}=1$

FIGURE 2

that $p(Q_j)\cap p(Q_k)=\emptyset$ for $j\neq k$, and any one of the relations of $G(Q_j)$ is a consequence of the others by the same reason as that of knots (see Crowell-Fox [2, Chap. VI. (2.5)]).

(1.2.3) Shrinking the tree T_i to one point $o_i\in T_i$, we have a wedge of β_i simple loops. Moreover, we stretch the wedge point o_i so that we obtain a β_i-leafed rose P^*_i, which consists of an oriented link of β_i components, say $L_i=K_{i1}\cup\cdots\cup K_{i\beta_i}$, and a star graph T^*_i, as shown in Figure 3. The orientation of each $K_{i\lambda}$ is, of course, coherent with that of $\Delta^1_{i\lambda}$, and we have the canonical fundamental cycles $k^*_i=\{k^*_{i1}, \cdots, k^*_{i\beta_i}\}$

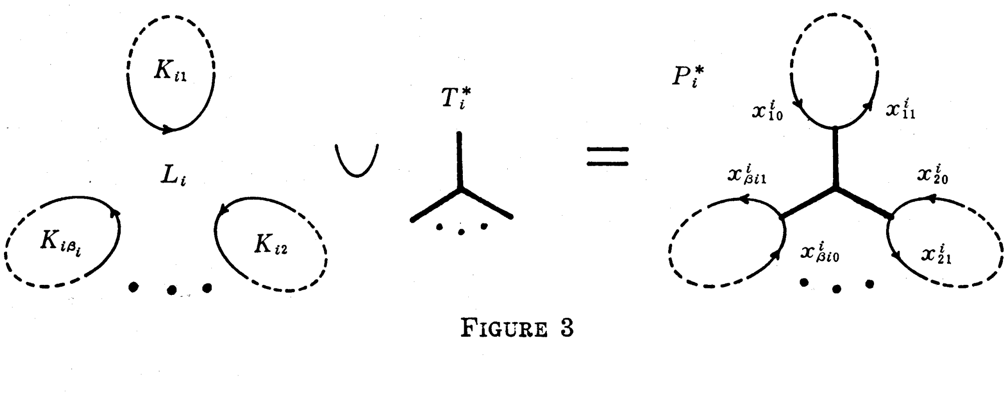


FIGURE 3

on P_i^* and $l_0^*=\{k_1^*, \cdots, k_\mu^*\}$ on $P^*=P_1^*\cup\cdots\cup P_\mu^*$. We call $(P^*\subset S^3)=(P_1^*\cup\cdots\cup P_\mu^*\subset S^3)$ the *associated rose*, and the link $(L\subset S^3)=(K_{11}\cup\cdots\cup K_{1\beta_1}\cup\cdots\cup K_{\mu 1}\cup\cdots\cup K_{\mu\beta_\mu}\subset S^3)$ the *associated link*. Of course, the associated rose $(P^*\subset S^3)$ is not uniquely determined, however it holds that $M(P^*)\cong M(P)$ and so $G(P^*)\cong G(P)$.

Now let $m(i, \lambda)$ be the number of overcrossing points of $p(P^*)$ on the arc $p(K_i-T_i^*)$. Then we have the following presentation of $G(P^*)$:

$$(1.2.4)\qquad \left\langle \begin{matrix} x_{\lambda 0}^i, x_{\lambda 1}^i, \cdots, x_{\lambda m(i,\lambda)}^i \\ (i=1, \cdots, \mu; \lambda=1, \cdots, \beta_i) \end{matrix} \middle| \begin{matrix} r_1, \cdots, r_m \\ \Omega_1^*, \cdots, \Omega_\mu^* \end{matrix} \right\rangle ,$$

where the relation Ω_i^* corresponding to $p(T_i^*)$ may be assumed of the form

$$\Omega_i^*: (x_{10}^i)(x_{11}^i)^{-1}(x_{20}^i)(x_{21}^i)^{-1}\cdots(x_{\beta_i 0}^i)(x_{\beta_i 1}^i)^{-1}=1 ,$$

and some c relations of these $m+\mu$ relations may be omitted provided that $(P^*\subset S^3)$ has c factors.

§2. Alexander ideals of graphs.

Let $t_{i\lambda}$ be an unknotted simple oriented loop in S^3-P for $i=1, \cdots, \mu$ and $\lambda=1, \cdots \beta_i$, such that $t_{i\lambda}$ bounds a disk $D_{i\lambda}$ in S^3 with $D_{i\lambda}\cap P=D_{i\lambda}\cap\Delta_{i\lambda}^1$ consists of one point and the linking number $lk(t_{i\lambda}, k_{i\lambda})=1=lk(t_{i\lambda}, k_{i\lambda}^*)$. Then, $\{t_{i\lambda}\,|\,i=1, \cdots, \mu; \lambda=1, \cdots, \beta_i\}$ forms a free abelian basis for each $H_1(M(P); \boldsymbol{Z})$, $H_1(M(P^*); \boldsymbol{Z})$ and $H_1(M(L); \boldsymbol{Z})$, free abelian groups of rank β, which is dual to the free abelian bases $\{k_{i\lambda}\,|\,i=1, \cdots, \mu; \lambda=1, \cdots, \beta_i\}$ for $H_1(P; \boldsymbol{Z})$, and $\{k_{i\lambda}^*\,|\,i=1, \cdots, \mu; \lambda=1, \cdots, \beta_i\}$ for $H_1(P^*; \boldsymbol{Z})$ and $H_1(L; \boldsymbol{Z})$. Let

$$\alpha: G(P)\to H(P)=H_1(M(P); \boldsymbol{Z}) ,$$
$$\alpha: G(P^*)\to H(P^*)=H_1(M(P^*); \boldsymbol{Z}) \quad\text{and}\quad \alpha: G(L)\to H(L)=H_1(M(L); \boldsymbol{Z})$$

be the abelianizers respectively defined by

$$\alpha(g)=\prod t_{i\lambda}^{lk(g,k_{i\lambda})} , \quad g\in G(P) ;$$
$$\alpha(g)=\prod t_{i\lambda}^{lk(g,k_{i\lambda}^*)} , \quad g\in G(P^*) \quad\text{or}\quad G(L) .$$

Using Fox's free differential calculus ([2], [3], [4]), we have an $(m+\mu)\times(m+\beta)$ Jacobian $A(G(P), \alpha)\equiv A(G(P), l_0)$ of $G(P)$ associated with the presentation (1.2.2) as follows:

$$A(G(P), l_0)=\left\|\begin{matrix} A(r) \\ A(\Omega) \end{matrix}\right\|, \ A(r)=\left\|\alpha\phi\left(\frac{\partial r_k}{\partial x_{\lambda\nu}^i}\right)\right\|, \ A(\Omega)=\left\|\alpha\phi\left(\frac{\partial \Omega_i}{\partial x_{\lambda\nu}^i}\right)\right\| ,$$

where $i=1, \cdots, \mu$; $k=1, \cdots, m$; $\lambda=1, \cdots, \beta_i$; $\nu=0, 1, \cdots m(i, \lambda)$. We call the matrix $A(G(P), l_0)$ the *Alexander matrix* of $(P\subset S^3)$ (associated with a fundamental cycle l_0).

For each nonnegative integer d, the d^{th} elementary ideal $E_d(A(G(P), l_0))\equiv E_d(P, l_0)$ of the Alexander matrix $A(G(P), l_0)$ is defined as Fox [4, § 4] and Kinoshita [11, § 1], and we call such the elementary ideal $E_d(P, l_0)$ the d^{th} *Alexander ideal* of $(P\subset S^3)$ associated with l_0.

The Alexander matrices $A(G(P^*), l_0^*)$ and $A(G(L), l_0^*)$ of the $(P^*\subset S^3)$ and $(L\subset S^3)$, respectively, associated with the induced fundamental cycle l_0^* are defined as the same way by using the above abelianizers, and so we have the d^{th} *Alexander ideals* $E_d(P^*, l_0^*)$ and $E_d(L, l_0^*)$ of $(P^*\subset S^3)$ and $(L\subset S^3)$, respectively, associated with l_0^* for each nonnegative integer d.

2.1. PROPOSITION. *Let $(P\subset S^3)$ be a graph, and let l_0 be a fundamental cycle on P as in* (1.2.1). *Let $(P^*\subset S^3)$ be an associated rose with the induced fundamental cycle l_0^*, and let $(L\subset S^3)$ be the associated link with the l_0^*. Then, for any nonnegative integer d, it holds that:*

$$E_d(P, l_0)=E_d(P^*, l_0^*)\subset E_d(L, l_0^*) \ .$$

PROOF. The first half of Proposition follows the construction of P^* in (1.2.1) and (1.2.3) and the above definition of the abelianizers. Now we assume that $G(P^*)$ has a presentation of the form (1.2.4). Then $G(L)$ has a presentation

$$(2.1.1)\quad \left\langle \begin{matrix} x_{\lambda 0}^i, x_{\lambda 1}^i, \cdots, x_{\lambda m(i,\lambda)}^i \\ (i=1, \cdots, \mu;\ \lambda=1, \cdots, \beta_i) \\ \ \end{matrix} \middle| \begin{matrix} r_1, \cdots, r_m\ , \\ x_{\lambda 0}^i=x_{\lambda 1}^i \\ (i=1, \cdots, \mu;\ \lambda=1, \cdots, \beta_i) \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} x_{\lambda 1}^i, \cdots, x_{\lambda m(i,\lambda)}^i \\ (i=1, \cdots, \mu;\ \lambda=1, \cdots, \beta_i) \end{matrix} \middle| \ r_1', \cdots, r_m' \right\rangle ,$$

where r_k' is obtained from r_k by substituting $x_{\lambda 1}^i$ for $x_{\lambda 0}^i$.

Under the equation $x_{\lambda 0}^i=x_{\lambda 1}^i$ $(i=1, \cdots, \mu; \lambda=1, \cdots, \beta_i)$, the relations $\Omega_1^*, \cdots, \Omega_\mu^*$ are trivial relations, so we have a homomorphism ξ of $G(P^*)$ onto $G(L)$ defined by

$$\begin{matrix} \xi(x_{\lambda 0}^i)=x_{\lambda 1}^i\ , \\ \xi(x_{\lambda \nu}^i)=x_{\lambda \nu}^i \end{matrix} \quad \begin{pmatrix} i=1, \cdots, \mu;\ \lambda=1, \cdots, \beta_i \\ \nu=1, \cdots, m(i, \lambda) \end{pmatrix} ,$$

which is consistent with the abelianizers. Now, Proposition follows from Kinoshita [11, Theorem 1]. □

2.2. **Theorem.** *In the notation of Proposition* 2.1, *it holds that*:

$$E_{d+\beta-\mu}(P,\, l_0)=E_{d+\beta-\mu}(P^*,\, l_0^*)\supset E_d(L,\, l_0^*)\ .$$

Proof. The Alexander matrix $A(G(P^*),\, l_0^*)$ associated with the presentation (1.2.4) is an $(m+\mu)\times(m+\beta)$ matrix of the form

$$\| A_1 | \cdots\cdots | A_i | \cdots\cdots | A_\mu \|$$

such that, for each $i=1,\cdots,\mu$,

$$A_i=\begin{array}{c} \\ r_1 \\ \vdots \\ r_m \\ \Omega_1^* \\ \vdots \\ \Omega_i^* \\ \vdots \\ \Omega_\mu^* \end{array}\left\|\begin{array}{c|c|c|c} x_{10}^i x_{20}^i \cdots x_{\beta_i 0}^i & x_{11}^i \cdots x_{1m(i,1)}^i & \cdots & x_{\beta_i 1}^i \cdots x_{\beta_i m(i,\beta_i)}^i \\ \alpha\phi\Big(\dfrac{\partial r_k}{\partial x_{\lambda 0}^i}\Big) & \alpha\phi\Big(\dfrac{\partial r_k}{\partial x_{1\nu}^i}\Big) & \cdots & \alpha\phi\Big(\dfrac{\partial r_k}{\partial x_{\beta_i \nu}^i}\Big) \\ \hline 0 & 0 & \cdots & 0 \\ \hline 1\ 1 \cdots 1 & -1\ 0 \cdots 0 & \cdots & -1\ 0 \cdots 0 \\ \hline 0 & 0 & \cdots & 0 \end{array}\right\| .$$

Add the $x_{\lambda 0}^i$-column to the $x_{\lambda 1}^i$-column for $i=1,\cdots,\mu$ and $\lambda=1,\cdots,\beta_i$, and subtract the x_{10}^i-column from each the x_{20}^i-, x_{30}^i-, $\cdots$, $x_{\beta_i 0}^i$-column. Then every entry of the Ω_i^*-row is zero except for the entry of the x_{10}^i-column where 1 stands, for $i=1,\cdots,\mu$. Developing this new matrix at the Ω_i^*-row for $i=1,\cdots,\mu$, $A(G(P^*),\, l_0^*)$ is elementary equivalent to an $m\times(m+\beta-\mu)$ matrix

$$A^0=\| A_1^0 | \cdots\cdots | A_i^0 | \cdots\cdots | A_\mu^0 \|$$

such that, for each $i=1,\cdots,\mu$,

$$A_i^0=\begin{array}{c} r_1 \\ \vdots \\ r_m \end{array}\left\|\ \begin{array}{c} x_{20}^i \cdots x_{\beta_i 0}^i \\ * \end{array} \left|\begin{array}{c:c} x_{11}^i \cdots\cdots & x_{1m(i,1)}^i \cdots \\ \alpha\phi\Big(\dfrac{\partial r_k}{\partial x_{10}^i}+\dfrac{\partial r_k}{\partial x_{11}^i}\Big) & \alpha\phi\Big(\dfrac{\partial r_k}{\partial x_{1\nu}^i}\Big) \end{array}\right| \cdots \left|\begin{array}{c:c} x_{\beta_i 1}^i \cdots\cdots & x_{\beta_i m(i,\beta_i)}^i \\ \alpha\phi\Big(\dfrac{\partial r_k}{\partial x_{\beta_i 0}^i}+\dfrac{\partial r_k}{\partial x_{\beta_i 1}^i}\Big) & \alpha\phi\Big(\dfrac{\partial r_k}{\partial x_{\beta_i \nu}^i}\Big) \end{array}\right.\right\| .$$

On the other hand, the Alexander matrix $A(G(L),\, l_0^*)$ of $G(L)$ associated with the presentation (2.1.1) is an $m\times m$ matrix

$$\| A_1' | \cdots\cdots | A_i' | \cdots\cdots | A_\mu' \|$$

such that, for each $i=1,\cdots,\mu$,

$$A'_i = \begin{matrix} & x^i_{11} \cdots x^i_{1m(i,1)} & \cdots & x^i_{\beta_i 1} \cdots x^i_{\beta_i m(i,\beta_i)} \\ \begin{matrix} r'_1 \\ \vdots \\ r'_m \end{matrix} & \left\| \alpha\phi\left(\frac{\partial r'_k}{\partial x^i_{1\nu}}\right) \right| & \cdots & \left| \alpha\phi\left(\frac{\partial r'_k}{\partial x^i_{\beta_i \nu}}\right) \right\| \end{matrix} .$$

Here, it is easy to see that:

$$\alpha\phi\left(\frac{\partial r'_k}{\partial x^i_{\lambda 1}}\right) = \alpha\phi\left(\frac{\partial r_k}{\partial x^i_{\lambda 0}} + \frac{\partial r_k}{\partial x^i_{\lambda 1}}\right), \quad \begin{pmatrix} k=1, \cdots, m \\ i=1, \cdots, \mu \\ \lambda=1, \cdots, \beta_i \end{pmatrix},$$

$$\alpha\phi\left(\frac{\partial r'_k}{\partial x^i_{\lambda\nu}}\right) = \alpha\phi\left(\frac{\partial r_k}{\partial x^i_{\lambda\nu}}\right), \quad \begin{pmatrix} \lambda=1, \cdots, \beta_i \\ \nu=2, \cdots, m(i, \lambda) \end{pmatrix},$$

and so $A(G(L), l^*_0)$ is a submatrix of A^0. The proof of Theorem 2.2 is now complete. □

2.3. In the notation of 1.2, we consider a set of integral 1-cycles $l=\{k'_{i\lambda} \mid i=1, \cdots, \mu; \lambda=1, \cdots, \beta_i\}$ on P such that $k'_{i1}, \cdots, k'_{i\beta_i}$ are integral 1-cycles on P_i, $i=1, \cdots, \mu$, and let $l^*=\{k'^*_{i\lambda} \mid i=1, \cdots, \mu; \lambda=1, \cdots, \beta_i\}$ be the induced set of 1-cycles on P^* and L. Then $k'_{i\lambda}$ and $k'^*_{i\lambda}$ can be expressed as

$$(2.3.1) \qquad k'_{i\lambda} = \sum_{u=1}^{\beta_i} c(i, \lambda, u) k_{iu}, \quad k'^*_{i\lambda} = \sum_{u=1}^{\beta_i} c(i, \lambda, u) k^*_{iu}$$

for some integers $c(i, \lambda, u)$, for $i=1, \cdots, \mu; \lambda=1, \cdots, \beta_i$.

Let H_β be a multiplicative free abelian group of rank β generated by $\{t'_{i\lambda} \mid i=1, \cdots, \mu; \lambda=1, \cdots, \beta_i\}$. Associating with the l, we define a homomorphism $\theta: G(P) \to H_\beta$ by

$$(2.3.2) \qquad \theta(g) = \prod_{i=1}^{\mu} \prod_{\lambda=1}^{\beta_i} (t'_{i\lambda})^{lk(g, k'_{i\lambda})}, \quad g \in G(P),$$

and we have the d^{th} elementary ideal $E_d(G(P), \theta) \equiv E_d(P, l)$ of the Jacobian $A(G(P), \theta) \equiv A(G(P), l)$ of $G(P)$ at θ in the sense of Kinoshita [11, §1]. It should be noted that:

$$(2.3.3) \qquad lk(g, k'_{i\lambda}) = \sum_{u=1}^{\beta_i} c(i, \lambda, u) lk(g, k_{iu}).$$

Let $\sigma: H(P) \to H_\beta$ be a canonical homomorphism defined by

$$(2.3.4) \qquad \sigma(t_{i\lambda}) = \prod_{u=1}^{\beta_i} (t'_{iu})^{c(i,\lambda,u)}, \qquad i=1, \cdots, \mu, \ \lambda=1, \cdots, \beta_i .$$

Then, $\theta=\sigma\alpha$; $G(P) \to H_\beta$, and as Kinoshita [11, §5] we have:

2.4. THEOREM. *In the above notation, it holds that:*

$$E_d(P, l)(t'_{11}, \cdots, t'_{i\nu}, \cdots, t'_{\mu\beta_\mu})$$
$$=E_d(P, l_0)\left(\prod_{u=1}^{\beta_1}(t'_{1u})^{c(1,1,u)}, \cdots, \prod_{u=1}^{\beta_i}(t'_{iu})^{c(i,\nu,u)}, \cdots, \prod_{u=1}^{\beta_\mu}(t'_{\mu u})^{c(\mu,\beta_\mu,u)}\right)$$

for each nonnegative integer d. Therefore, under the canonical homomorphism $\theta^*: G(P^*) \to H_\beta$, *Proposition* 2.1 *and Theorem* 2.2 *hold for elementary ideals* $E_d(P, l)$, $E_d(P^*, l^*)$ *and* $E_d(L, l^*)$, *i.e.*

$$E_d(P, l) = E_d(P^*, l^*) \subset E_d(L, l^*) ,$$
$$E_{d+\beta-\mu}(P, l) = E_{d+\beta-\mu}(P^*, l^*) \supset E_d(L, l^*) .$$

2.5. REMARK. Let $c(i, \lambda) = \sum_{u=1}^{\beta_i} c(i, \lambda, u)$ for $i=1, \cdots, \mu$; $\lambda=1, \cdots, \beta_i$. Let τ be a homomorphism of $H(P)$ onto the multiplicative infinite cyclic group H generated by t such that

$$\tau(t_{i\lambda}) = t^{c(i,\lambda)} , \qquad i=1, \cdots, \mu;\ \lambda=1, \cdots, \beta_i .$$

Then, associated with a set of integral 1-cycles l on P, we have a homomorphism $\psi = \tau\alpha$ of $G(P)$ onto H defined by

$$\psi(g) = t^{lk(g,l)} , \quad g \in G(P) .$$

Moreover, let H_μ be a free abelian group of rank μ generated by $\{t_1, \cdots, t_\mu\}$, and let ρ be a homomorphism of $H(P)$ onto H_μ defined by

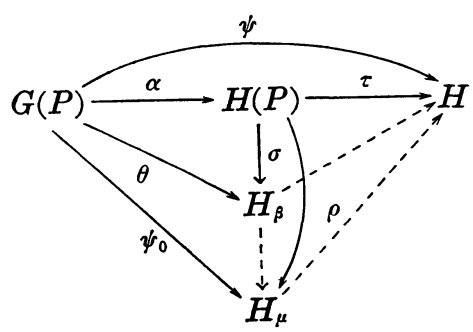
$$\rho(t_{i\lambda}) = t_i^{c(i,\lambda)} , \qquad i=1, \cdots, \mu;\ \lambda=1, \cdots, \beta_i .$$

Then we have an onto homomorphism $\psi_0 = \rho\alpha: G(P) \to H_\mu$ defined by

$$\psi_0(g) = \prod_{i=1}^{\mu} t_i^{lk(g,k'_i)} , \quad g \in G(P) ,$$

where $k'_i = \{k'_{i1}, \cdots, k'_{i\beta_i}\}$, the set of integral 1-cycles on P_i, $i=1, \cdots, \mu$.

Kinoshita [11, §§2, 3] discussed mainly the elementary ideals $E_d(G(P), \psi)$ and $E_d(G(P), \psi_0)$ of Jacobians $A(G(P), \psi)$ and $A(G(P), \psi_0)$, respectively, and Suzuki [19, §§4, 5] $E_d(G(P), \psi)$ alone.



2.6. EXAMPLE (Alford [1]). Alford [1] showed that the knot in Figure 4 has two minimal spanning surfaces S_1 and S_2 such that S^3-S_1 and S^3-S_2 are not homeomorphic. We will show this by using the Alexander ideals. The spine of S_1 [1, Figure 2] is a graph P_1 as shown in Figure 5, and the spine of S_2 [1, Figure 6] is a graph P_2 as shown in Figure 6. As shown in Figure 5 and Figure 6, we choose maximal trees $T_1\subset P_1$ and $T_2\subset P_2$, respectively, and let $l_0^1=\{k_1^1, k_2^1\}$ and $l_0^2=\{k_1^2, k_2^2\}$ be fundamental cycles on P_1 and P_2. Then

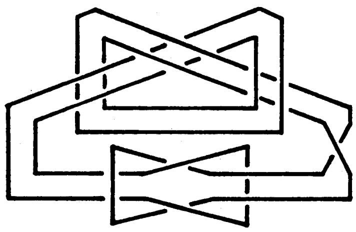


FIGURE 4

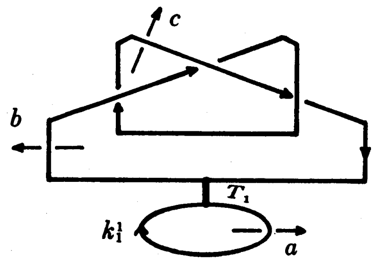


FIGURE 5

$$G(P_1)=\langle a, b, c\,|\,bcb=cbc\rangle ,$$
$$G(P_2)=\langle A, B, O, Q\,|\,ABA=BAB , \quad O^2=QBAB^{-1}ABA^{-1}QBAB^{-1}AB\rangle ,$$

where A, B, O and Q are given in Figure 7 (see [1, Theorem]).

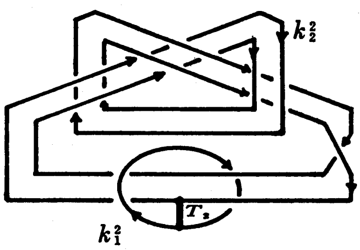


FIGURE 6

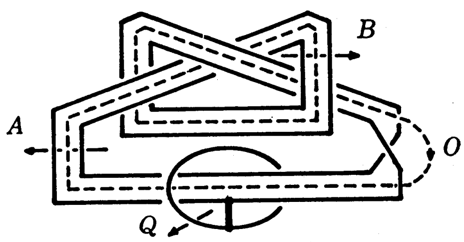


FIGURE 7

Let $\{s, t\}$ be the free abelian basis for $H(P_1)$ and $H(P_2)$, which is dual to free abelian bases $\{k_1^1, k_2^1\}$ for $H_1(P_1; \boldsymbol{Z})$ and $\{k_1^2, k_2^2\}$ for $H_1(P_2; \boldsymbol{Z})$, respectively. We have:

$$A(G(P_1), l_0^1)\sim\|\, t^2-t+1 \quad 0 \quad 0\,\| ,$$

therefore

$$E_d(P_1, l_0^1)=\begin{cases}(0) & d=0, 1 ,\\ (t^2-t+1) & d=2 ,\\ (1) & 3\leqq d .\end{cases}$$

Since $\alpha\phi(A)=\alpha\phi(B)=t^2$, $\alpha\phi(O)=st^5$ and $\alpha\phi(Q)=s$,

$$A(G(P_2), l_0^2)\sim\|t^4-t^2+1 \quad 0 \quad 0\| ,$$

therefore

$$E_d(P_2, l_0^2)=\begin{cases}(0) & d=0, 1 ,\\ (t^4-t^2+1) & d=2 ,\\ (1) & 3\leqq d .\end{cases}$$

This implies that for any set of integral 1-cycles l on P_2, $E_2(P_2, l)$ can not be equal to (t^2-t+1) by Theorem 2.4, and so we conclude that $G(P_1)\not\cong G(P_2)$ and $S^3-S_1\cong S^3-P_1\not\cong S^3-P_2\cong S^3-S_2$.

§3. Retractible and boundary-retractible cubes-with-holes.

For a connected graph $(P\subset S^3)$, the exterior $M(P)$ is called a *cube-with-holes* of genus β (Lambert [14], Jaco-McMillan [9]). We call a cube-with-holes $M(P)$ of genus β *retractible* iff $M(P)$ can be retracted onto a wedge W_β of β simple loops in $M(P)$. If such a wedge W_β can be chosen in $\partial M(P)$, then $M(P)$ is *boundary-retractible* (Jaco-McMillan [9]).

3.1. THEOREM. *Let $(P\subset S^3)$ be a connected graph. If $M(P)$ is retractible, then for any set of integral 1-cycles l on P, $E_d(P, l)=0$ for $0\leqq d\leqq\beta-1$, and $E_\beta(P, l)$ is a principal ideal $(\Delta(t_1, \cdots, t_\mu))$ with $\Delta(1, \cdots, 1)=1$.*

PROOF. By Theorem 2.4, it suffices to prove that $E_d(P, l_0)=(0)$ for $0\leqq d\leqq\beta-1$, and $E_\beta(P, l_0)$ is principal for a fundamental cycle l_0 on P. Now, Theorem follows from Proposition 1.1, Jaco-McMillan [9, Corollary to Theorem 2] and Hillman [8, Theorem VI and p. 42]. □

The following theorem responds partially to Question 1 of McMillan [16].

3.2. THEOREM. *Let $(P\subset S^3)$ be a connected graph. If $M(P)$ is boundary-retractible, then there exists a boundary link $K=K_1\cup\cdots\cup K_\beta$ in $\partial M(P)\subset S^3$ satisfying the following: for any fundamental 1-cycle k_0^* on K, there exists a set of integral 1-cycles l on P such that $E_d(P, l)=E_d(K, k_0^*)=(0)$ for $0\leqq d\leqq\beta-1$, and $E_\beta(P, l)=E_\beta(K, k_0^*)$ is principal $(\Delta(t_1, \cdots, t_\beta))$ with* (i) *$\Delta(1, \cdots, 1)=1$,* (ii) *$\Delta(t_1, \cdots, t_\beta)=\Delta(t_1^{-1}, \cdots, t_\beta^{-1})$.*

PROOF. Since $M(P)$ is boundary-retractible, there is a wedge of β simple loops, say $W_\beta=J_1\vee\cdots\vee J_\beta$, on $\partial M(P)=\partial N(P; S^3)$ so that $M(P)$

retracts on W_β. An intermediate step in the proof of Jaco-McMillan [9, Theorem 3] asserts that there exist mutually disjoint connected orientable surfaces $F_1, \cdots, F_\beta$ properly embedded in $M(P)$, each with connected boundary $\partial F_\lambda = K_\lambda$, so that $K_\lambda \cap J_\lambda$ consists of one crossing point and $K_\lambda \cap J_\nu = \emptyset$ for $\lambda \neq \nu$ (see also Lambert [14, Theorem 2]). We will show that $K = K_1 \cup \cdots \cup K_\beta$ is a required link.

Let $r: M(P) \to W_\beta$ be a retraction. Then the homomorphism $r_*: G(P) \to \pi_1(W_\beta)$ induced by r induces an isomorphism of $H(P)$ onto the free abelian group $H_1(W_\beta; Z)$ and so oriented $\{J_1, \cdots, J_\beta\}$ forms a free abelian basis for $H(P)$. Let k_λ be a fundamental 1-cycle on K_λ, $\lambda = 1, \cdots, \beta$, so that the intersection number of k_λ with J_λ is 1, and let $k_0^* = \{k_1, \cdots, k_\beta\}$. Since $N(P; S^3)$ collapses to P, we have the set of integral 1-cycles $l = \{k_1', \cdots, k_\beta'\}$ on P such that k_λ' is induced from k_λ, $\lambda = 1, \cdots, \beta$.

We define Y to be the space obtained from $M(P)$ by cutting along the $F_1 \cup \cdots \cup F_\beta$ (see Levine [15], Gutiérrez [6], etc.). Then Y is a compact 3-manifold with boundary $F_{10} \cup F_{11} \cup \cdots \cup F_{\lambda 0} \cup F_{\lambda 1} \cup (\partial M(P) - K)$, where $F_{\lambda 0} \cong F_{\lambda 1} \cong F_\lambda$ for $\lambda = 1, \cdots, \beta$. For each $g \in H(P)$, let $Y(g)$ be a copy of Y.

Let $\tilde{M}$ be the universal abelian covering space of $M(P)$ associated to the commutator subgroup $G(P)'$ of $G(P)$. Since the sequence

$$1 \longrightarrow G(P)' \longrightarrow G(P) \longrightarrow H(P) \cong H_1(W_\beta; Z) \longrightarrow 0 \tag{3.2.1}$$

is exact, it follows that $\tilde{M}$ is obtained from $\cup_{g \in H(P)} Y(g)$ by identifying $F_{\lambda 0}(g + J_\lambda)$ to $F_{\lambda 1}(g)$. Now Theorem 3.2 follows from the same argument as that of boundary links by Gutiérrez [7, §§1, 2], (see also Smythe [18]). □

3.3. **Remark.** In the proof of Theorem 3.2, each J_λ of W_β is not always contractible in $N(P; S^3)$. If every J_λ is contractible in $N(P; S^3)$, then we can take a fundamental cycle l_0 on P, so that the associated link is of the same type as the boundary link K.

3.4. **Proposition.** *Let $\Delta(t_1, \cdots, t_\beta)$ be an integral polynomial in β variables satisfying* (i) *and* (ii) *in Theorem* 3.2. *Then, there exists a connected graph $(P \subset S^3)$ and a fundamental cycle l_0 on P such that $E_d(P, l_0) = (0)$ for $0 \leqq d \leqq \beta - 1$, and $E_\beta(P, l_0) = (\Delta(t_1, \cdots, t_\beta))$.*

Proof. By Gutiérrez [7, Theorem 4], there exists a boundary link $(L \subset S^3)$ with β components $K_1, \cdots, K_\beta$ such that $E_d(L, l_0) = (0)$ for $0 \leqq d \leqq \beta - 1$, and $E_\beta(L, l_0) = (\Delta(t_1, \cdots, t_\beta))$ for a fundamental cycle l_0 on L. Let $F_1, \cdots, F_\beta$ be mutually disjoint connected oriented surfaces in S^3

with $\partial F_\lambda = K_\lambda$, $\lambda=1, \cdots, \beta$. We connect $K_1, \cdots, K_\beta$ by mutually disjoint simple arcs $e_1, \cdots, e_{\beta-1}$ with $(e_1 \cup \cdots \cup e_{\beta-1}) \cap (\mathrm{Int}(F_1) \cup \cdots \cup \mathrm{Int}(F_\beta)) = \emptyset$. Now it is easily seen that $P = L \cup e_1 \cup \cdots \cup e_{\beta-1}$ is a desired graph with a desired fundamental cycle l_0. (See Example 4.4 below.) □

§ 4. Examples.

In the remainder of the paper, we shall calculate Alexander ideals of some examples of graphs, which may be of interest to some readers.

4.1. EXAMPLE (Figure 8, Jaco-McMillan [9]). Let $(P \subset S^3)$ be the graph of $\beta=3$ as shown in Figure 8 used in Theorem 6 of Jaco-McMillan [9]. $G(P)$ has a presentation

$$\left\langle \begin{matrix} a, b, c, d \\ x, y, v, w \end{matrix} \middle| a^2b^3 = x^2y^3 , \begin{matrix} a = c^{-2}d^{-1}, \ b = dc^2dc^{-1}d^{-1} \\ x = v^{-2}w^{-1}, \ y = wv^2wv^{-1}w^{-1} \end{matrix} \right\rangle$$
$$= \langle a, b, x, y \mid a^2b^3 = x^2y^3 \rangle .$$

Choose a fundamental cycle $l_0 = \{k_1, k_2, k_3\}$ as shown in the figure. Then we have $\alpha\phi(a) = s^{-3}u^{-1}$, $\alpha\phi(b) = s^2u$, $\alpha\phi(x) = t^{-3}u^{-1}$ and $\alpha\phi(y) = t^2u$, where $\{s, t, u\}$ is the free abelian basis for $H(P)$ with $lk(s, k_1) = lk(t, k_2) = lk(u, k_3) = 1$. Hence, we have;

$$A(P, l_0) \sim \| s^2-s+1 \ \ t^2-t+1 \ \ u-1 \ \ 0 \| ,$$
$$E_3(P, l_0) = (s^2-s+1, t^2-t+1, u-1) \qquad \text{(not principal)} .$$

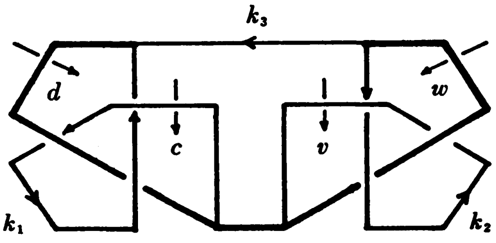


FIGURE 8

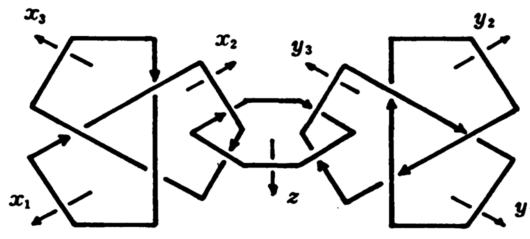


FIGURE 9

This implies that for any set of integral 1-cycles l (except for some trivial cases) on P the 3rd elementary ideal $E_3(P, l)$ is not principal by Theorem 2.4, therefore Theorem 3.1 gives an alternate proof that $M(P)$ is *not* retractible.

Figure 9 shows the associated oriented link $L = K_1 \cup K_2 \cup K_3$. $G(L)$ has a presentation

$$\left\langle \begin{matrix} x_1, x_2, x_3 \\ y_1, y_2, y_3, z \end{matrix} \middle| \begin{matrix} x_3x_1 = x_2x_3, \ x_2x_3 = x_1x_2, \ x_2zx_1^{-1} = zx_1^{-1}x_3 \\ y_1y_3 = y_3y_2, \ y_2y_1 = y_3y_2, \ y_2y_1z^{-1} = y_1z^{-1}y_3 \end{matrix} \right\rangle$$

and so

$$A(L, l_0^*) \sim \left\| \begin{matrix} s^2-s+1 & 0 & 0 & 0 \\ 0 & t^2-t+1 & 0 & 0 \\ 0 & 0 & u-1 & 0 \end{matrix} \right\| .$$

4.2. EXAMPLE (Figure 10; Lambert [14]): Let $(P \subset S^3)$ be the graph as shown in Figure 10, whose exterior $M(P)$ is *not* boundary-retractible by Lambert [14, Theorem 1] and Jaco-McMillan [9, p. 155 Example], and Jaco-McMillan showed that $M(P)$ is retractible. For the fundamental cycle l_0 shown in Figure 10, $G(P)$ has a presentation $\langle c, g, x \mid [c, [g, x]] = x \rangle$ with $\alpha\phi(c)=s$, $\alpha\phi(g)=t$ and $\alpha\phi(x)=1$. Thus, we have:

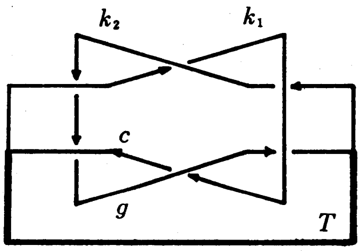


FIGURE 10

$$A(P, l_0) = \| 0 \ \ 0 \ \ s+t-1 \| ,$$

therefore $E_2(P, l_0) = (s+t-1)$ (principal). This implies that for any set of integral 1-cycles l on P, the 2[nd] elementary ideal $E_2(P, l)$ is principal, but it can not satisfy the condition (ii) in Theorem 3.2 (except for some trivial cases) by Theorem 2.4. Hence, Theorem 3.2 gives an alternate proof that $M(P)$ is not boundary-retractible.

4.3. EXAMPLE (Figure 11; Kinoshita [11]): Let $(P \subset S^3)$ be the graph as shown in Figure 11, which was examined in Kinoshita [12] and [13]. $G(P)$ has the following presentation:

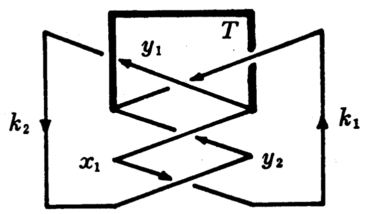


FIGURE 11

$$\langle x_1, y_1, y_2 \mid y_2x_1y_1y_2^{-1}x_1y_2y_1y_2^{-1}x_1^{-1}y_2y_1^{-1}x_1^{-1}y_2^{-1}x_1y_2^{-1}x_1^{-1} = 1 \rangle .$$

For the fundamental cycle $l_0=\{k_1, k_2\}$ shown in the figure, we have:

$$A(P, l_0)\sim\| st+t+1 \;\; 2 \;\; 0 \| ,$$

therefore $E_2(P, l_0)=(st+t+1, 2)$, (not principal). This means that for any set of integral 1-cycles l (except for some trivial cases) on P, the 2[nd] elementary ideal $E_2(P, l)$ is not principal, and we conclude that $M(P)$ is *not* retractible by Theorem 3.1. (We refer to Jaco [10] for a graph $(P\subset S^3)$ of $\beta=2$ with $M(P)$ non-retractible.)

4.4. EXAMPLE (Figure 13): The link $(L\subset S^3)$ shown in Figure 12 is a well-known boundary link. We give an orientation (i.e. a fundamental cycle $l_0^*=\{k_1, k_2\}$) on L as shown in the figure, and let $(P\subset S^3)$ be a graph obtained from $(L\subset S^3)$ as shown in Figure 13 with the fundamental cycle l_0 induced from l_0^*, so that $M(P)$ is boundary-retractible. Then, $G(P)$ has a presentation

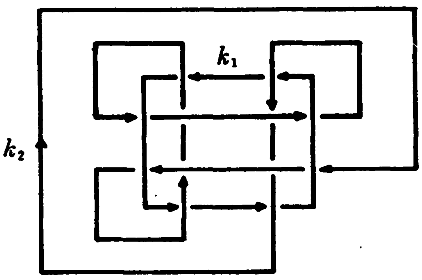


FIGURE 12

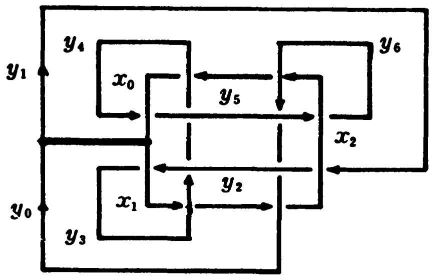


FIGURE 13

$$\left\langle \begin{array}{l|l} x_0,\ x_1,\ x_2 & y_6x_2y_6^{-1}y_4=y_4x_0,\ y_3x_1y_3^{-1}y_0=y_0x_2 \\ y_0,\ y_1,\ y_2,\ y_3 & y_2y_3y_2^{-1}y_5=y_5y_4,\ y_4x_0=x_0y_5,\ x_1y_2=y_3x_1 \\ y_4,\ y_5,\ y_6 & x_2y_5=y_6x_2,\ y_1x_2=x_2y_2,\ y_0y_1^{-1}x_0x_1^{-1}=1 \end{array} \right\rangle ,$$

with $\alpha\phi(x_i)=s$ $(i=0, 1, 2)$ and $\alpha\phi(y_j)=t$ $(j=0, 1, \cdots, 6)$. Thus,

$$A(P, l_0)\sim\left\| \begin{array}{cccc} st-s+1 & 0 & 0 & 0 \\ 0 & st-t+1 & 0 & 0 \end{array} \right\| ,$$

therefore $E_2(P, l_0)=E_2(L, l_0^*)=((st-s+1)(st-t+1))$ (principal). It should be noticed that:

$$(st-s+1)(st-t+1)=s^2t^2\left(\frac{1}{st}-\frac{1}{s}+1\right)\left(\frac{1}{st}-\frac{1}{t}+1\right) .$$

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