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# On a Global Solution of Some Quasilinear Hyperbolic Equation

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## Introduction

Our concern is a quasilinear hyperbolic equation of the form

$$u_{tt}(x, t)-a\left(\int_{\Omega} |\nabla u(y, t)|^2 dy\right)\Delta u(x, t)=f(x, t) \tag{1}$$

for $(x, t)\in \Omega\times[0, T]$, $\Omega\subset R^n$, which arises in mathematical physics as a limit case of nonlinear vibration of an elastic string (cf. Nishida [17]).

In this paper we treat the initial-boundary value problem with null Dirichlet condition (in this case $\Omega$ is a bounded domain with smooth boundary $\partial\Omega$) and also the Cauchy problem for the equation of type (1) where $a(\cdot)$ is defined on $[0, \infty)$, locally Lipschitz continuous and bounded from below by a positive constant. In many previous works $a(\cdot)$ has been assumed to be in $C^1$-class. Our results are obtained under a less restrictive condition on the regularity of $a(\cdot)$ that is merely locally Lipschitz continuous.

Pohozaev showed in [21] that the mixed problem above mentioned has a unique global solution $u\in C^1([0, T]; H^\infty(\Omega))$ with $u_{tt}$ being a generalized derivative under some qualitative conditions on smoothness for the data. Actually, if $\Omega$ is analytic, Pohozaev's conditions say equivalently that

(i) the initial data are real analytic in $\bar{\Omega}$

(ii) the forcing term $f$ is continuous in $t$ with values in the space of real analytic functions

satisfying the appropriate compatibility conditions (cf. Lions [14], Lions and Magenes [15]).

Our main purpose is to show existence and uniqueness of a classical solution to the problem which values for each $t$ in some function space $\tilde{P}$ when the initial data belong to $\tilde{P}$ and the forcing term $f$ to $C([0, T]; \tilde{P})$.

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$\tilde{P}$ is the space lying between the analytic class and $\bigcap_{s>1} G_s$, where $G_s$ is the Gevrey class of order $s$, which will be specified later. This generalizes Pohozaev's results.

For the Cauchy problem, Perla [19] proved existence and uniqueness of a local classical solution to the equation (1). Also Yamada [23] obtained a unique global solution to (1) in $\bigcap_{i=0}^{2} C^i([0, T]; H^\infty)$ with $f=0$ under the more restricted data than Perla's, which are rather similar to Pohozaev's one. Here we shall show global existence of a unique solution to (1) under some conditions on the data weaker than Yamada's but stronger than Perla's.

Finally we mention about the related works by the other authors. Dickey [6, 9] and Revera [22] proved the local existence and uniqueness theorem for the mixed or the Cauchy problem. See also Greenberg and Hu [10]. Dickey [7, 8], Medeiros [16] Perla [20], Brito [4], Yamada [24], Nishihara [20] and others investigated the equation (1) with such an additional term as $\Delta^2 u$ or $\lambda u_t$ and proved the global existence and uniqueness of solutions of some regularity to the mixed or the Cauchy problem.

## §1. Formulation of the problem and the main theorem.

Let $H$ be a real, separable Hilbert space with norm $\|\cdot\|$ and scalar product $(\cdot, \cdot)$ and also $A$ be a linear operator with a dense domain $D(A)$ in $H$.

We consider a differential equation

$$\frac{d^2}{dt^2}u(t)+a(\|A^{1/2}u(t)\|^2)Au(t)=f(t) \quad \text{in} \quad H \tag{2}$$

with the 'boundary' condition

$$u(t) \in D(A) \quad \text{for any} \quad t \in [0, T] \tag{3}$$

and the initial conditions

$$u(0)=u_0 \in D(A^{1/2}) , \qquad \frac{du}{dt}(0)=u_1 \in H , \tag{4}$$

where $T>0$ and $f \in C([0, T]; H)$ are given.

We assume the followings on $a(\cdot)$ and $A$:

(H1) $A$ is a self-adjoint, positive definite operator with discrete spectrum and $A^{-1}$ is compact.

(H2) $a(s)$ is a real-valued, locally Lipschitz continuous defined on $s \geqq 0$, and that $a(s) \geqq a_0$ for some constant $a_0 > 0$.

The condition (H1) implies that $A$ has an infinite sequence of eigenvalues $\{\lambda_j^2\}$ with

$$0 < \lambda_1^2 \leqq \cdots \leqq \lambda_j^2 \leqq \cdots, \quad \lim_{j\to\infty} \lambda_j^2 = \infty$$

and that there exists a complete orthonormal system (C.O.N.S.) $\{w_j\}$ in $H$, each $w_j$ being an eigenvector corresponding to $\lambda_j^2$.

We use the method of Fourier series expansion with respect to the C.O.N.S. $\{w_j\}$. Let $u_j(t)$, $j=1, 2, \cdots$, be the Fourier coefficient for $u(t) \in H$. Then we have an expansion:

$$u(t) = \sum_{j=1}^{\infty} u_j(t) w_j \quad \text{with} \quad \|u(t)\| = \Big(\sum_{j=1}^{\infty} u_j(t)^2\Big)^{1/2} .$$

Let $u(t)$ be in $D(A^{k/2})$, then there holds

$$A^{k/2} u(t) = \sum_{j=1}^{\infty} \lambda_j^k u_j(t) w_j \quad \text{with} \quad \|A^{k/2} u(t)\| = \Big(\sum_{j=1}^{\infty} \lambda_j^{2k} u_j(t)^2\Big)^{1/2} .$$

Now let $M(\lambda)$ be a nonnegative continuous function, strictly increasing, defined on $\lambda \geqq 0$, such that $M(0)=0$ and the mapping $\lambda \to M(\lambda^{1/2})$ is convex. Let $F(\lambda) = \sqrt{M(\lambda)}/\lambda$. Then to the function $F$ there corresponds a linear operator $F(A^{1/2})$ in $H$, defined by

$$F(A^{1/2}) u(t) = \sum_{j=1}^{\infty} F(\lambda_j) u_j(t) w_j$$

for $u(t) = \sum_{j=1}^{\infty} u_j(t) w_j$ provided that

$$\begin{aligned} \|F(A^{1/2}) u(t)\| &= \Big(\sum_{j=1}^{\infty} F(\lambda_j)^2 u_j(t)^2\Big)^{1/2} \\ &= \Big(\sum_{j=1}^{\infty} \lambda_j^{-2} M(\lambda_j) u_j(t)^2\Big)^{1/2} < +\infty . \end{aligned} \tag{5}$$

The domain of $F(A^{1/2})$ is the set of all functions $u(t) = \sum_{j=1}^{\infty} u_j(t) w_j \in H$ satisfying the condition (5).

We assume further on the data $u_0$, $u_1$ and $f$ that

$$\begin{aligned} &F(A^{1/2}) A u_0 \in H , \qquad F(A^{1/2}) A^{1/2} u_1 \in H \quad \text{and} \\ &\int_0^T \|F(A^{1/2}) A^{1/2} f(s)\|^2 ds < +\infty \end{aligned} \tag{D1}$$

and

(D2) $t \longrightarrow F(A^{1/2})f(t)$ is continuous from $[0, T]$ to $H$.

EXAMPLE. For $M(\lambda)=\lambda^{2k+2}$, so that $F(\lambda)=\lambda^k$, we have $F(A^{1/2})=A^{k/2}$.

THEOREM 1. *Under the assumptions* (H1), (H2), (D1) *and* (D2), *there exist a positive constant* $T_M$ *and a unique solution* $u(t)$ *on* $[0, T_M]$ *to the problem* (2), (3) *and* (4) *such that*

$$u \in C^2([0, T_M]; H)$$

*and*

$$F(A^{1/2})Au\ ,\quad F(A^{1/2})A^{1/2}u'\ ,\quad F(A^{1/2})u'' \in C([0, T_M]; H)\ ,$$

*where* $'=d/dt$.

REMARK 1. The constant $T_M$ depends on the function $M$ and so on $F$. However, if

$$\int_c^{\infty} \frac{ds}{s(1+d_0M^{-1}(s))} = \infty \tag{6}$$

for some positive constants $c$ and $d_0$, then we can take $T_M=T$, so that in this case our solution is a global one. The condition (6) holds for $M(\lambda)=O(\exp(\delta\lambda/\log\lambda))$, $O(\exp(\delta\lambda^p))$, $p\geqq 1$, as $\lambda\to\infty$ and etc..

## §2. Approximate equations and a priori estimates.

We first define an approximation $a_\varepsilon(\cdot)$ to $a(\cdot)$ for small $\varepsilon>0$. Since $a(\cdot)$ is absolutely continuous by (H2), there exists the derivative $a'(s)$ for a.a. $s\geqq 0$, which is finite, locally integrable and has the representation

$$a(s)=a(0)+\int_0^s a'(\tau)d\tau\ .$$

Also for each constant $K>0$ there is a constant $L>0$ such that ess. $\sup\{|a'(s)|; 0\leqq s\leqq K\}\leqq L$, which implies $a' \in L^\infty_{\mathrm{loc}}(0, \infty)$. We shall fix a constant $K$ so that

$$K \geqq (2e^T/a_0)\Big(\|u_1\|^2+\hat{a}(\|A^{1/2}u_0\|^2)+\frac{a_0}{2}\|A^{1/2}u_0\|^2+\int_0^T \|f(t)\|^2dt\Big) \tag{7}$$

where $\hat{a}(s)=\int_0^s a(\tau)d\tau$. We note $K\geqq \|A^{1/2}u_0\|^2$. Let

$$p(s)=\begin{cases} 0 & s\geqq K \\ a'(s) & 0<s<K \\ 0 & s\leqq 0 \end{cases}$$

and

$$p_\varepsilon(s)=\rho_\varepsilon * p(s)=\int_{-\infty}^{\infty}\rho_\varepsilon(s-\tau)p(\tau)d\tau$$

where $\rho_\varepsilon$ is the Friedrichs mollifier. Thus we define $a_\varepsilon(\cdot)$ by

$$a_\varepsilon(s)=a(0)+\int_0^s p_\varepsilon(\tau)d\tau\ . \tag{8}$$

For the properties of $a_\varepsilon(\cdot)$ we have

LEMMA 1. *Let $K$ and $a_\varepsilon(\cdot)$ be defined above. Then $a_\varepsilon(\cdot)$ is in $C^\infty(-\infty,\infty)$ and there is a positive constant $\varepsilon_0$ such that for any $\varepsilon$ with $0<\varepsilon\leqq\varepsilon_0$,*

$$a_\varepsilon(s)\leqq a(s)+a_0/2 \quad \textit{for} \quad s\in[0,K] \tag{9}$$

$$a_\varepsilon(s)\geqq a_0/2 \quad \textit{for} \quad s\in[0,\infty) \tag{10}$$

*and*

$$\max_{0\leqq s\leqq K}|a_\varepsilon'(s)|\leqq \operatorname*{ess.\,sup}_{0<s<K}|a'(s)|\equiv L\ . \tag{11}$$

PROOF. Since it holds that

$$\begin{aligned}\max_{0\leqq s\leqq K}|a_\varepsilon(s)-a(s)|&=\max_{0\leqq s\leqq K}\left|\int_0^s (p_\varepsilon(\tau)-a'(\tau))d\tau\right|\\ &\leqq\int_0^K|\rho_\varepsilon * p(\tau)-p(\tau)|\,d\tau\longrightarrow 0 \quad \text{as} \quad \varepsilon\longrightarrow 0\ ,\end{aligned}$$

both (9) and (10) for $s\leqq K$ are valid for $\varepsilon>0$ sufficiently small because $a(s)\geqq a_0$. When $s>K$,

$$a_\varepsilon(s)=a_\varepsilon(K)+\int_K^s p_\varepsilon(\tau)d\tau$$

implies (10). Also we have (11) as follows:

$$\begin{aligned}\max_{0\leqq s\leqq K}|a_\varepsilon'(s)|&=\max_{0\leqq s\leqq K}\left|\int_{-\varepsilon}^{\varepsilon}p(s-\tau)\rho_\varepsilon(\tau)d\tau\right|\\ &\leqq\operatorname*{ess.\,sup}_{-\varepsilon<s<K+\varepsilon}|p(s)|=\operatorname*{ess.\,sup}_{0<s<K}|a'(s)|=L\ .\end{aligned}$$

Thus the lemma is proved. Q.E.D.

Now we consider the initial value problems for $0<\varepsilon\leqq\varepsilon_0$ in $H$:

$$v''(t)+a_\varepsilon(\|A^{1/2}v(t)\|^2)Av(t)=f(t) \tag{12$_\varepsilon$}$$

(13) $$v(t) \in D(A)$$

(14) $$v(0)=u_0\,, \qquad v'(0)=u_1\,,$$

which approximates the problem (2), (3) and (4).

We shall employ the Galerkin method. Define for each $m$ and $\varepsilon$

$$v_{m,\varepsilon}(t)=\sum_{j=1}^{m} g_{jm,\varepsilon}(t)w_j$$

(we may drop the suffix $\varepsilon$ of $v_{m,\varepsilon}$ and $g_{jm,\varepsilon}$ without confusions) as the solution of

(15) $$(v''_m(t)+a_\varepsilon(\|A^{1/2}v_m(t)\|^2)Av_m(t),\, w)=(f(t),\, w) \quad \text{for} \quad w \in V_m$$

(16) $$\begin{aligned} v_m(0)&=\sum_{j=1}^{m}\alpha_j w_j=P_m u_0 \\ v'_m(0)&=\sum_{j=1}^{m}\beta_j w_j=P_m u_1 \end{aligned}$$

where $V_m$ is an $m$-dimensional vector space spanned by $\{w_1, \cdots, w_m\}$ and $P_m$ denotes the orthogonal projection in $H$ onto $V_m$. Then (15) with (16) turns to certain nonlinear differential system for the $g_{jm}$'s, which guarantees a unique solution defined on some interval $[0, t_{m,\varepsilon})$. Note that each $g_{jm}$ is in $C^2$ provided $f \in C([0, T]; H)$.

Putting $w=2v'_m(t)$ in (15) and integrating both sides over $[0, t]$, $0<t<t_{m,\varepsilon}$, we have

(17) $$\begin{aligned} &\|v'_m(t)\|^2+\hat{a}_\varepsilon(\|A^{1/2}v_m(t)\|^2) \\ &=\sum_{j=1}^{m}\beta_j^2+\hat{a}_\varepsilon\Big(\sum_{j=1}^{m}\lambda_j^2\alpha_j^2\Big)+2\int_0^t (f(s),\, v_m(s))ds \\ &\leqq \|u_1\|^2+\hat{a}_\varepsilon(\|A^{1/2}u_0\|^2)+\int_0^T \|f(s)\|^2 ds+\int_0^t \|v'_m(s)\|^2 ds \end{aligned}$$

where $\hat{a}_\varepsilon(s)=\int_0^s a_\varepsilon(\tau)d\tau$. Hence, by (9), (10) and Gronwall's inequality,

$$\begin{aligned} &\|v'_m(t)\|^2+\frac{a_0}{2}\|A^{1/2}v_m(t)\|^2 \\ &\leqq e^T\cdot\Big(\|u_1\|^2+\hat{a}(\|A^{1/2}u_0\|^2)+\frac{a_0}{2}\|A^{1/2}u_0\|^2+\int_0^T \|f(s)\|^2 ds\Big)\,, \end{aligned}$$

which yields the inequalities

(18) $$\|v'_m(t)\|^2=\sum_{j=1}^{m} g'_{jm}(t)^2 \leqq a_0 K/2$$

and

$$\|A^{1/2}v_m(t)\|^2=\sum_{j=1}^{m}\lambda_j^2 g_{jm}(t)^2\leqq K\,. \tag{19}$$

Those estimates (18), (19) ensure that all the intervals $[0, t_{m,\varepsilon})$, on which the $v_m(t)$ are defined, can be extended to the whole interval $[0, T]$.

For further a priori estimates we need two lemmas.

Let $\phi$ be a nonnegative, strictly monotonic, continuous function defined on $[0, \infty)$, and fix $2m$ nonnegative constants $\lambda_j$ and $q_j (j=1,2,\cdots,m)$ with $\sum_{j=1}^{m} q_j=1$. Then for the quantity $\mathfrak{M}_\phi$ defined by

$$\mathfrak{M}_\phi=\phi^{-1}\Big(\sum_{j=1}^{m} q_j\phi(\lambda_j)\Big)\,,$$

the following lemma holds:

LEMMA 2 (Hardy, Littlewood and Polya [11], p.p. 75-76). *Let $\phi$ and $\psi$ be both strictly monotonic, continuous and $\psi$ increasing. Then a necessary and sufficient condition that $\mathfrak{M}_\phi\leqq\mathfrak{M}_\psi$ for all $\lambda=(\lambda_j)$, $q=(q_j)$, $\sum q_j=1$, is that $\chi(\cdot)=\psi\circ\phi^{-1}(\cdot)=\psi(\phi^{-1}(\cdot))$ should be convex. Moreover suppose that $\chi(\cdot)$ possesses a second derivative $\chi''(\cdot)$ in the open interval $(0, \infty)$. Then a necessary and sufficient condition that $\chi(\cdot)$ should be convex in the interval is that $\chi''(\cdot)\geqq 0$.*

Also the generalized Gronwall inequality, due to Bihari and Langenhop, is needed (cf. Beckenbach and Bellman [3]).

LEMMA 3. *Let $f\colon [0, \infty)\to[0, \infty)$ be continuous, $g\colon (0, \infty)\to(0, \infty)$ be continuous and nondecreasing and let $c$ be a positive constant. Then the inequality*

$$f(t)\leqq c+\int_0^t g(f(s))ds\,,\quad 0\leqq t<\infty$$

*implies that*

$$f(t)\leqq G^{-1}(G_0)<+\infty\,,\quad 0\leqq t\leqq G_0$$

*for any fixed number $G_0$ less than $G(\infty)$, where*

$$G(t)=\int_c^t \frac{ds}{g(s)}\qquad \text{for}\quad t\geqq c\,.$$

*Moreover, if $G(\infty)=\infty$, then the inequality*

$$f(t)\leqq G^{-1}(t)$$

*is valid for all* $t \geqq 0$.

PROOF. Set $h(t)=c+\int_0^t g(f(s))ds$. *Then*

$$\frac{d}{dt}G(h(t))=\frac{g(f(t))}{g(h(t))}\leqq 1$$

and hence $G(h(t))\leqq t$ because $G(h(0))=G(c)=0$. Thus it follows

$$f(t)\leqq h(t)\leqq G^{-1}(t) \qquad \text{for any } t<G(\infty)$$

which means

$$f(t)\leqq G^{-1}(G_0) \qquad \text{for } t\leqq G_0<G(\infty)$$

since $G^{-1}$ is monotonically increasing. Q.E.D.

Since $\psi\circ\phi^{-1}(\lambda)=M(\lambda^{1/2})$ is convex by our assumption, Lemma 2 for $\phi(t)=t^2$ and $\psi(s)=M(s)$ implies $\mathfrak{M}_\phi\leqq\mathfrak{M}_\psi$, that is,

$$\left(\sum_{j=1}^m g_j\lambda_j^2\right)^{1/2}\leqq M^{-1}\left(\sum_{j=1}^m q_j M(\lambda_j)\right). \tag{20}$$

Substitute $2\sum_{j=1}^m M(\lambda_j)g'_{jm}(t)w_j$ for $w$ in (15). Then we have

$$\begin{aligned}\frac{d}{dt}E_M^m(t)=2&\left(f(t), \sum_{j=1}^m M(\lambda_j)g'_{jm}(t)w_j\right)\\ &+2a'_\epsilon\left(\sum_{j=1}^m\lambda_j^2 g_{jm}(t)^2\right)\left(\sum_{j=1}^m\lambda_j^2 g_{jm}(t)g'_{jm}(t)\right)\left(\sum_{j=1}^m\lambda_j^2 M(\lambda_j)g_{jm}(t)^2\right)\end{aligned} \tag{21}$$

where

$$E_M^m(t)=\sum_{j=1}^m M(\lambda_j)g'_{jm}(t)^2+a_\epsilon\left(\sum_{j=1}^m\lambda_j^2 g_{jm}(t)^2\right)\sum_{j=1}^m\lambda_j^2 M(\lambda_j)g_{jm}(t)^2. \tag{22}$$

We proceed to estimate the last term in (21). From (11) and (19) it follows that

$$\left|a'_\epsilon\left(\sum_{j=1}^m\lambda_j^2 g_{jm}(t)^2\right)\right|\leqq L. \tag{23}$$

Also combining the Schwarz inequality with (19) yields

$$\left|\sum_{j=1}^m\lambda_j^2 g_{jm}(t)g'_{jm}(t)\right|\leqq K^{1/2}\left(\sum_{j=1}^m\lambda_j^2 g'_{jm}(t)^2\right)^{1/2}. \tag{24}$$

When $\sum_{j=1}^m g'_{jm}(t)^2\geqq 1$, since $M^{-1}$ is increasing, by using (18) and (20) we have

$$
\begin{aligned}
\Big(\sum_{j=1}^{m} g'_{jm}(t)^2\lambda_j^2\Big)^{1/2} &= \Big(\sum_{i=1}^{m} g'_{im}(t)^2\Big)^{1/2}\Bigg(\sum_{j=1}^{m} \frac{g'_{jm}(t)^2}{\sum_{i=1}^{m} g'_{im}(t)^2}\lambda_j^2\Bigg)^{1/2} \\
&\leqq (a_0K/2)^{1/2}M^{-1}\Bigg(\sum_{j=1}^{m} \frac{g'_{jm}(t)^2}{\sum_{i=1}^{m} g'_{im}(t)^2}M(\lambda_j)\Bigg) \qquad (25)\\
&\leqq (a_0K/2)^{1/2}M^{-1}\Big(\sum_{j=1}^{m} g'_{jm}(t)^2 M(\lambda_j)\Big)\,.
\end{aligned}
$$

On the other hand, when $\sum_{j=1}^{m} g'_{jm}(t)^2<1$, putting $q_j=g'_{jm}(t)^2$ $(j=1, 2, \cdots, m)$, $q_0=1-\sum_{j=1}^{m} q_j$ and $\lambda_0=0$, we also have

$$
\begin{aligned}
\Big(\sum_{j=1}^{m} g'_{jm}(t)^2\lambda_j^2\Big)^{1/2} &= \Big(\sum_{j=0}^{m} q_j\lambda_j^2\Big)^{1/2} \qquad (26)\\
&\leqq M^{-1}\Big(\sum_{j=0}^{m} q_j M(\lambda_j)\Big) = M^{-1}\Big(\sum_{j=1}^{m} g'_{jm}(t)^2 M(\lambda_j)\Big)
\end{aligned}
$$

since $M(0)=0$. From the estimates (23)-(26) it follows

$$
\begin{aligned}
&|\text{the last term in (21)}| \\
&\quad \leqq 2LK^{1/2}\max((a_0K/2)^{1/2}, 1)M^{-1}(E_M^m(t))\cdot\frac{2}{a_0}E_M^m(t) \\
&\quad \equiv d_0E_M^m(t)M^{-1}(E_M^m(t))
\end{aligned}
$$

(Note the constant $d_0$ is independent of $\varepsilon$). Thus we have from (21)

$$
\frac{d}{dt}E_M^m(t)\leqq\sum_{j=1}^{m} M(\lambda_j)\gamma_j(t)^2+E_M^m(t)(1+d_0M^{-1}(E_M^m(t))) \qquad (27)
$$

where each $\gamma_j(t)$ is the Fourier coefficient of $f(t)$ corresponding to $w_j$. Integrating (27) over $[0, t]$ and using (D1) we get

$$
E_M^m(t)\leqq c+\int_0^t E_M^m(s)(1+d_0M^{-1}(E_M^m(s)))ds \qquad (28)
$$

(Note the constant $c$ is independent of $\varepsilon$ by virtue of (9)).

Here we can apply Lemma 3 to (28) for $g(s)=s(1+d_0M^{-1}(s))$ and $G(t)=\int_0^t (1/s(1+d_0M^{-1}(s)))ds$. Taking a number $T_M$ (cf. Remark 1) so that

$$
T_M=T \qquad \text{if} \quad G(\infty)>T
$$

and

$$
T_M<G(\infty) \qquad \text{if} \quad G(\infty)\leqq T\,,
$$

we have for each $t\in[0, T_M]$

$$E_M^m(t) \leqq G^{-1}(T_M) < +\infty$$

which gives the a priori estimates

$$(29) \qquad \begin{aligned} &\sum_{j=1}^m M(\lambda_j) g'_{jm}(t)^2 \leqq G^{-1}(T_M) \\ &\sum_{j=1}^m \lambda_j^2 M(\lambda_j) g_{jm}(t)^2 \leqq (2/a_0) G^{-1}(T_M) , \end{aligned}$$

that is,

$$(30) \qquad \|F(A^{1/2})A^{1/2}v'_m(t)\| \leqq C , \qquad \|F(A^{1/2})Av_m(t)\| \leqq C$$

(We shall often use the same $C$ for various constants, independent of $m$, $\varepsilon$ and $t \in [0, T_M]$).

Finally substituting $\sum_{j=1}^m M(\lambda_j)\lambda_j^{-2} g''_{jm}(t) w_j$ for $w$ in (15), we have

$$\begin{aligned} \sum_{j=1}^m M(\lambda_j)\lambda_j^{-2} g''_{jm}(t)^2 &= \sum_{j=1}^m \gamma_j(t) M(\lambda_j)\lambda_j^{-2} g''_{jm}(t) \\ &\quad - a_\varepsilon(\|A^{1/2}v_m(t)\|^2) \sum_{j=1}^m \lambda_j^2 g_{jm}(t) \cdot M(\lambda_j)\lambda_j^{-2} g''_{jm}(t) \\ &\leqq \Big(\sum_{j=1}^m M(\lambda_j)\lambda_j^{-2}\gamma_j(t)^2\Big)^{1/2} \Big(\sum_{j=1}^m M(\lambda_j)\lambda_j^{-2} g''_{jm}(t)^2\Big)^{1/2} \\ &\quad + L\Big(\sum_{j=1}^m \lambda_j^2 M(\lambda_j) g_{jm}(t)^2\Big)^{1/2} \Big(\sum_{j=1}^m M(\lambda_j)\lambda_j^{-2} g''_{jm}(t)^2\Big)^{1/2} . \end{aligned}$$

Hence it follows from (D2) and (30) that

$$(31) \qquad \sum_{j=1}^m M(\lambda_j)\lambda_j^{-2} g''_{jm}(t)^2 \leqq C ,$$

than is,

$$(32) \qquad \|F(A^{1/2})u''_m(t)\| \leqq C .$$

## §3. Passage to the limit and proof of theorem 1.

For any $\varepsilon$, $\delta$ $(0<\varepsilon, \delta \leqq \varepsilon_0)$ and positive integers $m$, $k$ $(k \leqq m)$ we have the approximate solutions $v_{m,\varepsilon}(t) \in V_m$ to $(12)_\varepsilon$ and $v_{k,\delta}(t) \in V_k$ to $(12)_\delta$ under the same conditions (13), (14). We observe that those solutions satisfy the apriori estimate (18), (19), (29) and (31) from the preceding section.

We put

$$\mu_{m,\varepsilon}(t) = \|A^{1/2}v_{m,\varepsilon}(t)\|^2 , \qquad \mu_{k,\delta}(t) = \|A^{1/2}v_{k,\delta}(t)\|^2$$

and

$$d(t) \equiv d_{\varepsilon,\delta}^{m;k}(t) = v_{m,\varepsilon}(t) - v_{k,\delta}(t) \equiv \sum_{j=1}^{m} d_j(t) w_j \in V_m$$

where $g_{jk,\delta}=0$ for $j=k+1, \cdots, m$. Then

$$0 \leqq \mu_{m,\varepsilon}(t),\ \ \mu_{k,\delta}(t) \leqq K$$

and

$$d(0) = \sum_{j=k+1}^{m} \alpha_j w_j\ , \qquad d'(0) = \sum_{j=k+1}^{m} \beta_j w_j\ .$$

Equations $(12)_\varepsilon$ and $(12)_\delta$ lead to

$$(33) \qquad \begin{aligned} &(d''(t) + a_\varepsilon(\mu_{m,\varepsilon}(t)) A d(t), w) = \Big( \sum_{j=k+1}^{m} \gamma_j(t) w_j, w \Big) \\ &- \{a_\varepsilon(\mu_{m,\varepsilon}(t)) - a_\delta(\mu_{k,\delta}(t))\} (A v_{k,\delta}(t), w) \qquad \text{for} \quad w \in V_m\ . \end{aligned}$$

Taking $w = 2 \sum_{j=1}^{m} M(\lambda_j) \lambda_j^{-2} d_j'(t) w_j$ in (33) we get

$$\begin{aligned} \frac{d}{dt} D(t) = 2 \sum_{j=k+1}^{m} M(\lambda_j) \lambda_j^{-2} \gamma_j(t) d_j'(t) + \frac{d}{dt} a_\varepsilon(\mu_{m,\varepsilon}(t)) \cdot \sum_{j=1}^{m} M(\lambda_j) d_j(t)^2 \\ - \{a_\varepsilon(\mu_{m,\varepsilon}(t)) - a_\delta(\mu_{k,\delta}(t))\} \cdot \sum_{j=1}^{m} M(\lambda_j) g_{jk}(t) d_j'(t) \end{aligned}$$

where

$$D(t) = \sum_{j=1}^{m} M(\lambda_j) \lambda_j^{-2} d_j'(t)^2 + a_\varepsilon(\mu_{m,\varepsilon}(t)) \sum_{j=1}^{m} M(\lambda_j) d_j(t)^2\ .$$

Since

$$\begin{aligned} \left| \frac{d}{dt} a_\varepsilon(\mu_{m,\varepsilon}(t)) \right| &\leqq 2 |a_\varepsilon'(\mu_{m,\varepsilon}(t))| \, \|A^{1/2} v_{m,\varepsilon}(t)\| \, \|A^{1/2} v_{m,\varepsilon}'(t)\| \\ &\leqq 2LKC \end{aligned}$$

and since

$$\begin{aligned} &|a_\varepsilon(\mu_{m,\varepsilon}(t)) - a_\delta(\mu_{k,\delta}(t))| \\ &\quad = \left| \int_0^{\mu_{m,\varepsilon}(t)} (p_\varepsilon(s) - p_\delta(s)) ds + \int_{\mu_{k,\delta}(t)}^{\mu_{m,\varepsilon}(t)} p_\delta(s) ds \right| \\ &\quad \leqq \int_0^K |p_\varepsilon(s) - p_\delta(s)| \, ds + L |\mu_{m,\varepsilon}(t) - \mu_{k,\delta}(t)| \\ &\quad \leqq \sigma(\varepsilon, \delta) + 2LK \|A^{1/2} d(t)\|\ , \end{aligned}$$

we obtain

$$\frac{d}{dt}D(t) \leqq \sum_{j=k+1}^{m} M(\lambda_j)\lambda_j^{-2}\gamma_j(t)^2 + \sum_{j=1}^{m} M(\lambda_j)\lambda_j^{-2}d_j'(t)^2 + C\sum_{j=1}^{m} M(\lambda_j)d_j(t)^2$$

$$(34) \qquad + (\sigma(\varepsilon,\delta) + C\|A^{1/2}d(t)\|)\Big(\sum_{j=1}^{m} M(\lambda_j)\lambda_j^2 g_{j,k}(t)^2\Big)^{1/2}\Big(\sum_{j=1}^{m} M(\lambda_j)\lambda_j^{-2}d_j'(t)^2\Big)^{1/2}$$

$$\leqq \sigma(\varepsilon,\delta)^2 + \sum_{j=k+1}^{m} M(\lambda_j)\lambda_j^{-2}\gamma_j(t)^2 + C\cdot D(t)\ ,$$

where $\sigma(\varepsilon, \delta) \to 0$ as $\varepsilon, \delta \to 0$. Integrating (34) from 0 to $t \leqq T_M$ we have

$$D(t) \leqq \rho(m, k) + T_M\sigma(\varepsilon,\delta)^2 + C\int_0^t D(s)ds$$

where

$$\rho(m, k) = \sum_{j=k+1}^{m} M(\lambda_j)\Big\{\lambda_j^{-2}\beta_j^2 + a_\varepsilon(\|A^{1/2}u_0\|^2)\alpha_j^2 + \lambda_j^{-2}\int_0^T \gamma_j(s)^2 ds\Big\}$$
$$\longrightarrow 0 \quad \text{as} \quad k, m \longrightarrow \infty$$

uniformly in $\varepsilon$ and $\delta$ by virtue of (9). This means $\{F(A^{1/2})A^{1/2}v_{m,\varepsilon}\}$ and $\{F(A^{1/2})v_{m,\varepsilon}'\}$ are the Cauchy sequences in $C([0, T_M]; H)$. Hence there exists some $u \in C^1([0, T_M]; H)$ such that

$$(35) \qquad v_{m,\varepsilon} \longrightarrow u \quad \text{in} \quad C^1([0, T_M]; H)$$

$$(36) \qquad F(A^{1/2})A^{1/2}v_{m,\varepsilon} \longrightarrow F(A^{1/2})A^{1/2}u \quad \text{in} \quad C([0, T_M]; H)$$

$$(37) \qquad F(A^{1/2})v_{m,\varepsilon}' \longrightarrow F(A^{1/2})u' \quad \text{in} \quad C([0, T_M]; H)$$

as $m \to \infty$ and $\varepsilon \to 0$.

We shall prove the stronger results in convergence than (35), (36), (37), that is, as $m \to \infty$ and $\varepsilon \to 0$,

$$(38) \qquad F(A^{1/2})Av_{m,\varepsilon} \longrightarrow F(A^{1/2})Au$$

$$(39) \qquad F(A^{1/2})A^{1/2}v_{m,\varepsilon}' \longrightarrow F(A^{1/2})A^{1/2}u'$$

and

$$(40) \qquad F(A^{1/2})v_{m,\varepsilon}'' \longrightarrow F(A^{1/2})u''$$

in $C([0, T_M]; H)$. In fact, the convergence (35) means for each $j$

$$g_{jm,\varepsilon} \longrightarrow u_j \quad \text{and} \quad g_{jm,\varepsilon}' \longrightarrow u_j'$$

in $C[0, T_M]$ as $m \to \infty$ and $\varepsilon \to 0$, where each $u_j(t)$ is the Fourier coefficient of $u(t)$. Putting $w = 2M(\lambda_j)g_{jm,\varepsilon}'(t)w_j$ in (15), integrating both sides from 0 to $t$, taking (9), (10), (11) and (29) into consideration and then using

Gronwall's inequality we get

$$M(\lambda_j)g'_{jm}(t)^2+M(\lambda_j)\lambda_j^2 g_{jm}(t)^2\leqq Cs_j^2 ,$$

where

$$s_j^2=M(\lambda_j)\beta_j^2+M(\lambda_j)\lambda_j^2\alpha_j^2+M(\lambda_j)\int_0^T \gamma_j(s)^2ds .$$

For an arbitrary positive number $\theta$ we fix an integer $N$ such that $\sum_{j=N+1}^{\infty} s_j^2\leqq\theta$, which is possible by (D1). The estimates in (30) imply that $F(A^{1/2})Au(t)\in H$ and $F(A^{1/2})A^{1/2}u'(t)\in H$ for almost all $t\in[0, T_M]$. Therefore, for $m>N$,

$$\begin{aligned}&\|F(A^{1/2})Av_{m,\varepsilon}(t)-F(A^{1/2})Au(t)\|^2\\ &=\Big(\sum_{j=1}^{N}+\sum_{j=N+1}^{\infty}\Big)(M(\lambda_j)\lambda_j^2|g_{jm,\varepsilon}(t)-u_j(t)|^2)\\ &\leqq\lambda_N^2\sum_{j=1}^{N}M(\lambda_j)|g_{jm,\varepsilon}(t)-u_j(t)|^2+4C\sum_{j=N+1}^{\infty}s_j^2\end{aligned}$$

and

$$\overline{\lim_{m\to\infty,\varepsilon\to 0}}\ \|F(A^{1/2})Av_{m,\varepsilon}(t)-F(A^{1/2})Au(t)\|^2\leqq 4C\theta ,$$

which concludes (38) as $\theta\to 0$. Similarly we have (39). It holds (40) from the equation

$$F(A^{1/2})v''_m=F(A^{1/2})P_mf-a(\|A^{1/2}v_{m,\varepsilon}(\cdot)\|^2)F(A^{1/2})Av_{m,\varepsilon}$$

and (38).

By (38), (39) and (40) this $u$ satisfies (2), (3) and (4) and this completes the existence part in Theorem 1.

The uniqueness of solution to the problem (2), (3), (4) shall be shown in the class of functions $u$ such that $Au$, $A^{1/2}u'$ and $u''$ belong to $C([0, T]; H)$. Let $u^1$ and $u^2$ be two solutions to the problem (2), (3), (4). Then $w\equiv u^1-u^2$ satisfies the equation

$$\begin{aligned}&w''(t)+a(\|A^{1/2}u^1(t)\|^2)Aw(t)\\ &=-\{a(\|A^{1/2}u^1(t)\|^2)-a(\|A^{1/2}u^2(t)\|^2)\}Au^2(t)\end{aligned}\tag{41}$$

with null initial data: $w(0)=w_t(0)=0$. We fix a constant $K$ larger than $\max\{\|A^{1/2}u^i(t)\|^2; 0\leqq t\leqq T\}$ $(i=1, 2)$, and define the approximating functions $a_\varepsilon(\cdot)$ of $a(\cdot)$ for $K$, as in section 2. Then $b_\varepsilon(s)\equiv a_\varepsilon(s)-a(s)$ satisfy

$$\lim_{\varepsilon\to 0}\max_{0\leqq s\leqq K}|b_\varepsilon(s)|=0 .$$

Since $a(\cdot)$ is locally Lipschitz continuous,

$$\begin{aligned}|a(\|A^{1/2}u^1(t)\|^2)-a(\|A^{1/2}u^2(t)\|^2)| \\ &\leqq L(\|A^{1/2}u^1(t)\|+\|A^{1/2}u^2(t)\|)\,\|A^{1/2}w(t)\| \\ &\leqq 2LK\|A^{1/2}w(t)\| .\end{aligned}$$

Taking scalar product of $2w'(t)$ and each term of (41) and integrating it from 0 to $t$ we obtain for small $\varepsilon>0$

$$\begin{aligned}&\|w'(t)\|^2+(a_0/2)\,\|A^{1/2}w(t)\|^2 \\ &\quad\leqq \|w'(t)\|^2+a_\varepsilon(\|A^{1/2}u^1(t)\|^2)\,\|A^{1/2}w(t)\|^2 \\ &\quad\leqq \int_0^t\Big[2LK\|A^{1/2}w(s)\|\,\|w'(s)\|+\Big|\frac{d}{ds}a_\varepsilon(\|A^{1/2}u^1(s)\|^2)\Big|\,\|A^{1/2}w(s)\|^2 \\ &\qquad\quad +|b_\varepsilon(\|A^{1/2}u^1(s)\|^2)(Aw(s),\, w'(s))|\Big]ds \\ &\quad\leqq C\int_0^t(\|w'(s)\|^2+\|A^{1/2}w(s)\|^2)ds+CT\cdot\max_{0\leqq s\leqq K}|b_\varepsilon(s)| .\end{aligned}$$

Hence

$$\|w'(t)\|^2+\|A^{1/2}w(t)\|^2\leqq CT\cdot\max_{0\leqq s\leqq K}|b_\varepsilon(s)|\to 0 \quad \text{as} \quad \varepsilon\to 0$$

which implies $w\equiv 0$.

Thus the proof of Theorem 1 is completed.

COROLLARY 1. *Let $a\in C^1[0,\infty)$ and $F(\lambda)/\lambda\to c_0$ or $+\infty$ as $\lambda\to\infty$, where $c_0$ is some positive constant. We assume $A^{-1/2}F(A^{1/2})f'\in C([0, T]; H)$. Then the solution $u(t)$ in Theorem 1 satisfies*

$$A^{-1/2}F(A^{1/2})u'''\in C([0, T_M]; H) .$$

PROOF. Since $F(A^{1/2})A^{1/2}u'$ is in $C([0, T_M]; H)$, $Au'$ is in $C([0, T_M]; H)$. Differentiating (2) in $t$ and applying $A^{-1/2}F(A^{1/2})$ to it, we have

$$\begin{aligned}A^{-1/2}F(A^{1/2})u'''&=A^{-1/2}F(A^{1/2})f'-a(\|A^{1/2}u(\cdot)\|^2)F(A^{1/2})A^{1/2}u' \\ &\quad -2a'(\|A^{1/2}u(\cdot)\|^2)(A^{1/2}u(\cdot),\, A^{1/2}u'(\cdot))F(A^{1/2})A^{1/2}u .\end{aligned}$$

Since, by Theorem 1, each term in the right hand side belongs to $C([0, T_M]; H)$, so does $A^{-1/2}F(A^{1/2})u'''$. Q.E.D.

COROLLARY 2 (Local existence of solutions). *Assume* (H1) *and* (H2). *Let $u_0\in D(A)$, $u_1\in D(A^{1/2})$, $f\in C([0, T]; H)$ and $\int_0^T\|A^{1/2}f(t)\|^2dt<+\infty$. Then there exist a positive constant $T_0$ and a unique solution $u(t)$ on $[0, T_0]$ to the problem* (2), (3), (4) *satisfying*

$$u \in C^2([0, T_0]; H) \cap C^1([0, T_0]; D(A^{1/2})) \cap C([0, T_0]; D(A)) .$$

PROOF. Take $M(\lambda)=\lambda^2$ or $F(A^{1/2})=I$, and apply Theorem 1. Q.E.D.

COROLLARY 3 (Regularity of local solutions). *Assume* $u_0 \in D(A^k)$, $u_1 \in D(A^{k-1/2})$, $A^{k-1}f \in C([0, T]; H)$ *and* $\int_0^T \|A^{k-1/2}f(t)\|^2 dt < +\infty$, *in addition to the assumptions of Corollary* 2. *Then there exist a positive constant* $T_1$ *and the solution* $u(t)$ *on* $[0, T_1]$ *to the problem* (2), (3), (4) *satisfies*

$$u \in \bigcap_{i=0}^{2} C^i([0, T_1]; D(A^{k-1/2})) .$$

PROOF. Take $M(\lambda)=\lambda^{4k-2}(k \geqq 1)$ or $F(A^{1/2})=A^{k-1}$, and apply Theorem 1. Q.E.D.

## §4. Convex function and global solution.

We use the properties of $N$-functions (cf. Krasnoselskii and Rutickii [13]). A function $M(\lambda)$ is called an $N$-function if it admits the representation

$$M(\lambda)=\int_0^{|\lambda|} m(t)dt$$

where the function $m(t)$ is right-continuous for $t \geqq 0$, positive nondecreasing for $t>0$ and satisfies the conditions

$$m(0)=0 , \qquad m(\infty)=\lim_{t\to\infty} m(t)=\infty .$$

Then $M(\lambda)$ is an even, continuous, increasing and convex function satisfying the properties

$$\lim_{\lambda\to 0} M(\lambda)/\lambda=0 , \qquad \lim_{\lambda\to\infty} M(\lambda)/\lambda=\infty .$$

We write $M_1(\lambda) \succ M_2(\lambda)$ for two $N$-functions $M_1(\lambda)$, $M_2(\lambda)$ if $\lim_{\lambda\to\infty} M_1(\lambda)/M_2(\lambda)$ exists and positive or $\infty$. We also write $M_1(\lambda) \sim M_2(\lambda)$ if $M_1(\lambda) \succ M_2(\lambda)$ and $M_2(\lambda) \succ M_1(\lambda)$. $Q(\lambda)$ is called the principal part of the $N$-function $M(\lambda)$ if $Q(\lambda)=M(\lambda)$ for large values of $\lambda$. Note that for a convex function $Q(\lambda)$ satisfying the $\lim_{\lambda\to\infty} Q(\lambda)/\lambda=\infty$, there exists some $N$-function $M(\lambda)$ such that $Q(\lambda)$ is its principal part.

LEMMA 4. *Let* $Q(\lambda)=\exp(\delta\lambda/\log(\lambda+1))$ *for a positive constant* $\delta$. *Then there are positive constants* $\lambda_0$, $\mu$ *and* $\sigma(>2)$ *such that an N-function* $M(\lambda)$ *defined by*

(42) $$M(\lambda)=\begin{cases}\mu\lambda^{\sigma} & (0\leqq\lambda\leqq\lambda_0)\\ Q(\lambda)+(\mu\lambda_0^{\sigma}-Q(\lambda_0)) & (\lambda\geqq\lambda_0)\end{cases}$$

*belongs to* $C^2[0,\infty)$ *and a mapping* $\lambda\to M(\lambda^{1/2})$ *is convex.*

LEMMA 5. *For the N-function* $M(\lambda)$ *defined in Lemma* 4, *the function*

$$G(t)=\int_c^t \frac{ds}{s(1+d_0M^{-1}(s))}$$

*satisfies* $G(\infty)=\infty$. *Here both c and* $d_0$ *are positive constants.*

The proofs are elementary and are omitted.

THEOREM 2. *Let* $M(\lambda)$ *be given in Lemma* 4. *Then the unique solution* $u(t)$ *in Theorem* 1 *is global.*

REMARK 2. The same assertion as Lemma 4 is valid for many other $Q(\lambda)$'s, for $Q(\lambda)=\exp(\delta\lambda)$, $\exp(\delta\lambda^s)$, $\exp(\delta\lambda^{1/s})$, $s>1$, and etc.. If $Q(\lambda)$ is a monomial of order $k\geqq 2$, then we can take as $M(\lambda)=Q(\lambda)$ for all $\lambda\geqq 0$.

When $M(\lambda)\succ\exp(\delta\lambda/\log(\lambda+1))$, Lemma 5 and hence Theorem 2 are valid. However for $M(\lambda)$ such as $M(\lambda)\prec\exp(\delta\lambda^{1/s})$, $s>1$, Lemma 5 does not hold and we do not know whether the global solutions exist or not in this case.

## §5. Application to the mixed problem.

We shall now apply the results obtained above to the equation

(1) $$u_{tt}(x,t)-a\Big(\int_\Omega |\nabla u(y,t)|^2dy\Big)\Delta u(x,t)=f(x,t) \qquad \text{in } \Omega\times[0,T]$$

with the initial conditions

(43) $$u(x,0)=u_0(x)\,,\qquad u_t(x,0)=u_1(x) \qquad \text{in } \Omega$$

and the boundary condition

(44) $$u(x,t)=0 \qquad \text{on } \partial\Omega\times[0,T]\,.$$

Here $\Omega$ is a bounded domain with the analytic boundary $\partial\Omega$. Let an operator $A$ in $L^2(\Omega)$ be $(-\Delta)$ with $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$. Then we know $A$ satisfies (H1). Thus we have the eigenvalues $\{\lambda_j^2\}$ and the corresponding eigenfunctions $\{w_j\}$, $j=1,2,\cdots$. Then Theorem 2 for $A=-\Delta$ simply gives

**THEOREM 3.** *Let*

$$(45)\qquad u_i(x)=\sum_{j=1}^{\infty} u_{ij}w_j(x)\ ,\qquad u_{ij}=(u_i(x),\ w_j(x))_{L^2(\Omega)}\ ,$$

*satisfy*

$$(46)\qquad \sum_{j=1}^{\infty} |u_{ij}|^2 \exp(\delta\lambda_j/\log(\lambda_j+1))<+\infty$$

*for* $i=0, 1$ *and let*

$$f(x,\ t)=\sum_{j=1}^{\infty} \gamma_j(t)w_j(x)\ ,\qquad \gamma_j(t)=(f(x,\ t),\ w_j(x))_{L^2(\Omega)}\ ,$$

*satisfy the conditions*

$$(\mathrm{i})\qquad \sum_{j=1}^{\infty}\left(\int_0^T |\gamma_j(t)|^2 dt\right)\exp(\delta\lambda_j/\log(\lambda_j+1))<+\infty\ ,$$

$$(\mathrm{ii})\qquad \sum_{j=1}^{\infty}|\gamma_j(t)|^2 \quad \textit{and} \quad \sum_{j=1}^{\infty}|\gamma_j(t)|^2\exp(\delta\lambda_j/\log(\lambda_j+1))$$

*are convergent and both continuous on* $[0,\ T]$ .

*Then under the assumption* (H2) *there exists a unique global solution* $u(x,\ t)$ *on* $\Omega\times[0,\ T]$ *of* (1), (43) *and* (44) *which is a* $C^2$*-function in* $x,\ t$ *and a* $C^\infty$*-function in* $x$ *for each fixed* $t$.

By Theorem 3 combined with Corollary 1 we immediately have

**COROLLARY 4.** *Let* $a(\cdot)$ *be in* $C^1[0,\ \infty)$. *If*

$$\sum_{j=1}^{\infty}|\gamma'_j(t)|^2 \exp(\delta\lambda_j/\log(\lambda_j+1))$$

*is continuous on* $[0,\ T]$, *then the solution in Theorem* 3 *belongs to* $C^3(\bar{\Omega}\times[0,\ T])$.

**REMARK 3.** As is well known, the fact that $v$ can be expanded as

$$(47)\qquad v(x)=\sum_{j=1}^{\infty} v_j \exp(-\delta\lambda_j)w_j(x)\ ,\qquad \sup_{1\le j<\infty}|v_j|<+\infty\ ,$$

is equivalent to that $v(x)$ is analytic in $\bar{\Omega}$ when the domain $\Omega$ is analytic or parallelepiped (cf. Kotake and Narashimhan [12], Arosio [1]). It is also known that $v(x)$ is in Gevrey class of order $s>1$ if

$$v(x)=\sum_{j=1}^{\infty} v_j \exp(-\delta\lambda_j^{-1/s})w_j(x)\ ,\qquad \sup_{1\le j<\infty}|v_j|<+\infty$$

(cf. Colombini, De Giorgi and Spagnolo [5]). Therefore our class of the initial data in Theorem 3 includes the analytic class and is included in the Gevrey class $G_s$ for any $s>1$ with the compatibility conditions.

## §6. Cauchy problem.

In this section we consider the Cauchy problem on $\boldsymbol{R}^n$:

$$u_{tt}(x, t)-a\left(\int_{\boldsymbol{R}^n} |\nabla u(y, t)|^2 dy\right)\Delta u(x, t)=f(x, t) \tag{48}$$

$$u(x, 0)=u_0(x)\ , \qquad u_t(x, 0)=u_1(x) \tag{49}$$

where data are complex-valued.

When $a(\cdot)$ is in $C^1[0, \infty)$, under the appropriate conditions on $u_0$, $u_1$ and $f$, Perla [19] has showed that existence and uniqueness of the classical solutions for (48), (49) in a small time interval $[0, t_1]$, $t_1>0$, employing the Fourier transformation and the energy method. Now we shall extend the theory of Perla to the global one by imposing some stronger conditions on $u_0$, $u_1$ and $f$ than his.

Denote $\hat{u}$ by

$$\hat{u}(y, t)=(2\pi)^{-n/2}\int_{\boldsymbol{R}^n} u(x, t)e^{-ix\cdot y}dx\ .$$

Then we have

THEOREM 4. *Let* $a(\cdot)$ *satisfy* (H2), $M(\lambda)$ *be given in Lemma* 4 *and assume the conditions*:

1) $f: \boldsymbol{R}^n\times\boldsymbol{R}\to\boldsymbol{C}$ *(the set of all complex numbers) is continuous and* $f(\cdot, t)\in L^2(\boldsymbol{R}^n)$ *for each* $t\in[0, T]$
2) $u_0\in C^2(\boldsymbol{R}^n)\cap L^2(\boldsymbol{R}^n)$, $u_1\in C^1(\boldsymbol{R}^n)\cap L^2(\boldsymbol{R}^n)$
3) *a map* $t\to f(\cdot, t)\in L^2(\boldsymbol{R}^n)$ *is continuous on* $[0, T]$ *and*

$$\int_0^T\int_{\boldsymbol{R}^n}[|\hat{f}(y, t)|^2+M(|y|)|\hat{f}(y, t)|^2]dydt<+\infty$$

4) *each one of the mappings*

$$t\longrightarrow\|\hat{f}(\cdot, t)\|_{L^\infty}$$

$$t\longrightarrow\||\cdot|\hat{f}(\cdot, t)\|_{L^\infty}$$

$$t\longrightarrow\||\cdot|^{(n+1)/2}\hat{f}(\cdot, t)\|_{L^\infty}$$

$$t\longrightarrow\||\cdot|\hat{f}(\cdot, t)\|_{L^2}$$

$$t \longrightarrow \| |\cdot|^{(n+8)/2}\hat{f}(\cdot, t)\|_{L^2}$$

*is continuous for* $t \in [0, T]$

5) $$\int_{R^n} |y|^2 M(|y|)|\hat{u}_0(y)|^2 dy < +\infty$$

*and*

$$\int_{R^n} M(|y|)|\hat{u}_1(y)|^2 dy < +\infty \ .$$

*Then there exists a unique solution* $u \in C^2(\boldsymbol{R}^n \times [0, T])$ *satisfying* (48), (49).

REMARK 4. We may say that there exists a global solution for the Cauchy problem (48), (49) (with $f \equiv 0$) when the initial data $u_0$, $u_1$ belong to the class

$$\{u \in L^2(\boldsymbol{R}^n);\ \hat{u}(y) = O(e^{-\delta|y|/\log(|y|+1)}) \quad \text{as} \quad |y| \longrightarrow \infty$$
$$\text{for some positive constant } \delta\} \ .$$

In order to obtain a priori estimates we need the following lemma corresponding to Lemma 2.

LEMMA 6 (Hardy, Littlewood and Polya [11], p. 169). *Let* $\phi$ *and* $\psi$ *be continuous and strictly increasing. We define* $\mathfrak{M}_\phi(f)$ *by*

$$\mathfrak{M}_\phi(f) = \phi^{-1}\Big(\int \phi(f(x))q(x)dx\Big)$$

*where* $f$ *and* $q$ *are the nonnegative functions such that* $\int q(x)dx = 1$ *and* $\int \phi(f(x))q(x)dx$ *exists. Then in order that* $\mathfrak{M}_\phi(f) \leqq \mathfrak{M}_\psi(f)$ *for all* $f$, *it is necessary and sufficient that* $\psi \circ \phi^{-1}$ *should be convex.*

Along the proof of Perla we transform (48), (49) to the problem

$$\hat{u}_{tt}(y, t) + a\Big(\int_{R^n} |\xi|^2 |\hat{u}(\xi, t)|^2 d\xi\Big)|y|^2 \hat{u}(y, t) = \hat{f}(y, t)$$

$$\hat{u}(y, 0) = \hat{u}_0(y) \ , \qquad \hat{u}_t(y, 0) = \hat{u}_1(y) \ ,$$

and consider the following approximate and truncated problem for $|y| \leqq r$:

(50) $$v_{tt}(y, t) + a_\varepsilon\Big(\int_{|\xi| \leqq r} |\xi|^2 |v(\xi, t)|^2 d\xi\Big)|y|^2 v(y, t) = \hat{f}(y, t)$$

(51) $$v(y, 0) = \hat{u}_0(y) \ , \qquad v_t(y, 0) = \hat{u}_1(y)$$

letting $v=v_{r,\varepsilon}\equiv 0$ for $|y|>r$. The constant $K$ appeared in the definition of $a_\varepsilon(\cdot)$ be taken as in (7). Because of our conditions on $f$ and $a_\varepsilon(\cdot)$, the sequence of Picard's successive iteration for (50), (51) converges. Thus we have a local solution $v$ to the problem (50), (51). A priori estimates

$$\int_{|y|\leq r}|y|^2|v(y,t)|^2dy\leqq K\,,\qquad \int_{|y|\leq r}|v_t(y,t)|^2\,dy\leqq C \tag{52}$$

are easily derived and these imply that the intervals of the existence of $v$ are extended to the whole $[0, T]$. (We denote by $C$ each constant independent of $r$, $\varepsilon$ and $t\in[0, T]$.)

We shall prove the estimate

$$\int_{|y|\leq r}|y|^2|v_t(y,t)|^2dy\leqq C$$

for any $t\in[0, T]$. Adding the equation obtained by multiplying (50) by $M(|y|)\bar{v}_t$ to the one obtained by taking a complex conjugate to (50) and multiplied by $M(|y|)v_t$, we have

$$\begin{aligned}\frac{d}{dt}E_M(t)&=2\int M(|y|)\mathrm{Re}[\hat{f}(y,t)\bar{v}_t(y,t)]dy\\&\quad+2a_\varepsilon'\Big(\int|y|^2|v(y,t)|^2dy\Big)\Big(\int|y|^2\mathrm{Re}[v(y,t)\bar{v}_t(y,t)]dy\Big)\\&\quad\times\int|y|^2M(|y|)\,|v(y,t)|^2dy\end{aligned}\tag{53}$$

where

$$E_M(t)=\int M(|y|)\,|v_t(y,t)|^2dy+a_\varepsilon\Big(\int|y|^2|v(y,t)|^2dy\Big)\int|y|^2M(|y|)\,|v(y,t)|^2dy\,.$$

(For simplicity here and below we abbreviate denoting the domain in integral sign.) Schwarz's inequality and (52) imply

$$\begin{aligned}\Big|\int|y|^2\,\mathrm{Re}[v\bar{v}_t]dy\Big|&\leqq\Big(\int|y|^2|v|^2dy\Big)^{1/2}\Big(\int|y|^2|v_t|^2dy\Big)^{1/2}\\&\leqq C^{1/2}\Big(\int|y|^2|v_t|^2dy\Big)^{1/2}\,.\end{aligned}\tag{54}$$

If $\int|v_t|^2dy\geqq 1$, then from Lemma 6 and $M^{-1}$ being increasing it follows that

$$\Big(\int|y|^2|v_t|^2dy\Big)^{1/2}=\Big(\int|v_t|^2dy\Big)^{1/2}\Bigg(\int\frac{|v_t|^2}{\int|v_t|^2d\xi}|y|^2dy\Bigg)^{1/2}$$

$$\leqq C^{1/2} M^{-1}\left(\int \frac{|v_t|^2}{\int |v_t|^2 d\xi} M(|y|) dy\right) \tag{55}$$

$$\leqq C^{1/2} M^{-1}\left(\int |v_t|^2 M(|y|) dy\right) .$$

While, in the case $\int |v_t|^2 dy < 1$, we use Friedrichs' mollifier $\rho_\theta$ $(0<\theta<\min(1, r))$ on $R^n_y$. Putting $q_0(y, t)=\left(1-\int |v_t(\xi, t)|^2 d\xi\right)\rho_\theta(y)$ we have $0<\int q_0(y, t)dy \leqq 1$ and $\int (|v_t(y, t)|^2+q_0(y, t))dy=1$. Then we obtain

$$\left(\int |y|^2 |v_t|^2 dy\right)^{1/2} \leqq \left(\int |y|^2(|v_t|^2+q_0(y, t))dy\right)^{1/2}$$

$$\leqq M^{-1}\left(\int M(|y|)(|v_t|^2+q_0(y, t))dy\right)$$

$$\leqq M^{-1}\left(\int M(|y|)|v_t|^2 dy+M(\theta)\right)$$

by Lemma 6 and by the fact $M(\lambda)$ is increasing and $\operatorname{supp} q_0 \subset \{y; |y| \leqq \theta\}$. Letting $\theta \to 0$, we obtain

$$\int |y|^2 |v_t|^2 dy \leqq M^{-1}\left(\int M(|y|)|v_t|^2 dy\right) . \tag{56}$$

From the estimates (53)-(56) it follows

$$E_M(t) \leqq E_M(0)+\int_0^T \int_{R^n} M(|y|)|\hat{f}(y, s)|^2 dy ds$$

$$+\int_0^t E_M(s)(1+d_0 M^{-1}(E_M(s)))ds .$$

Making use of Lemma 1, 3, 5 and the conditions on $u_0$, $u_1$ and $f$ we conclude

$$\begin{aligned} &\int_{|y|\leqq r} M(|y|)|v_t(y, t)|^2 dy \leqq C \\ &\int_{|y|\leqq r} |y|^2 M(|y|)|v(y, t)|^2 dy \leqq C . \end{aligned} \tag{57}$$

The inequality (57) combined with (52) gives

$$\int_{|y|\leqq r} |y|^2 |v_t(y, t)|^2 dy \leqq C .$$

Thus we obtained the boundedness on $[0, T]$ for second derivatives of the original solution $u$. We need to show that the sequences of the truncated solutions $v=v_{r,\varepsilon}$ tends to some $w$ as $r\to\infty$ and $\varepsilon\to 0$, whose Fourier inverse transform is the solution $u$ on $[0, T]$ to (48), (49). For the details of the proof see Perla [19].

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