TOKYO J. MATH.
VOL. 10, NO. 2, 1987

On a p-Adic Interpolation of the Generalized Euler Numbers and Its Applications

Hirofumi TSUMURA

Tokyo Metropolitan University
(Communicated by K. Ogiue)

Introduction

The Euler numbers E_n are defined by

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}\ .$$

They are classical and important in number theory. Frobenius ([4]) extended E_n to the Euler numbers $H^n(u)$ belonging to an algebraic number u (see also §1), and many authors (e.g. [2], [4] and [8]) investigated their properties. Recently Shiratani-Yamamoto ([10]) constructed a p-adic interpolation $G_p(s, u)$ of the Euler numbers $H^n(u)$, and as its application, they obtained an explicit formula for $L'_p(0, \chi)$ with any Dirichlet character χ, including Ferrero-Greenberg's formula ([5]), and gave an explanation of Diamond's formula ([3]).

In the present paper, we shall define the generalized Euler numbers $H^n_\chi(u)$ for any Dirichlet character χ, which are analogous to the generalized Bernoulli numbers (see §1), and we shall construct their p-adic interpolation (see §2), which is an extension of Shiratani-Yamamoto's p-adic interpolation $G_p(s, u)$ of $H^n(u)$. The function $G_p(s, u)$ interpolates the n-th Euler number for $n\geqq 0$ with $(p-1)|n$, but our function interpolates the n-th generalized Euler number for any n. As applications, we shall obtain some congruences for the generalized Euler numbers (see §3), which improve the congruences for the Euler numbers in [2], [4] and [8]. In the last section, we shall define an element of a group ring. By using it, we shall reconstruct a p-adic interpolation of the Euler numbers in the Iwasawa method which makes use of the formal power series (cf. [12], §7.2).

The author wishes to express his sincere gratitude to Professor H. Miki who suggested him to study the application of p-adic Γ-transforms

Received September 18, 1986

and gave him many valuable suggestions and comments. The author also wishes to express his sincere gratitude to Professor M. Ishida and Professor K. Shiratani for their valuable comments.

NOTATIONS. $\boldsymbol{Q}$: the field of rational numbers. $\bar{\boldsymbol{Q}}$: the algebraic closure of $\boldsymbol{Q}$. $\boldsymbol{Z}$: the ring of rational integers. $\boldsymbol{N}$: the set of positive integers. $\boldsymbol{R}$: the field of real numbers. $\boldsymbol{C}$: the field of complex numbers. Let p be an odd prime number. $\boldsymbol{Q}_p$: the field of p-adic numbers. $\boldsymbol{Z}_p$: the ring of p-adic integers. $\boldsymbol{Z}_p^*$: the group of p-adic units in $\boldsymbol{Z}_p$. $\boldsymbol{C}_p$: the p-adic completion of the algebraic closure $\bar{\boldsymbol{Q}}_p$ of $\boldsymbol{Q}_p$. $|\cdot|$: the p-adic absolute value on $\boldsymbol{C}_p$ normalized by $|p|=1/p$. V: the group $\{x\in\boldsymbol{Q}_p: x^{p-1}=1\}$. Then $\boldsymbol{Z}_p^*=V\times(1+p\boldsymbol{Z}_p)$; $a=w(a)\langle a\rangle$ where $w(a)$ (resp. $\langle a\rangle$) denotes the projection of a onto V (resp. onto $1+p\boldsymbol{Z}_p$).

§1. Definition of the generalized Euler numbers.

Let $u\neq 0$ be an algebraic number. We fix an embedding $\bar{\boldsymbol{Q}}\to\boldsymbol{C}$, $\bar{\boldsymbol{Q}}\to\boldsymbol{C}_p$, so that we take u as an element of $\boldsymbol{C}$, and $\boldsymbol{C}_p$. The number $H^n(u)$ defined by

$$(1)\qquad \frac{1-u}{e^t-u}=\sum_{n=0}^{\infty}H^n(u)\frac{t^n}{n!}$$

is called the n-th Euler number belonging to u. The polynomial $E_n(u, x)\in\boldsymbol{Q}(u)[x]$ defined by

$$(2)\qquad \frac{(1-u)e^{xt}}{e^t-u}=\sum_{n=0}^{\infty}E_n(u, x)\frac{t^n}{n!}$$

is called the n-th Euler polynomial belonging to u. As is well known,

$$(3)\qquad E_n(u, 1-x)=(-1)^nE_n(u^{-1}, x)\ ,$$

and

$$(4)\qquad E_n(u, x)=\sum_{i=0}^{n}\binom{n}{i}H^i(u)x^{n-i}\ .$$

Let χ be a primitive Dirichlet character with conductor f. We define the n-th generalized Euler number*) $H_\chi^n(u)$ belonging to u by

$$(5)\qquad \sum_{a=0}^{f-1}\frac{(1-u^f)\chi(a)e^{at}u^{f-a-1}}{e^{ft}-u^f}=\sum_{n=0}^{\infty}H_\chi^n(u)\frac{t^n}{n!}\ .$$

*) This definition is slightly different from the original one due to the author. This modification by H. Miki enables us to simplify the following argument.

Note that when $\chi=1$, we have

$$H_1^n(u)=H^n(u) \qquad \text{for} \quad n\geqq 0 .$$

By using (1), (2) and (4), we can easily see that

$$(6) \qquad H_\chi^n(u)=f^n\sum_{a=0}^{f-1}\chi(a)u^{f-a-1}E_n\left(u^f, \frac{a}{f}\right)$$
$$=\sum_{a=0}^{f-1}\chi(a)u^{f-a-1}\sum_{i=0}^{n}\binom{n}{i}H^i(u^f)a^{n-i}f^i .$$

§2. *p*-adic interpolation of the generalized Euler numbers.

From now on, we fix a primitive Dirichlet character χ with conductor f, and we assume the following:

$$(7) \qquad |1-u^{fp^N}|\geqq 1 \qquad \text{for} \quad N\geqq 0 .$$

This assumption is an analogue of that in [8]. Then we have the following formula which is an extension of that in [8].

LEMMA 1. *For integer $n\geqq 0$,*

$$\frac{u}{1-u^f}H_\chi^n(u)=\lim_{N\to\infty}\sum_{b=0}^{fp^N-1}\chi(b)b^n\frac{u^{fp^N-b}}{1-u^{fp^N}} ,$$

where the limit in the right hand side is the p-adic one.

PROOF. It is proved in the same way as in [8]. By (5), we have

$$\left(\sum_{a=0}^{f-1}(1-u^f)\chi(a)e^{at}u^{f-a}\right)e^{kt-1}=\left(\sum_{n=0}^{\infty}H_\chi^n(u)\frac{t^n}{n!}\right)(e^{ft}-u^f)e^{kt}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}H_\chi^i(u)\{(f+k)^{n-i}-u^fk^{n-i}\}\right)\frac{t^n}{n!}$$

for $k\geqq 0$. Hence we have

$$\sum_{a=0}^{f-1}(1-u^f)\chi(a)u^{f-a-1}(a+k)^n$$
$$=\sum_{i=0}^{n}\binom{n}{i}H_\chi^i(u)\{(f+k)^{n-i}-u^fk^{n-i}\}$$

for $k, n\geqq 0$. Put $k=0, f, \cdots, (p^N-1)f$, and multiply the both sides by $u^{f(p^N-1)-k}$ for each case, then we obtain by summing up

$$\sum_{a=0}^{f-1}(1-u^f)\chi(a)u^{f-a-1}\sum_{j=0}^{p^N-1}(a+fj)^n u^{f(p^N-1)-fj}$$
$$=\sum_{i=0}^{n}\binom{n}{i}H_\chi^i(u)(p^N f)^{n-i}-H_\chi^n(u)u^{fp^N}\ .$$

Hence we obtain

$$(1-u^{fp^N})H_\chi^n(u)+p^N\sum_{i=0}^{n-1}\binom{n}{i}H_\chi^i(u)p^{N(n-1-i)}f^{n-i}$$
$$=(1-u^f)\sum_{b=0}^{fp^N-1}\chi(b)b^n u^{fp^N-b-1}\ .$$

Let N tend to the infinity, then by (7) we have the assertion.

Now we define some notations. Let $\pi_{N,M}$: $\boldsymbol{Z}/fp^N\boldsymbol{Z}\to\boldsymbol{Z}/fp^M\boldsymbol{Z}$ with $a \bmod fp^N\boldsymbol{Z}\mapsto a \bmod fp^M\boldsymbol{Z}$ for $a\in\boldsymbol{Z}$ and $N\geqq M$, then $\{\boldsymbol{Z}/fp^N\boldsymbol{Z}, \pi_{N,M}\}$ is an inverse system. Let X be the inverse limit of this system: $X=\varprojlim \boldsymbol{Z}/fp^N\boldsymbol{Z}$, and for $a\in\boldsymbol{Z}$, let $X_N(a)$ denote the set of $x\in X$ which maps to $a \bmod fp^N\boldsymbol{Z}$ under the canonical map $X\to\boldsymbol{Z}/fp^N\boldsymbol{Z}$ for $N\geqq 0$. Put $X^*=\cup X_N(a)$ where the union is taken over all a with $0<a<fp$, $(a, fp)=1$. Let π: $X\to\boldsymbol{Z}_p$ be the continuous homomorphism induced by the map $\boldsymbol{Z}/fp^N\boldsymbol{Z}\to\boldsymbol{Z}/p^N\boldsymbol{Z}$ with $a \bmod fp^N\boldsymbol{Z}\mapsto a \bmod p^N\boldsymbol{Z}$, for all $N\geqq 0$. Finally let $\alpha_u=\alpha_{u,X}$ be the measure on X defined by

$$\alpha_u(X_N(a))=\frac{u^{fp^N-a}}{1-u^{fp^N}} \tag{8}$$

with $0\leqq a<fp^N$ and $N\geqq 0$, which is an analogue of the Koblitz measure ([7]).

REMARK. In the case where $\chi=1$, α_u is the Shiratani-Yamamoto measure μ_u which was defined in [10].

Now we fix a natural embedding $\boldsymbol{Z}\to X$ induced by $a\mapsto(a \bmod fp^N)$. For a $\boldsymbol{C}_p$-valued continuous function g on X, we define the p-adic integration by

$$\int_X g(x)d\alpha_u(x)=\lim_{N\to\infty}\sum_{a=0}^{fp^N-1}g(a)\alpha_u(X_N(a))$$

and

$$\int_{X^*} g(x)d\alpha_u(x)=\lim_{N\to\infty}\sum_{\substack{a=0\\(a,fp)=1}}^{fp^N-1}g(a)\alpha_u(X_N(a))\ .$$

It follows from Lemma 1 that

$$\int_X \pi(x)^n \chi(x) d\alpha_u(x) = \frac{u}{1-u^f} H_\chi^n(u) \tag{9}$$

for any $n \geqq 0$.

Let $k \geqq 1$ be an integer with $f|k$, and assume that $|1-u^{kp^N}| \geqq 1$ for all $N \geqq 0$. Then we can easily see that

$$\int_Y \bar{\pi}(y)^n \chi(y) d\alpha_{u,Y}(y) = \int_X \pi(x)^n \chi(x) d\alpha_{u,X}(x)$$

where $Y = \varprojlim \boldsymbol{Z}/kp^N\boldsymbol{Z}$ is the inverse limit of the inverse system $\{\boldsymbol{Z}/kp^N\boldsymbol{Z}, \bar{\pi}_{N,M}\}$ with $\bar{\pi}_{N,M}: a + kp^N\boldsymbol{Z} \mapsto a + kp^M\boldsymbol{Z}$ for $a \in \boldsymbol{Z}$ and $N \geqq M$, and $\bar{\pi}: X \to \boldsymbol{Z}_p$ be the continuous homomorphism induced by the map $\boldsymbol{Z}/kp^N\boldsymbol{Z} \to \boldsymbol{Z}/p^N\boldsymbol{Z}$ with $a \bmod kp^N\boldsymbol{Z}$ to $a \bmod p^N\boldsymbol{Z}$ for all $N \geqq 0$.

Finally, for any $s \in \boldsymbol{Z}_p$, we define the p-adic interpolation $l_p(u, s, \chi)$ of the generalized Euler numbers by

$$l_p(u, s, \chi) = \int_{X^*} \langle \pi(x) \rangle^{-s} \chi(x) d\alpha_u(x) . \tag{10}$$

By the definition of X^*, we can see that $X^* \simeq (\boldsymbol{Z}/dp\boldsymbol{Z})^* \times (1 + p\boldsymbol{Z}_p)$ where $f = dp^\kappa$ with $(d, p) = 1$ and $\kappa \geqq 0$. Hence it follows from [12] (Theorem 12.4) that $l_p(u, s, \chi)$ is an Iwasawa function, especially analytic in s.

REMARK. Our function $l_p(u, s, \chi)$ is one of the extension of the Shiratani-Yamamoto function $G_p(s, u)$ in [10]. Indeed $l_p(u, s, 1) = G_p(s, u)$. If $(p, f) = 1$, then some calculations show that

$$l_p(u, s, \chi) = \frac{\tau(\chi)}{f} \sum_{j=1}^{f} \bar{\chi}(j) G_p(s, u\zeta_f^j)$$

where $\tau(\chi) = \sum_{a=1}^{f} \chi(a) \zeta_f^a$ is the normalized Gauss sum attached to χ, and $\zeta_f = \exp(2\pi\sqrt{-1}/f)$.

By using the fact that

$$\alpha_u(X_N(pa)) = \alpha_{u^p}(X_{N-1}(a))$$

and (10), we have the following

THEOREM 1. *For any integer* $n \geqq 0$,

$$l_p(u, -n, \chi\omega^n) = \frac{u}{1-u^f} H_\chi^n(u) - \frac{\chi(p) p^n u^p}{1-u^{pf}} H_\chi^n(u^p) .$$

Especially when $p|f$,

$$l_p(u, -n, \chi\omega^n)=\frac{u}{1-u^f}H_\chi^n(u)\ .$$

REMARK. Since

$$\frac{2}{e^{2t}-1}=\frac{1}{e^t-1}-\frac{1}{e^t+1}\ ,$$

we have

$$\frac{1}{2}E_n=(1-2^{n+1})\frac{B_{n+1}}{n+1}$$

for $n\geqq 0$, where $E_n=H^n(-1)$ is the classical Euler number (see Introduction). More generally, for $c>0$ with $(c, fp)=1$, we have

$$\sum_{j=1}^{c-1}\frac{\zeta_c^j}{1-\zeta_c^{jf}}H_\chi^n(\zeta_c^j)=(c^{n+1}\chi(c)-1)\frac{B_{n+1,\chi}}{n+1} \tag{11}$$

with $\zeta_c=\exp(2\pi\sqrt{-1}/c)$. Hence it follows from Theorem 1 that

$$\sum_{j=1}^{c-1}l_p(\zeta_c^j, s, \chi)=(1-\langle c\rangle^{1-s}\chi\omega(c))L_p(s, \chi\omega)$$

where $L_p(s, \chi)$ is the Kubota-Leopoldt p-adic L-function (see [12], Theorem 5.11).

§3. Some congruences for the generalized Euler numbers.

In this section, by considering the expansion of $l_p(u, s, \chi)$ defined in §2 at $s=1$, we have some congruences for the generalized Euler numbers in a method similar to [12], §5.3.

Let Θ be the ring of integers in a finite extension of $\boldsymbol{Q}_p$ in $\boldsymbol{C}_p$, and let α be an Θ-valued measure on $X=\varprojlim \boldsymbol{Z}/fp^N\boldsymbol{Z}$. For an Θ-valued continuous function g on X, we put

$$F(s)=\int_{X^*}\langle\pi(x)\rangle^{-s}g(x)d\alpha(x)\ .$$

Then we have the following

THEOREM 2.

$$F(s)=\sum_{n=0}^{\infty}a_n(s-1)^n$$

with $|a_0|\leqq 1$ *and with* $p|a_n$ *for* $n\geqq 1$.

PROOF. By using the formula

$$\langle y\rangle^{1-s}=\exp((1-s)\log_p\langle y\rangle) \qquad \text{for} \quad s,\, y\in \boldsymbol{Z}_p\ ,$$

where exp (resp. $\log_p$) is the p-adic exponential (resp. logarithm) function, we have

$$F(s)=\sum_{n=0}^{\infty}(1-s)^n\int_{X^*}\frac{(\log_p\langle\pi(x)\rangle)^n}{n!}\,\frac{g(x)}{\langle\pi(x)\rangle}d\alpha(x)\ .$$

Hence for $n\geqq 0$,

$$a_n=(-1)^n\int_{X^*}\frac{(\log_p\langle\pi(x)\rangle)^n}{n!}\,\frac{g(x)}{\langle\pi(x)\rangle}d\alpha(x)\ .$$

Suppose $n\geqq 1$. Since for any $b\in\boldsymbol{Z}$ with $(p, b)=1$,

$$|\log_p\langle b\rangle|\leqq p^{-1}<p^{-1/(p-1)}$$

and $|n!|>p^{-n/(p-1)}$, it follows that

$$\frac{(\log_p\langle b\rangle)^n}{n!}\equiv 0 \qquad (\mathrm{mod}\ p)\ .$$

Thus we obtain $p|a_n$ for $n\geqq 1$. $|a_0|\leqq 1$ is obvious. Q.E.D.

By Theorem 2, we have the Kummer congruences for the generalized Euler numbers, which were proved for the ordinary Euler numbers $H^n(u)$ in [4] and [8].

COROLLARY 1. *For integers* m, n *with* $0\leqq m\leqq n$ *and* $m\equiv n$ $(\mathrm{mod}\ (p-1)p^a)$,

$$\frac{u}{1-u^f}H_\chi^m(u)\equiv\frac{u}{1-u^f}H_\chi^n(u) \qquad (\mathrm{mod}\ p^M)$$

with $M=\min(m, a+1)$ *or* $a+1$ *according as* $(p, f)=1$ *or not.*

PROOF. Put $g=\chi$ and $\alpha=\alpha_u$ in Theorem 2, then it follows that

$$l_p(u, -m, \chi\omega^m)\equiv l_p(u, -n, \chi\omega^n) \qquad (\mathrm{mod}\ p^{a+1})$$

when $m\equiv n$ $(\mathrm{mod}\ (p-1)p^a)$. By using Theorem 1, we have the assertion.

The next result is an analogue of that in [11] which was for the Bernoulli numbers.

COROLLARY 2. *Let i be an integer with $1<i<p$. If there exists a minimal integer $k=k_i\geqq 0$ such that*

$$\frac{u}{1-u^f}H_\chi^{ip^k-1}(u)\not\equiv 0 \qquad (\mathrm{mod}\ p^{k+1}),$$

then

$$k_i=v_p(l_p(u,1,\chi\omega^{i-1}))$$

where v_p is the p-adic valuation on C_p normalized by $v_p(p)=1$.

PROOF. In the case where $k=0$, it is obvious. Suppose $k\geqq 1$. Put $F(s)=l_p(u,s,\chi)$ in Theorem 2, then we have

$$\begin{aligned} l_p(u,1,\chi\omega^{i-1}) &\equiv l_p(u,1-ip^k,\chi\omega^{ip^k-1}) && (\mathrm{mod}\ p^{k+1}) \\ &\equiv \frac{u}{1-u^f}H_\chi^{ip^k-1}(u)\not\equiv 0 && (\mathrm{mod}\ p^{k+1}) \end{aligned}$$

by the assumption on k. On the other hand, it follows from the minimality of k that

$$\begin{aligned} l_p(u,1,\chi\omega^{i-1}) &\equiv l_p(u,1-ip^{k-1},\chi\omega^{ip^{k-1}-1}) && (\mathrm{mod}\ p^k) \\ &\equiv \frac{u}{1-u^f}H_\chi^{ip^{k-1}}(u)\equiv 0 && (\mathrm{mod}\ p^k). \end{aligned}$$

Thus we have the assertion.

By using Theorem 2 for $F(s)=l_p(u,s,\chi)$, we have the following "generalized Frobenius' congruences" which were proved when $\chi=1$, and $n\equiv 0\ (\mathrm{mod}\ p-1)$ in [2], [4] and [8]. Now we can remove the assumption $n\equiv 0\ (\mathrm{mod}\ p-1)$ by using Dirichlet characters.

COROLLARY 3. *For an integer $n\geqq 1$ with $n\equiv i\ (\mathrm{mod}\ (p-1)p^a)$ and $0\leqq i<p-1$,*

$$\frac{u}{1-u^f}H_\chi^n(u)\equiv\sum_{k=0}^{g-1}\chi\omega^i(k)\left(\frac{u^{g-k}}{1-u^g}-\frac{\chi\omega^i(p)u^{p(g-k)}}{1-u^{pg}}\right) \qquad (\mathrm{mod}\ p^M)$$

where f (resp. g) is the conductor of χ (resp. $\chi\omega^i$), and $M=a+1$ or 1 according as $i=0$ or not.

PROOF. By Theorem 2, we have

$$\begin{aligned} l_p(u,-i,\chi\omega^i) &\equiv l_p(u,-n,\chi\omega^n) && (\mathrm{mod}\ p^{a+1}) \\ &\equiv \frac{u}{1-u^f}H_\chi^n(u) && (\mathrm{mod}\ p^{a+1}). \end{aligned}$$

On the other hand,

$$l_p(u,\ -i,\ \chi\omega^i)\equiv l_p(u,\ 0,\ \chi\omega^i) \qquad (\mathrm{mod}\ p^L)$$

with L arbitrary or 1 according as $i=0$ or not, and

$$\begin{aligned} l_p(u,\ 0,\ \chi\omega^i) &= \frac{u}{1-u^g}H^0_{\chi\omega^i}(u)-\frac{\chi\omega^i(p)u^p}{1-u^{gp}}H^0_{\chi\omega^i}(u^p) \\ &= \sum_{k=0}^{g-1}\frac{\chi\omega^i(k)u^{g-k}}{1-u^g}-\chi\omega^i(p)\sum_{k=0}^{g-1}\frac{\chi\omega^i(k)u^{p(g-k)}}{1-u^{pg}}\ . \end{aligned}$$

Thus we have the assertion.

REMARK. In the special case where $\chi=1$ and $n\equiv 0\ (\mathrm{mod}\ (p-1)p^a)$, we have the following ordinary Frobenius' congruences:

$$\frac{u}{1-u}H^n(u)\equiv\frac{u}{1-u}\,\frac{1-u^{p-1}}{1-u^p} \qquad (\mathrm{mod}\ p^{a+1})\ .$$

Similarly when $\chi=1$ and $n\equiv i\ (\mathrm{mod}\ p-1)$ with $1\leqq i\leqq p-1$, we have

$$\frac{u}{1-u}H^n(u)\equiv\frac{u^p}{1-u^p}\sum_{k=1}^{p-1}\omega^i(k)u^{-k} \qquad (\mathrm{mod}\ p)\ .$$

By the definition of $H^n_\chi(u)$, we have

$$H^1_\chi(u)=\sum_{k=0}^{f-1}\chi(k)u^{f-k-1}\left(k-\frac{f}{1-u^f}\right)\ .$$

By Theorem 2, we have

$$l_p(u,\ -i,\ \chi\omega^i)\equiv l_p(u,\ -1,\ \chi\omega^i) \qquad (\mathrm{mod}\ p^L)$$

with L arbitrary or 1 according as $i=1$ or not, hence we have the following which can be proved in the same way as Corollary 3.

COROLLARY 4. *For an integer $n\geqq 1$ with $n\equiv i\ (\mathrm{mod}\ (p-1)p^a)$ and $0\leqq i<p-1$,*

$$\begin{aligned} \frac{u}{1-u^f}H^n_\chi(u) &\equiv \sum_{k=0}^{g-1}\chi\omega^{i-1}(k)\left\{k\left(\frac{u^{g-k}}{1-u^g}-\frac{\chi\omega^{i-1}(p)pu^{p(g-k)}}{1-u^{pg}}\right)\right. \\ &\quad \left. -g\left(\frac{u^{g-k}}{(1-u^g)^2}-\frac{\chi\omega^{i-1}(p)pu^{p(g-k)}}{(1-u^{pg})^2}\right)\right\} \qquad (\mathrm{mod}\ p^M) \end{aligned}$$

where f (resp. g) is the conductor of χ (resp. $\chi\omega^{i-1}$), and $M=a+1$ or 1 according as $i=1$ or not.

§4. The Iwasawa construction of $l_p(u, s, \chi)$.

We fix an integer i with $0 \leqq i \leqq p-2$. By (11), we have

$$\sum_{j=1}^{c-1} \frac{\zeta_c^j}{1-\zeta_c^{jp}} H_{\omega^{-i}}^0(\zeta_c^j) = (c\omega^{-i}(c)-1)B_{1,\omega^{-i}} \tag{12}$$

for $c>1$ with $(c, p)=1$. On the other hand, by (5), we have

$$\frac{\zeta_c^j}{1-\zeta_c^{jp}} H_{\omega^{-i}}^0(\zeta_c^j) = \frac{\zeta_c^j}{1-\zeta_c^{jp}} \sum_{a=1}^{p-1} \omega^{-i}(a)\zeta_c^{j(p-a)} . \tag{13}$$

By (12) and (13), we can describe $B_{1,\omega^{-i}}$ using the character ω, which is stated in [6] without using the Euler numbers. Furthermore let $G = G(\boldsymbol{Q}(\zeta_p)\,|\,\boldsymbol{Q}) = \{\sigma_a \,|\, \zeta_p \mapsto \zeta_p^a,\ (a, p)=1\}$, and we define

$$\xi(\zeta_c) = \sum_{a=1}^{p-1} \left(\sum_{j=1}^{c-1} \frac{\zeta_c^{j(p-a)}}{1-\zeta_c^{jp}} \right) \sigma_a^{-1}$$

for $c>1$ with $c\,|\,p-1$. By (12) and (13), we have

$$\xi(\zeta_c)\varepsilon_i = (c\omega^{-i}(c)-1)B_{1,\omega^{-i}}\varepsilon_i \tag{14}$$

for $1 \leqq i \leqq p-2$, where $\varepsilon_k = (1/(p-1)) \sum_{a=1}^{p-1} \omega^k(a)\sigma_a^{-1}$ is the orthogonal idempotent of $\boldsymbol{Z}_p[G]$ for $0 \leqq k \leqq p-2$. Let A be the p-Sylow subgroup of the ideal class group of $\boldsymbol{Q}(\zeta_p)$. Since ε_0 is the norm, $\varepsilon_0 A = 0$. Since the right hand side of (14) annihilates $\varepsilon_i A$ (see [12], §6.3), $\xi(\zeta_c)$ annihilates A.

Now we reconstruct $l_p(u, s, \chi)$ in the Iwasawa method (cf. [12], §7.2). The following lemma can be proved in the same way as (6).

LEMMA 1. *Let F be any multiple of f. Then*

$$H_\chi^m(u) = F^m \frac{1-u^f}{1-u^F} \sum_{k=0}^{F-1} \chi(k) u^{F-k-1} E_m\left(u^F, \frac{k}{F}\right)$$

for any $m \geqq 0$.

We define some notations following [12], §7.2. Put $f = dp^\kappa$ where $(d, p)=1$ and $\kappa \geqq 0$. Put $K_n = \boldsymbol{Q}(\zeta_{dp^{n+1}})$ and $G_n = G(K_n\,|\,\boldsymbol{Q}) = \{\sigma_a \,|\, \zeta_{dp^{n+1}} \mapsto \zeta_{dp^{n+1}}^a,\ (a, dp)=1\}$ for any $n \geqq 0$. Then we have

$$G_n \simeq \Delta \times \Gamma_n$$

where $\Delta = (\boldsymbol{Z}/dp\boldsymbol{Z})^*$ and $\Gamma_n = \boldsymbol{Z}/p^n\boldsymbol{Z}$. Corresponding to this decomposition, we write

$$\sigma_a = \delta(a)\gamma_n(a)$$

with $\delta(a)\in\Delta$ and $\gamma_n(a)\in\Gamma_n$. Regarding χ as a character of G_n, we may uniquely write

$$\chi=\theta\psi$$

with $\theta\in\hat{\Delta}$ where $\hat{\Delta}$ is a character group of Δ, and with $\psi\in\hat{\Gamma}_n$. Finally we define

$$\xi_n(u)=\sum_{\substack{a=1\\(a,dp)=1}}^{dp^{n+1}}\frac{u^{dp^{n+1}-a}}{1-u^{dp^{n+1}}}\delta(a)^{-1}\gamma_n(a)^{-1}\in\Theta[G_n] \tag{15}$$

where Θ is an integer ring of $\boldsymbol{Q}_p(u,\chi)$. $\xi_n(u)$ is an analogue of the Stickelberger element. Let

$$\varepsilon_\rho=\frac{1}{|\Delta|}\sum_{\delta\in\Delta}\rho(\delta)\delta^{-1}$$

be the idempotent for $\rho\in\hat{\Delta}$. By using the fact $\varepsilon_\rho\delta(a)=\rho(a)\varepsilon_\rho$, we have

$$\varepsilon_{\bar\theta}\xi_n(u)=\xi_{n,\theta}(u)\varepsilon_{\bar\theta}$$

where

$$\xi_{n,\theta}(u)=\sum_{\substack{a=1\\(a,dp)=1}}^{dp^{n+1}}\frac{u^{dp^{n+1}-a}}{1-u^{dp^{n+1}}}\theta(a)\gamma_n(a)^{-1}\ , \tag{16}$$

and $\bar\theta=\theta^{-1}$. We can see that $\xi_{n,\theta}(u)\in\Theta[\Gamma_n]$.

LEMMA 2. *Let $\rho_{m,n}$ be the natural map $\Theta[\Gamma_m]\to\Theta[\Gamma_n]$ with $\gamma_m(a)\mapsto\gamma_n(a)$ for $m\geqq n$. Then $\rho_{m,n}(\xi_{m,\theta}(u))=\xi_{n,\theta}(u)$ for $m\geqq n$.*

PROOF. Put $\xi'_{n,\theta}(u)=\rho_{m,n}(\xi_{m,\theta}(u))$, then

$$\xi'_{n,\theta}(u)=\sum_{\substack{a=1\\(a,dp)=1}}^{dp^{m+1}}\frac{u^{dp^{m+1}-a}}{1-u^{dp^{m+1}}}\theta(a)\gamma_n^{-1}(a)\ .$$

Put $a=j+kdp^{n+1}$ with $0\leqq j<dp^{n+1}$ and $(j,dp)=1$, and with $0\leqq k<p^{m-n}$. By using the fact that $\theta(j+kdp^{n+1})=\theta(j)$ and $\gamma_n(j+kdp^{n+1})=\gamma_n(j)$, we can see that

$$\begin{aligned}\xi'_{n,\theta}(u)&=\sum_{\substack{j=1\\(j,dp)=1}}^{dp^{n+1}}\left\{\frac{u^{dp^{m+1}-j}}{1-u^{dp^{m+1}}}\sum_{k=0}^{p^{m-n}-1}u^{-kdp^{n+1}}\right\}\theta(j)\gamma_n^{-1}(j)\\&=\xi_{n,\theta}(u)\ .\end{aligned}$$

Thus we have the assertion.

Put $\Theta[[\Gamma]]=\varprojlim \Theta[[\Gamma_n]]$ where the limit is the projective limit with the homomorphism $\rho_{m,n}$ in Lemma 2. By Lemma 2, we put

$$\xi^{\theta}(u)=\varprojlim \xi_{n,\theta}(u) \in \Theta[[\Gamma]] .$$

It follows from [12] (Theorem 7.1) that

$$\Theta[[\Gamma]]\simeq\Theta[[T]]\simeq\varprojlim \Theta[[T]]/((1+T)^{p^n}-1)$$

with $\gamma_n(a)\mapsto(1+T)^{i(a)} \bmod ((1+T)^{p^n}-1)$ for $a\in \boldsymbol{Z}$, where

$$i(a)=\log_p\langle a\rangle/\log_p(1+dp) .$$

Let $e_u(T, \theta)$ be the image of $\xi^{\theta}(u)$ with the above isomorphism. Then by (16), we have

$$e_u(T, \theta)\equiv \sum_{\substack{a=1\\(a,dp)=1}}^{dp^{n+1}} \frac{u^{dp^{n+1}-a}}{1-u^{dp^{n+1}}}\theta(a)(1+T)^{-i(a)} \qquad (\bmod ((1+T)^{p^n}-1)) \tag{17}$$

for any $n\geqq 0$. Put $\zeta_{\psi}=\psi(1+dp)^{-1}$, and put $T=\zeta_{\psi}(1+dp)^{-m}-1$ for $m\equiv 0 \pmod{p-1}$ in (17). Then we have

$$\begin{aligned} &e_u(\zeta_{\psi}(1+dp)^{-m}-1, \theta)\\ &\equiv \sum_{\substack{a=1\\(a,dp)=1}}^{dp^{n+1}} \frac{u^{dp^{n+1}-a}}{1-u^{dp^{n+1}}}\theta(a)\psi(a)\langle a\rangle^m \qquad (\bmod p^{n+1}\Theta)\\ &\equiv \sum_{\substack{a=1\\(a,dp)=1}}^{dp^{n+1}} \frac{u^{dp^{n+1}-a}}{1-u^{dp^{n+1}}}\chi(a)a^m \qquad (\bmod p^{n+1}\Theta)\end{aligned} \tag{18}$$

for $n\gg 0$. Put $F=dp^{n+1}$ in Lemma 1, we have

$$H_{\chi}^m(u)=(dp^{n+1})^m\frac{1-u^f}{1-u^{dp^{n+1}}}\sum_{k=0}^{dp^{n+1}-1}\chi(k)u^{dp^{n+1}-k-1}E_m\left(u^{dp^{n+1}}, \frac{k}{dp^{n+1}}\right)$$

for $m\equiv 0 \pmod{p-1}$. By (3), we can see that

$$(dp^{n+1})^m E_m\left(u^{dp^{n+1}}, \frac{k}{dp^{n+1}}\right)\equiv k^m \qquad (\bmod p^{n+1}\Theta) .$$

Hence we have

$$\frac{u}{1-u^f}H_{\chi}^m(u)\equiv \sum_{k=0}^{dp^{n+1}-1}\chi(k)k^m\frac{u^{dp^{n+1}-k}}{1-u^{dp^{n+1}}} \qquad (\bmod p^{n+1}\Theta) \tag{19}$$

for $m\equiv 0 \pmod{p-1}$. It follows from (18), (19) and Theorem 1 that

$$l_p(u, -m, \chi) \equiv e_u(\zeta_\psi(1+dp)^{-m}-1, \theta) \pmod{p^{n+1}\Theta}$$

for $n \gg 0$. Let n tend to the infinity, then for $m \equiv 0 \pmod{p-1}$,

$$l_p(u, -m, \chi) = e_u(\zeta_\psi(1+dp)^{-m}-1, \theta) .$$

Since $(p-1)\boldsymbol{Z}$ is dense in $\boldsymbol{Z}_p$, we obtain the following

PROPOSITION. *Let $\chi = \theta\psi$ be the Dirichlet character where θ is a character of the first kind and ψ is a character of the second kind, and let $f = dp^\kappa$ be the conductor of χ with $(d, p)=1$ and $\kappa \geqq 0$. Put $\zeta_\psi = \psi(1+dp)^{-1}$ and let Θ be an integer ring of $\boldsymbol{Q}_p(u, \chi)$. Then there exists a formal power series $e_u(T, \theta) \in \Theta[[T]]$ such that*

$$l_p(u, s, \chi) = e_u(\zeta_\psi(1+dp)^s - 1, \theta)$$

for any $s \in \boldsymbol{Z}_p$.

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Present Address:
DEPARTMENT OF MATHEMATICS
TOKYO METROPOLITAN UNIVERSITY
FUKAZAWA, SETAGAYA-KU, TOKYO 158