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A Generalization of a Lemma by Schmitt and Vogel

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Abstract. We prove a generalization of a lemma by Schmitt and Vogel which will allow us to compute the arithmetical rank of new classes of monomial ideals.

Introduction

Given a commutative ring with identity R, the *arithmetical rank* of an ideal I of R, denoted $\operatorname{ara} I$, is defined as the minimum number of elements which generate I *up to radical*, i.e., generate an ideal which has the same radical as I. Determining this number is, in general, a very hard open problem; a trivial lower bound is given by the height of I, but this is the actual value of $\operatorname{ara} I$ only in special cases (I is then called a *set-theoretic complete intersection*). There are, however, techniques which allow us to provide upper bounds. Some results in this direction have been recently proved by the author, as, e.g., in [1] and [2]. These apply especially to the case where R is a polynomial ring over a field, and I is a *monomial* ideal (i.e., an ideal generated by products of indeterminates) and are essentially based on the following criterion by Schmitt and Vogel (see [9], p. 249).

LEMMA 1. *Let P be a finite subset of elements of R. Let $P_0, \ldots, P_r$ be subsets of P such that*

(i) $\bigcup_{i=0}^{r} P_i = P$;

(ii) *P_0 has exactly one element*;

(iii) *if p and p'' are different elements of P_l $(0 < i \leq r)$ there is an integer i' with $0 \leq i' < i$ and an element $p' \in P_{i'}$ such that $pp'' \in (p')$.*

Let $0 \leq i \leq r$, and, for any $p \in P_i$, let $e(p) \geq 1$ be an integer. We set $q_l = \sum_{p\in P_i} p^{e(p)}$. We will write (P) for the ideal of R generated by the elements of P. Then we get

$$\sqrt{(P)} = \sqrt{(q_0, \ldots, q_r)}\,.$$

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If, in the construction given in the claim, we take all exponents $e(p)$ to be equal to 1, then $q_0, \ldots, q_r$ are sums of generators; in this paper we present a new method, which gives rise to elements of the same form, but applies under a different assumption. It will enable us to determine the arithmetical rank of certain monomial ideals which could not be treated by the above lemma.

For the determination of the arithmetical rank, every ideal can be replaced by its radical. In the sequel we will therefore throughout consider radical (or reduced) monomial ideals, i.e., ideals generated by squarefree monomials.

The problem of the arithmetical rank of monomial ideals has been intensively studied by several other authors over the past three decades: see [4], [5], [7], [8], [9] and [10].

1. The main result and some applications

We consider the ring $R = K[x_1, \ldots, x_N]$, where K is a field. We will determine the arithmetical rank of certain ideals generated by monomials. We prove a result, based on combinatorial considerations, which generalizes Lemma 1 for this class of ideals and which will allow us to prove the set-theoretic intersection property in various examples.

PROPOSITION 1. *Let $G \subset R$ be a set of monomials. Suppose that there are subsets $S_0, \ldots, S_r$ of G such that*

(i) $\bigcup_{i=0}^{r} S_i = G$;

(ii) *S_0 has exactly one element*;

(iii) *the following recursive procedure can always be performed and always comes to an end regardless of the choice of the indeterminate z and the index j at each step.*

0. *Set $T = S_0$.*
1. *Pick an indeterminate z dividing the only element of T.*
2. *Cancel all monomials divisible by z in every S_i.*
3. *If no element of G is left, then end. Else pick an index j such that there is exactly one element left in S_j and set $T = S_j$.*
4. *Go to* 1.

For all $i = 0, \ldots, r$ we set $q_i = \sum_{\mu \in S_i} \mu$. Then we get

$$\sqrt{(G)} = \sqrt{(q_0, \ldots, q_r)}\,.$$

PROOF. It suffices to show that $(G) \subset \sqrt{(q_0, \ldots, q_r)}$. We proceed by induction on $r \geq 0$. For $r = 0$ we have that $(G) = (S_0) = (q_0)$, so that the claim is trivially true. Now suppose that $r > 0$ and that the claim is true for all smaller r. According to Hilbert's Nullstellensatz (see [6], Theorem 5.4), it suffices to show that, whenever all q_i vanish at some $\mathbf{x} \in K^N$, the same is true for all $\mu \in G$. In the sequel, as long as this does not cause any ambiguity, we will denote a polynomial and its value at $\mathbf{x}$ by the same symbol. So assume that $q_i = 0$ for all $i = 0, \ldots, r$. From $q_0 = 0$ we deduce that one of the indeterminates dividing the only element of S_0, say the indeterminate z, vanishes. Then all $\mu \in G$ that are divisible by z vanish. Let $\bar{G}$ be the set of $\mu \in G$ that are not divisible by z. We have to show that all $\mu \in \bar{G}$ vanish. If $\bar{G} = \emptyset$, then there is nothing to be proven. Otherwise, for all $i = 1, \ldots, r$, set $\bar{S}_i = S_i \cap \bar{G}$. By assumption we have that $|\bar{S}_j| = 1$ for some index j; up to a change of indices we may assume that $j = 1$. Then $\bar{G}$ and its subsets $\bar{S}_1, \ldots, \bar{S}_r$ fulfil the assumption of the proposition. For all $i = 1, \ldots, r$ we set $\bar{q}_i = \sum_{\mu \in \bar{S}_i} \mu$. Then by induction $\sqrt{(\bar{G})} = \sqrt{(\bar{q}_1, \ldots, \bar{q}_r)}$. Since, by assumption, all $\bar{q}_i$ vanish, this implies that all $\mu \in \bar{G}$ vanish, as required. This completes the proof.

REMARK 1. Proposition 1 generalizes Lemma 1 for ideals generated by monomials over a field. In fact, if P is a set of monomials and and $P_0, \ldots, P_r$ are as in the assumption of Lemma 1, and we set $G = P$ and $S_i = P_i$ for all $i = 0, \ldots, r$, then the assumption of Proposition 1 is fulfilled, as we are going to show next. Let z be any indeterminate dividing the only element of S_0. As in the proof of Proposition 1, for all $i = 1, \ldots, r$, let $\bar{S}_i$ denote the set of monomials in S_i which are not divisible by z. Let j be the smallest index $j > 0$ such that $\bar{S}_j$ is not empty. Then the product of each two distinct monomials of S_j is divisible by a monomial of some S_i, $i < j$, which is divisible by z. Hence all but possibly one of the monomials of S_j are divisible by z. Therefore $|\bar{S}_j| \leq 1$. We conclude by finite descending induction.

The above result can be used for computing the arithmetical rank of some monomial ideals.

EXAMPLE 1. In the polynomial ring $R = K[x_1, \ldots, x_5]$, where K is any field, consider the ideal

$$I = (x_1x_3,\ x_1x_4,\ x_2x_4,\ x_2x_5,\ x_3x_5)\,,$$

It is of pure height 3. Then by Proposition 1 we have

$$I = \sqrt{(x_1x_3,\ x_1x_4 + x_2x_5,\ x_2x_4 + x_3x_5)}\,.$$

Hence ara $I = 3$, and I is a set-theoretic complete intersection. Note that three elements of R generating I up to radical cannot be found by applying Lemma 1 to the set of minimal monomial generators of I.

EXAMPLE 2. In $R = K[x_1, \ldots, x_6]$ consider the ideal

$$I = (x_1x_3,\ x_1x_4,\ x_1x_6,\ x_2x_4,\ x_2x_5,\ x_3x_5,\ x_3x_6,\ x_4x_6)\,,$$

It has pure height 4. It is a set-theoretic complete intersection, since, as we will see,

$$I = \sqrt{(x_1x_6,\ x_3x_5,\ x_1x_3 + x_2x_4 + x_3x_6,\ x_1x_4 + x_2x_5 + x_4x_6)}\,. \tag{1}$$

We only need to prove the inclusion $\subset$. Note that, according to Proposition 1,

$$\sqrt{(x_3x_5,\ x_3x_6 + x_2x_4,\ x_4x_6 + x_2x_5)} = (x_3x_5,\ x_3x_6,\ x_2x_4,\ x_4x_6,\ x_2x_5)\,.$$

Therefore, to prove (1) it suffices to show that x_1x_3, x_1x_4 belong to the right-hand side of (1). This is true, because, firstly

$$\begin{aligned} x_1^3x_3^3 = {} & x_3^2(x_2x_4 - x_1x_3)x_1x_6 + x_1x_2^2x_3 \cdot x_3x_5 \\ & + x_1^2x_3^2(x_1x_3 + x_2x_4 + x_3x_6) - x_1x_2x_3^2(x_1x_4 + x_2x_5 + x_4x_6)\,, \end{aligned}$$

which proves the claim for x_1x_3, and secondly,

$$\begin{aligned} x_1^2x_4^2 = {} & (x_3x_5 - x_4^2)x_1x_6 + x_1^2 \cdot x_3x_5 \\ & - x_1x_5(x_1x_3 + x_2x_4 + x_3x_6) + x_1x_4(x_1x_4 + x_2x_5 + x_4x_6)\,. \end{aligned}$$

Finally we apply Proposition 1 to a class of ideals which extends Example 1. We will determine the arithmetical rank of these ideals and show that they are not set-theoretic complete intersections except for the ideal studied in that example.

EXAMPLE 3. Let $m \geq 2$ be an integer, and let I_m be the reduced monomial ideal of $R = K[x_1, \ldots, x_{3m+2}]$ generated by the following monomials:

$$\begin{aligned} r_1 = x_1x_2\,, \quad s_n &= x_{3n-2}x_{3n+2} \\ t_n &= x_{3n}x_{3n+1} \\ u_n &= x_{3n+1}x_{3n+2} \\ v_n &= x_{3n-1}x_{3n} \qquad (n = 1, \ldots, m)\,. \end{aligned}$$

We prove that

$$\operatorname{ara} I_m = 2m + 1\,. \tag{2}$$

Let J_m be the ideal of R generated by the following $2m+1$ elements:

$$x_1x_2\,, \quad s_n + t_n\,, \quad u_n + v_n \quad (n = 1, \ldots, m)\,.$$

We first show that

$$\operatorname{ara} I_m \leq 2m + 1 \tag{3}$$

by proving that

$$I_m = \sqrt{J_m}\,.$$

It suffices to show that $I_m \subset \sqrt{J_m}$, i.e., that $s_n, t_n, u_n, v_n \in \sqrt{J_m}$ for $1 \leq n \leq m$. We proceed by finite induction on n, $1 \leq n \leq m$. For $n = 1$ the claim follows if we apply Proposition 1 to $S_0 = \{r_1\} = \{x_1x_2\}$, $S_1 = \{s_1 + t_1\} = \{x_1x_5 + x_3x_4\}$ and $S_2 = \{u_1 + v_1\} = \{x_4x_5 + x_2x_3\}$. Now suppose that $n > 1$ and that the claim is true for $n - 1$. Then, in particular, $u_{n-1} = x_{3n-2}x_{3n-1} \in \sqrt{J_m}$. We have that

$$S_0 = \{u_{n-1}\}\,, \quad S_1 = \{s_n + t_n\}\,, \quad S_2 = \{u_n + v_n\}$$

fulfil the assumption of Proposition 1. It follows that

$$s_n, t_n, u_n, v_n \in \sqrt{J_m}\,,$$

which achieves the induction step and proves (3).
We now show the opposite inequality. Let

$$S = \{x_1\} \cup \{x_{3n},\ x_{3n+2}, \mid n = 1, \ldots, m\}\,,$$

and set $P = (S)$. Then P is a prime ideal and $I_m \subset P$, since x_1 divides r_1 and, for all $n = 1, \ldots, m$, x_{3n+2} divides s_n and u_n, and x_{3n} divides t_n and v_n. Ideal P is in fact a minimal prime of I_m, because

$$r_1 \notin (S \setminus \{x_1\}),\ \text{and}\ u_n \notin (S \setminus \{x_{3n+2}\}),\ t_n \notin (S \setminus \{x_{3n}\}) \quad (n = 1, \ldots m)\,.$$

We have that height $P = 2m + 1$. By Krull's principal ideal theorem (see [6], Theorem 13.5), it follows that

$$2m + 1 \leq \operatorname{ara} I_m\,,$$

as required. This completes the proof of (2). Now let

$$T = \{x_2\} \cup \{x_{3n+1},\ x_{3n+2}, \mid n = 1, \ldots, m-1\} \cup \{x_{3m+1}\}\,,$$

and set $Q = (T)$. Then $I_m \subset Q$, since

- x_2 divides r_1, v_1;
- for $n = 1, \ldots, m-1$, x_{3n+2} divides s_n and u_n, and x_{3n+1} divides t_n;
- for $n = 2, \ldots, m$, $x_{3(n-1)+2} = x_{3n-1}$ divides v_n;
- $x_{3(m-1)+1} = x_{3m-2}$ divides s_m;
- x_{3m+1} divides u_m and t_m.

Ideal Q is a minimal prime of I_m, because

- $r_1 \notin (T \setminus \{x_2\})$;
- $t_n \notin (T \setminus \{x_{3n+1}\}) \quad (n = 1, \ldots, m)$;
- $v_{n+1} \notin (T \setminus \{x_{3n+2}\}) \quad (n = 1, \ldots, m-1)$.

We have that height $Q = 1 + 2(m-1) + 1 = 2m < 2m + 1$, so that the height of I_m is less than $2m + 1$. Hence I is not a set-theoretic complete intersection: as it is well known, any reduced monomial ideal which is a set-theoretic complete intersection is Cohen-Macaulay, and therefore, it has pure height (see [3], Corollary 5.1.5). Let us remark that for $m = 1$, ideal I_m is, up to a change of variables, the same as ideal I of Example 1.

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