Defining q as the set of conditional distributions

$$\{(P(\Theta = 0 \mid X = x) = 2/3,\ P(\Theta = 1 \mid X = x) = 1/3)):\ x = 1, 2, 3, \ldots\},$$

Kadane, Schervish, and Seidenfeld (1985) assert that it is "reasonable to claim that q is the posterior for θ given X once finite additivity is accepted," and that the example "makes clear the need for a less restrictive definition of posterior distribution that will allow inference even when a probability cannot be made conglomerable in a specific partition." For q,

$$(2) \qquad \frac{P(\Theta = 1 \mid X = x)}{P(\Theta = 0 \mid X = x)} = 1/2, \quad x = 1, 2, \ldots.$$

To ignore the conflict between (1) and (2), on the grounds that this is merely an expression of acceptable nonconglomerability, is to turn a blind eye to the problem that it raises in the use of P to approximate honest opinion about what odds to quote for Θ given $12 < X < 12^{144}$.

Of course, not all the applications of finitely noncountably additive probabilities are unattractive. As yet, there appear to be no axioms that will discriminate either the probabilities or the applications that are acceptable. It is not easy to see how the necessary weakening or replacement of monotone continuity might be engineered. There may be one or two clues in the work of Seidenfeld and Schervish (1983). Let us hope that Dr. Fishburn will return once again to the topic, in a survey that will remove the remaining obscurities.

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Comment

William D. Sudderth

Most of Professor Fishburn's interesting article treats axiom schemes for the relations *is more probable than* and *is at least as probable as*, and the question of when these schemes lead to a compatible probability measure. There are two other approaches to formulating axioms for probabilities interpreted as degrees of belief. The first is due to de Finetti (1937, 1949) and gives a direct economic interpretation to probability numbers. The second was developed by Cox (1961) and Jaynes to formulate axioms for rational beliefs and for how such beliefs should be modified. Perhaps some readers will be interested in a brief description of these two alternative routes.

One version of the de Finetti theory begins with a function P which assigns a real number $P(A)$ to certain events A. Think of $P(A)$ as your price in dollars for a ticket worth \$1 if A occurs and \$0 if not. You are required to be willing to buy or sell a finite number of tickets on any of the events in a given collection $\mathscr{A}$. (There is no need to assume $\mathscr{A}$ is an algebra.) Then de Finetti shows that you are *coherent* in the sense that you cannot be made a sure loser if and only if P is a finitely additive probability measure (or can be extended to be one if $\mathscr{A}$ is not an algebra). An advantage of this approach is that the conditional probability $P(A \mid B)$ can be defined directly as the price of a \$1 ticket on A with the provision that the transaction is called off if B does not occur. A requirement of coherence for these conditional transactions leads to the formula

$$P(AB) = P(B)P(A \mid B)$$

which in turn implies the finite form of Bayes' formula given in Section 7. All of this is explained in detail by de Finetti (1949). There are extensions of de Finetti's result which yield Bayes' formula for infinite partitions (cf. Heath and Sudderth (1978) and Lane and Sudderth (1984)). These extensions involve a strengthening of the coherence condition which is not acceptable to all of de Finetti's followers.

In the Cox-Jaynes theory it is assumed that the *plausibility* of A on the evidence B can be represented by a real number $(A \mid B)$. Qualitative arguments are given for a postulate stating that the plausibility number $(AB \mid C)$ should be some function F of $(B \mid C)$ and $(A \mid BC)$. Because AB is the same as BA, the function F is required to give the same answer if its arguments

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are $(A \mid C)$ and $(B \mid AC)$. This leads to a functional equation for F which under mild regularity assumptions can be solved. The solution, after being rescaled by a monotone transformation, gives the usual formula for the probability of an intersection. Further qualitative assumptions and another functional equation lead to finite additivity and Bayes' formula. Jaynes writes in the objective Bayesian tradition of Laplace and Jeffreys, but the approach should be of interest to subjectivists also.

Finally, I would like to comment briefly on countable additivity. The requirement of coherence does not imply countable additivity as de Finetti has often emphasized, nor do the Cox-Jaynes axioms. Even the objectivistic relative frequency interpretation of probability fails to support it. The axiom of monotone continuity may be an appealing way to reformulate countable additivity, but in general I agree with Kolmogorov (1933) that the assumption of countable additivity, although expedient, is arbitrary.

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Rejoinder

Peter C. Fishburn

It is a pleasure to thank the discussants for their contributions to issues of subjective probability and decision making in the face of uncertainty. I am especially grateful for their enhancement of the whole by their emphasis on topics covered only lightly if at all in my survey of axiomatics.

The diversity of the responses, ranging from Suppes's discussion of the foundational issues of uniqueness, exchangeability, and expectation to Stone's more pointed focus on finite versus countable additivity, is impressive. At the same time, three concerns were raised in common by several discussants, namely the problem of imprecision, the decisional bases of subjective probability, and the matter of finite versus countable additivity. I conclude with a few words on each of these primary issues.

The problem of imprecision or vagueness in judgment is raised by Berger, Good, Fine, and Seidenfeld. It has been a central theme in my own research, beginning with *Decision and Value Theory* (Wiley, 1964). Others who have mined it in past years include Good, Art Dempster, C. A. B. Smith, and, before them, John Maynard Keynes. A typical way of dealing with the problem is to admit a possibly convex family of representing measures (not necessarily additive), which might be characterized by intervals or upper and lower bounds on distributions. It remains a viable research topic as seen in the exciting work of Glen Shafer and the research papers of Fine and his coauthors.

The decisional bases of subjective probability are discussed by Berger, Sudderth, and Seidenfeld. Berger emphasizes the interface with statistical practice, while Sudderth and Seidenfeld recall the important works of Bruno de Finetti, C. A. B. Smith, and others that phrase axioms for subjective probability in terms of preferences or choices in the face of uncertainty. This too has been one of my own preoccupations in the tradition pioneered by Frank P. Ramsey, Jimmie Savage, and de Finetti, although it was mentioned only briefly in the survey. I am indebted to the discussants for reminding us of its centrality.

Finally, the matter of finite versus countable additivity, made prominent by de Finetti and Savage, is raised by Berger, Stone, and Sudderth. The present wisdom seems to be that countable additivity can keep one out of trouble that might arise in its absence even if it is arbitrary, or at best uncompelling, as a principle of rational choice. My own attitude toward the issue is pragmatic. Much like the Axiom of Choice in set theory, if I can do without countable additivity to get where I want to go, so much the better. But I will not hesitate to invoke it when its denial would create mathematical complexities of little interest to the topic at hand.