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The Axioms of Subjective Probability

Peter C. Fishburn

Abstract. This survey recounts contributions to the axiomatic foundations of subjective probability from the pioneering era of Ramsey, de Finetti, Savage, and Koopman to the mid-1980's. It is designed to be accessible to readers who have little prior acquaintance with axiomatics. At the same time, it provides a fairly complete picture of the present state of the measurement-theoretic foundations of subjective probability.

Key words and phrases: Axioms for comparative probability, numerical representations, additivity, countable additivity.

The *probability of any event* is the ratio between the value at which an expectation depending on the happening of the event ought to be computed, and the value of the thing expected upon its happening. (Bayes, 1763, page 376)

We are driven therefore to the second supposition that the degree of a belief is a causal property of it, which we can express vaguely as the extent to which we are prepared to act on it. (Ramsey, 1931, page 170)

... the degree of probability attributed by an individual to a given event is revealed by the conditions under which he would be disposed to bet on that event. (de Finetti, 1937; from Kyburg and Smokler, 1964, page 101)

The intuitive thesis in probability holds that ... probability derives directly from the intuition, and is prior to objective experience ... (Koopman, 1940a, page 269)

Personalistic views hold that probability measures the confidence that a particular individual has in the truth of a particular proposition, for example, the proposition that it will rain tomorrow. (Savage, 1954, page 3)

1. INTRODUCTION

The theory of subjective probability attempts to make precise the connection between coherent dispositions toward uncertainty and quantitative probability as axiomatized by Kolmogorov (1933) and others. It accommodates the classical interpretations of probability in Bayes (1763) and Laplace (1812), the intuitive views of Koopman (1940a, 1940b) and Good (1950), and the decision-oriented approach of Ramsey (1931), de Finetti (1931, 1937), and Savage (1954). The aim of this essay is to recount the axiomatic development of subjective or personal (Savage) probability from 1926, when Frank P. Ramsey wrote *Truth and Probability*, to the present. My hope is that this will not only provide a useful current perspective on subjective probability *per se* but that it will also promote appreciation of a vital part of the Bayesian approach to statistical decision theory pioneered by Good (1950) and Savage (1954) and further developed by Schlaifer (1959), Raiffa and Schlaifer (1961), DeGroot (1970), and Hartigan (1983) among others. A brief but very informative introduction to problems and perspectives of the Bayesian approach is given by Sudderth (1985).

The axioms of subjective probability refer to assumed properties of a binary relation *is more probable than*, or its nonstrict companion *is at least as probable as*, on a set of propositions or events. This relation, often referred to as a qualitative or *comparative probability* relation, can be taken either as an undefined primitive (intuitive views) or as a relation derived from a preference relation (decision-oriented approach). In the latter case, to say that you regard *rain* as more probable tomorrow than *shine*, or that you believe the pound sterling is more likely to *fall* than *rise* against the dollar next year means roughly that you would rather bet on the first-named event for a valuable prize that you will receive if your chosen event obtains.

Comparative probability axioms are usually thought of as criteria of consistency and coherence for a person's attitudes toward uncertainty. The prevailing view has been that these criteria do not purport to describe actual behavioral attitudes so much as they characterize the partial beliefs of a rational, idealized individual. Lately, however, some traditional axioms

Peter C. Fishburn is a member of the Mathematical Studies Department in the Mathematical Sciences Research Center, AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, New Jersey 07974.

have been weakened to yield theories that accommodate observed departures from those axioms, and the lines between the ideal and the actual have become blurred. I shall say more about this later.

Except when noted otherwise, I adopt the strict comparative probability relation $\succ$ as basic. Read $A \succ B$ as "event A is (regarded by the individual as) more probable than event B." When $\succ$ is basic, its symmetric complement $\sim$ and nonstrict companion $\succsim$ are *defined* by

$$A \sim B \quad \text{if neither} \quad A \succ B \quad \text{nor} \quad B \succ A,$$

$$A \succsim B \quad \text{if either} \quad A \succ B \quad \text{or} \quad A \sim B.$$

We usually view $\sim$ as a comparative equiprobability relation in either a precise sense or an approximate "no significant difference" sense.

On the other hand, when $\succsim$ ("is at least as probable as") is taken as basic, then $\sim$ and $\succ$ are defined from *it* by

$$A \sim B \quad \text{if} \quad A \succsim B \quad \text{and} \quad B \succsim A,$$

$$A \succ B \quad \text{if} \quad A \succsim B \quad \text{and not} \quad (B \succsim A).$$

This has the advantage of identifying A and B as *noncomparable* (Keynes, 1921) when neither $A \succsim B$ nor $B \succsim A$. In the $\succ$ basic case, noncomparability is embedded in $\sim$ and there is no obvious way to distinguish between it and comparative equiprobability when such a distinction is desired.

It is easily checked that, when $\succ$ is basic, the definitions in the preceding paragraphs agree with each other if and only if $\succ$ is *asymmetric*, i.e., whenever $A \succ B$ then not $(B \succ A)$. Alternatively, if $\succsim$ is basic, the definitions agree if and only if $\succsim$ is *complete*, i.e., $A \succsim B$ or $B \succsim A$ for every pair of events. We shall generally assume that $\succ$ is asymmetric when it is taken as basic, but will consider cases where $\succsim$ is not complete when *it* is taken as basic (see Section 3).

The set on which $\succ$ or $\succsim$ is defined is assumed to be a Boolean algebra $\mathscr{A}$ of subsets A, B, $\cdots$ of a universal set S. We refer to each A in $\mathscr{A}$ as an *event*. The *empty event* is $\varnothing$, the *universal event* is S, and $\varnothing \subseteq A \subseteq S$ for every A in $\mathscr{A}$. We recall that $\mathscr{A}$ is a *Boolean algebra* if $S \in \mathscr{A}$, $A \in \mathscr{A} \Rightarrow S \backslash A \in \mathscr{A}$, and A, $B \in \mathscr{A} \Rightarrow A \cup B \in \mathscr{A}$. The complement $S \backslash A$ of A in S will also be written as A^c.

A *probability measure* on $\mathscr{A}$ is a real valued function P on $\mathscr{A}$ such that $P(S) = 1$ and, for all A and B in $\mathscr{A}$, $P(A) \geq 0$, and $A \cap B = \varnothing \Rightarrow P(A \cup B) = P(A) + P(B)$. This last property implies that P is *finitely additive*: if $A_1, \cdots, A_n$ are mutually disjoint events in $\mathscr{A}$, then $P(\cup_j A_j) = \sum_j P(A_j)$. Countable additivity and σ-algebras, which are taken for granted in some standard works on probability (Loève, 1960; Feller, 1966), will be discussed later.

We say that P on $\mathscr{A}$

partially agrees with $\succ$ if $A \succ B \Rightarrow P(A) > P(B)$,

almost agrees with $\succsim$ if $A \succsim B \Rightarrow P(A) \geq P(B)$,

agrees with $\succ$ (or $\succsim$) if $A \succ B \Leftrightarrow P(A) > P(B)$,

for all A and B in $\mathscr{A}$. Partial agreement requires asymmetry for $\succ$, but almost agreement does not require completeness for $\succsim$, so $\succsim$ is the appropriate basis for consideration of almost agreement.

The most demanding axiomatizations of subjective probability specify conditions for $\succ$ on $\mathscr{A}$ that are necessary and sufficient, or perhaps only sufficient, for the existence of a probability measure that agrees with $\succ$. Less demanding theories seek only partial agreement or almost agreement. We also mention cases intermediate between partial and full agreement and note others in which additivity of P must be replaced by a weaker concept to obtain a suitable numerical representation.

What axioms for $\succ$ or $\succsim$ on $\mathscr{A}$ are so obvious and uncontroversial as to occasion no serious criticism? A few that might qualify for this distinction are

asymmetry: If $A \succ B$ then not $(B \succ A)$;

nontriviality: $S \succ \varnothing$;

nonnegativity: $A \succsim \varnothing$;

monotonicity: If $A \supseteq B$ then $A \succsim B$;

inclusion monotonicity:
If $(A \supseteq B, B \succ C)$ or $(A \succ B, B \supseteq C)$ then $A \succ C$.

One might also nominate

transitivity: If $A \succ B$ and $B \succ C$ then $A \succ C$;

additivity: If $A \cap C = \varnothing = B \cap C$ then $A \succ B \Leftrightarrow A \cup C \succ B \cup C$;

complementarity: If $A \succ B$ then not $(A^c \succ B^c)$,

but caution is advised here since these assumptions have not gone unchallenged. Examples appear in Section 3.

In any event, axioms like these form the foundations of theories of subjective probability and of our remarks about those theories. We begin with agreement when the event set is finite, then consider weaker finite representations, finitely and countably additive representations with infinite algebras, and conclude with comments on conditional probability.

Excellent sources for articles of historical interest for subjective probability are Kyburg and Smokler (1964) and Savage, Hacking, and Shimony (1967). Technical surveys are included in Luce and Suppes

(1965), Krantz et al. (1971), Fine (1973), and Roberts (1979). Other interpretations of probability are discussed in Savage (1954), Good (1959), Fishburn (1964), Fine (1973), de Finetti (1978), and Walley and Fine (1979).

2. FINITE AGREEMENT

This section and the next assume that $\mathscr{A}$ is finite. With no loss in generality, let $\mathscr{A}$ be the family of all subsets of $S = \{1, 2, \cdots, n\}$. Each i in S is a *state* and S is the set of states or the set of states of the world (Savage, 1954).

We take $\succ$ as basic in the present section. For convenience, let

$$p_i = P(\{i\}), \qquad i = 1, \cdots, n,$$

so that P agrees with $\succ$ if, for all A and B in $\mathscr{A}$,

$$(1) \qquad A \succ B \Leftrightarrow \sum_{i \in A} p_i > \sum_{i \in B} p_i.$$

This requires $\succ$ to be asymmetric and transitive and, since $A \sim B \Leftrightarrow P(A) = P(B)$, it also requires $\sim$ to be transitive. We refer to $\succ$ as a *weak order* when it has these properties.

What else besides weak order is needed for agreement? Some time ago de Finetti (1931) noted that (1) also entails nontriviality ($S \succ \varnothing$), nonnegativity ($A \succsim \varnothing$ for each A), and additivity ($A \cap C = \varnothing = B \cap C \Rightarrow [A \succ B \Leftrightarrow A \cup C \succ B \cup C]$). The question of whether these axioms are sufficient for agreement remained open until it was settled in the negative by Kraft, Pratt, and Seidenberg (1959). They constructed an example with $n = 5$ that satisfies de Finetti's basic axioms and includes the comparisons

$$\{4\} \succ \{1, 3\},$$

$$\{2, 3\} \succ \{1, 4\},$$

$$\{1, 5\} \succ \{3, 4\},$$

$$\{1, 3, 4\} \succ \{2, 5\}.$$

If (1) holds then

$$p_4 > p_1 + p_3,$$

$$p_2 + p_3 > p_1 + p_4,$$

$$p_1 + p_5 > p_3 + p_4,$$

$$p_1 + p_3 + p_4 > p_2 + p_5.$$

But these are inconsistent since addition and cancellation leaves us with $0 > 0$.

What Kraft, Pratt, and Seidenberg discovered is that a much stronger additivity condition is needed for (1). To motivate it, let $(A_1, \cdots, A_m) =_0 (B_1, \cdots, B_m)$ mean that the A_j and B_j are in $\mathscr{A}$ and, for each $1 \le i \le n$, the number of A_j that contain i equals the number of B_j that contain i. In other words, the sums of the indicator functions over the two event sequences are identical. Then, for any real numbers $p_1, \cdots, p_n$,

$$(A_1, \cdots, A_m) =_0 (B_1, \cdots, B_m) \Rightarrow \sum_{j=1}^{m} \sum_{i \in A_j} p_i = \sum_{j=1}^{m} \sum_{i \in B_j} p_i.$$

Consequently, if (1) holds and $(A_1, \cdots, A_m) =_0 (B_1, \cdots, B_m)$, then we cannot have $A_j \succsim B_j$ for every j along with $A_j \succ B_j$ for at least one j. In the preceding example,

$$(A_1, A_2, A_3, A_4) = (\{4\}, \{2, 3\}, \{1, 5\}, \{1, 3, 4\}),$$

$$(B_1, B_2, B_3, B_4), = (\{1, 3\}, \{1, 4\}, \{3, 4\}, \{2, 5\}).$$

We shall say that $\succ$ on $\mathscr{A}$ is *strongly additive* if, for all $m \ge 2$ and all A_j and B_j,

$$(2) \qquad [(A_1, \cdots, A_m) =_0 (B_1, \cdots, B_m), A_j \succsim B_j \text{ for each } j < m] \Rightarrow \text{not } (A_m \succ B_m).$$

As just noted, this is necessary for (1). Its strength is suggested by the fact that it implies weak order as well as additivity. For example,

$$(A, B) =_0 (B, A), \quad \text{so} \quad A \succ B \Rightarrow \text{not} \quad (B \succ A);$$

$$(A, B, C) =_0 (B, C, A), \quad \text{so} \quad (A \succsim B, B \succsim C) \Rightarrow A \succsim C.$$

But strong additivity implies much more. Namely (Fishburn, 1970, Chapter 4), $\succ$ on $\mathscr{A}$ is strongly additive *if and only if* there are real numbers $p_1, \cdots, p_n$ that satisfy (1). In the special case of subjective probability with $p_i \ge 0$ and $\sum p_i = 1$, it is enough to assume that $\succ$ is nontrivial and nonnegative as well as strongly additive. Proofs that these or equivalent axioms are necessary and sufficient for agreement appear in Kraft, Pratt, and Seidenberg (1959), Scott (1964), and Krantz et al. (1971).

Those proofs, and ones for theorems in the next section that use conditions similar to strong additivity, are based on solution theory for systems of linear inequalities (Kuhn and Tucker, 1956). Relevant theorems are referred to Farkas's lemma, the theorem of the alternative, Fan's theorem, and separating-hyperplane lemmas. The basic algebraic results of linear solution theory are also used by Heath and Sudderth (1972) and Buehler (1976) among others to expand on de Finetti's ideas on coherent systems of bets.

A crude but instructive comparison between betting and strong additivity can be made as follows. Suppose strong additivity fails with $(A_1, \cdots, A_m) =_0 (B_1, \cdots, B_m)$, $A_j \gtrsim B_j$ for all $j < m$, and $A_m \succ B_m$. Because of $A_m \succ B_m$, an individual with these comparisons would presumably be willing to pay some positive amount to engage in the following game. For each A_j that obtains he receives \$1, and for each B_j that obtains he pays out \$1. (Event A obtains if the true state is in A.) However, this would be foolish, for regardless of which state obtains, $(A_1, \cdots, A_m) =_0 (B_1, \cdots, B_m)$ ensures that his net take, exclusive of stake, will be precisely zero. I shall say more about decision models shortly.

Assuming agreement as in (1) with nonnegative p_i that sum to 1, it is trivial to note that the p_i are unique up to transformations that preserve nonnegativity, the sum of 1, and the weak order of partial sums induced by $\succ$. Of more interest is the question of when the p_i in (1) are unique. A partial answer, sufficient for uniqueness (Luce, 1967; Krantz et al., 1971) is that whenever events A, B, C, and D are such that $A \cap B = \varnothing$, $A \succ B$, and $B \gtrsim D$, then there are events C', D', and E such that

$$(3)\qquad \begin{aligned} &E \sim A \cup B, \\ &C' \cap D' = \varnothing, \\ &C' \cup D' \subseteq E, \\ &C' \sim C, \quad \text{and} \quad D' \sim D. \end{aligned}$$

This is a difficult condition to unravel and it is not necessary even in simple cases. An example not covered by the condition is, for $n = 5$,

$$\begin{aligned} &\{1, 2\} \sim \{4\}, \\ &\{1, 4\} \sim \{2, 3\}, \\ &\{2, 4\} \sim \{5\}, \\ &\{2, 5\} \sim \{1, 3, 4\}. \end{aligned}$$

Under (1) these yield $p_1 + p_2 = p_4$, $p_1 + p_4 = p_2 + p_3$, $p_2 + p_4 = p_5$, and $p_2 + p_5 = p_1 + p_3 + p_4$. The unique solution with nonnegative p_i that sum to 1 is

$$(p_1, p_2, p_3, p_4, p_5) = (2/22, 3/22, 4/22, 5/22, 8/22).$$

In general, given $p_i > 0$ for all i, the p_i are unique if and only if there are $n - 1$ event pairs (A_j, B_j) with $A_j \cap B_j = \varnothing$ and $A_j \sim B_j$ such that the corresponding $n - 1$ linear equations

$$\sum_{i \in A_j} p_i - \sum_{i \in B_j} p_i = 0, \qquad j = 1, \cdots, n - 1,$$

are linearly independent.

Other avenues to uniqueness are available in the finite states setting if $\mathscr{A}$ is embedded in or extended to a richer structure and something more than (1) is required of P that will ensure its uniqueness. For example, Suppes and Zanotti (1976) and Luce and Narens (1978) extend $\mathscr{A}$ to infinite structures with corresponding extensions of $\succ$ such that the representation required of the probability measure for the extended system implies that it is unique. If P for (1) is the restriction to $\mathscr{A}$ of this measure, then P is unique, but only because of the conditions imposed on its parent. In particular, the linear independence condition of the preceding paragraph need not hold for $\sim$ on $\mathscr{A}$. Extendability is treated in detail in Kaplan and Fine (1977).

A more traditional route to uniqueness arises when $\mathscr{A}$ is embedded in a decision structure and extraneous scaling probabilities (Anscombe and Aumann, 1963; Fishburn, 1967, 1970; Myerson, 1979) or canonical lotteries (Pratt, Raiffa, and Schlaifer, 1964) are used to derive a subjective expected utility model in which the state probabilities are unique. One might, for example, use probabilities associated with events for a random device to construct lotteries on a set of monetary outcomes. We then consider an individual's preference relation on the set $\{(l_1, \cdots, l_n)$: each l_i is a lottery$\}$, where $(l_1, \cdots, l_n)$ is a "lottery act" that selects lottery l_i if state i obtains or is the "true state." The selected lottery is then played to determine the final outcome. If the individual's preference relation on the set of lottery acts satisfies the expected utility axioms of von Neumann and Morgenstern (1944), then for each i in S there is a real valued utility function u_i on the outcome set M such that, for all pairs of lottery acts,

$$(l_1, \cdots, l_n) \text{ is preferred to } (l'_1, \cdots, l'_n)$$
$$\Leftrightarrow \sum_{i=1}^{n} \sum_{a \in M} l_i(a) u_i(a) > \sum_{i=1}^{n} \sum_{a \in M} l'_i(a) u_i(a),$$

where $l(a)$ is the probability that lottery l yields outcome a.

Subjective probabilities emerge from this utility representation (Fishburn, 1967) if we adopt an independence axiom which says that preferences over lotteries within each nonnull state are essentially identical. [State i is *null* if $(l_1, \cdots, l_n)$ is indifferent to $(l'_1, \cdots, l'_n)$ whenever $l_j = l'_j$ for all $j \neq i$.] We can then replace u_i by $p_i u$ under suitable rescaling of the u_i to obtain

$$(l_1, \cdots, l_n) \text{ is preferred to } (l'_1, \cdots, l'_n)$$
$$\Leftrightarrow \sum_{i=1}^{n} p_i \sum_{a \in M} l_i(a) u(a) > \sum_{i=1}^{n} p_i \sum_{a \in M} l'_i(a) u(a),$$

with the p_i unique and $p_i = 0$ if and only if state i is null.

Additional details of this and other lottery-based theories are given in Fishburn (1981, Section 8). Other decision-oriented theories that derive P on finite or infinite $\mathscr{A}$ as part of the utility representation for preferences are described there also.

3. OTHER FINITE MODELS

Finite models weaker than (1) arise when we relax strong additivity, including perhaps its implication of weak order. Plausible reasons for considering weaker models will be illustrated by examples before we get into specific representations. I omit asymmetry, nontriviality, and monotonicity from the examples.

3.1 Noncomparability (Incompleteness of $\succsim$)

A = Mexico City's population will exceed 20,000,000 by 1994;
B = The first card drawn from this old and probably incomplete bridge deck will be a heart.

As Keynes (1921) might have argued, A and B are sufficiently disparate to discourage a reasonable person from judging either $A \succsim B$ or $B \succsim A$.

3.2 Nontransitive $\sim$

For a slightly bent coin,
A = The next 101 flips will give at least 40 heads;
B = The next 100 flips will give at least 40 heads;
C = The next 1000 flips will give at least 460 heads.

The judgments $A \sim C$, $C \sim B$, and $A \succ B$ do not seem unreasonable.

3.3 Nontransitive $\succ$ (May, 1954; Tversky, 1969; Fishburn, 1983a)

Cyclic patterns could arise from comparisons between multidimensional events. Sue will meet Mike X. Smith at a party next Saturday. She knows nothing about him except that he has just published a hot new book on winning strategies in real estate. Her judgments on separate attributes for Mr. Smith include

Height: 6′-0″ $\succ$ 6′-1″ $\succ$ 6′-2″,
Age: 40 $\succ$ 50 $\succ$ 60,
Hair color: brunette $\succ$ red $\succ$ blonde.

Three composites for Mr. Smith are

A = 6′-0″ 60-year-old redhead;
B = 6′-1″ 40-year-old blonde;
C = 6′-2″ 50-year-old brunette.

Sue considers one of these more believable than another if the first is more probable on two of the three attributes. Hence $A \succ B$, $B \succ C$, and $C \succ A$.

3.4 Nonadditivity (Ellsberg, 1961)

An urn contains 90 colored balls, 30 of which are ecru. The other 60 are red and navy in unknown proportion. One ball will be chosen at random from the urn.

E = chosen ball is ecru;
R = chosen ball is red;
N = chosen ball is navy.

An individual can earn a valuable prize by guessing the color of the chosen ball. Two comparisons are made: E versus R, and $E \cup N$ versus $R \cup N$. Ellsberg's experiments indicate that $E \succ R$ and $R \cup N \succ E \cup N$ are likely. In each case the "more probable" event has greater specificity: exactly 30 balls correspond to E, and exactly 60 correspond to $R \cup N$. The noted judgments clearly violate additivity. They also violate complementarity.

Two relatives of $=_0$ in the preceding section will be used to express necessary and sufficient conditions for several weaker models. First,

$$(A_1, \cdots, A_m) \geqq_0 (B_1, \cdots, B_m)$$

means that, for each $1 \leq i \leq n$, the number of A_j that contain i equals or exceeds the number of B_j that contain i. Second,

$$(A_1, \cdots, A_m) >_0 (B_1, \cdots, B_m)$$

means that for *every* i the number of A_j that contain i exceeds the number of B_j that contain i. In both cases, the A_j and B_j are understood to be in $\mathscr{A}$.

We first consider almost agreement with $\succsim$ as basic. In this case (Kraft, Pratt, and Seidenberg, 1959) there is a probability measure P on $\mathscr{A}$ that almost agrees with $\succsim$ if and only if, for all $m \geq 1$ and all A_j and B_j,

$$[(A_1, \cdots, A_m) >_0 (B_1, \cdots, B_m),\ B_j \succsim A_j \text{ for each } j < m] \Rightarrow \text{not } (B_m \succsim A_m). \tag{4}$$

This is a very weak condition. Its only implication for $m = 1$ is not $(\varnothing \succsim S)$. For $m = 2$, since $(A, A^c) >_0 (\varnothing, \varnothing)$, it implies that we cannot have both $\varnothing \succsim A$ and $\varnothing \succsim A^c$. Indeed, the weakness of almost agreement is seen by the fact that it does not forbid cyclic $\succ$. For example, if $S = \{1, 2, 3\}$ with $\{1\} \succ \{2\} \succ \{3\} \succ \{1\}$, almost agreement could hold with $p_1 = p_2 = p_3 = 1/3$.

Partial agreement with $\succ$ as basic forbids cyclic $\succ$ and seems somewhat more demanding than almost agreement. In this case (Adams, 1965; Fishburn, 1969) there is a probability measure P on $\mathscr{A}$ that partially agrees with $\succ$ if and only if, for all $m \geq 1$ and all A_j and B_j,

$$[(A_1, \cdots, A_m) \geqq_0 (B_1, \cdots, B_m),\ B_j \succ A_j \text{ for all } j < m] \Rightarrow \text{not } (B_m \succ A_m). \tag{5}$$

This entails asymmetry, nonnegativity, monotonicity, and acyclicity, but not nontriviality, transitivity, or additivity. However, it does forbid additivity reversals like $E \succ R$ and $R \cup N \succ E \cup N$ in our fourth example.

Stronger versions of partial agreement have been developed to explicitly include discriminatory thresholds in "if and only if" representations. They were motivated by Luce's (1956) semiorder model and Fishburn's (1970, 1985) interval order model in preference theory. We call $\succ$ an *interval order* if it is asymmetric and satisfies ($A \succ B$ and $C \succ D$) $\Rightarrow$ ($A \succ D$ or $C \succ B$), for all A, B, C, and D. If, in addition, $(A \succ B \succ C) \Rightarrow$ ($A \succ D$ or $D \succ C$), for all A, B, C, and D, then $\succ$ is a *semiorder*. Examples like those given earlier show that these conditions can be violated by reasonable judgments in some circumstances. For instance, if $A \supset B$ and $C \supset D$ with A and B of a substantially different nature than C and D (c.f. A versus B in our first example), then $A \succ B$, $C \succ D$, $A \sim C \sim B$, and $A \sim D \sim B$ might obtain.

The semiorder partial agreement representation is a constant threshold model with

$$A \succ B \Leftrightarrow P(A) > P(B) + \delta, \qquad 0 \leq \delta < 1,$$

and the interval order partial agreement representation is a variable threshold model with

$$A \succ B \Leftrightarrow P(A) > P(B) + \sigma(B), \qquad \sigma \geq 0.$$

In both cases, P is a probability measure on $\mathscr{A}$. Necessary and sufficient axioms (Fishburn, 1969; Domotor and Stelzer, 1971) for the semiorder representation are asymmetry, nontriviality, inclusion monotonicity, and

$$\begin{aligned}[(A_1, \cdots, A_{2m}) &=_0 (B_1, \cdots, B_{2m}),\\ &B_j \succsim A_j \text{ for } j = 1, \cdots, m, \text{ and}\\ &A_j \succ B_j \text{ for } j = m+1, \cdots, 2m-1]\\ &\qquad\qquad \Rightarrow \text{not} \quad (A_{2m} \succ B_{2m}).\end{aligned}$$

Sufficient axioms (Fishburn, 1969) for the interval order representation are asymmetry, ($A \succ B$ and $C \succ D$) $\Rightarrow$ ($A \succ D$ or $C \succ B$), inclusion monotonicity (not strictly necessary), and

$$\begin{aligned}[(A_1, \cdots, A_m) &=_0 (B_1, \cdots, B_m),\ A_j \succ^* B_j\\ &\text{for all } j < m]\\ &\qquad \Rightarrow \text{not} \quad (A_m \succ^* B_m),\end{aligned}$$

where $A \succ^* B$ means that, for all C, $A \succ C$ if $B \succ C$, and for some C, $A \succ C$ and not ($B \succ C$). The displayed conditions essentially reflect the difference between the constant and variable threshold notions.

The interval order representation bears comparison to models of upper and lower probabilities that arise in Koopman (1940a), Smith (1961), Good (1962), Dempster (1967, 1968), Suppes (1974), Williams (1976), Walley and Fine (1979), and Kumar (1982) among others. See also Shafer (1976) on the concept of belief functions. One model of this type has

$$A \succ B \Leftrightarrow P_*(A) > P^*(B),$$

where P_* and P^* are monotonic but not necessarily additive measures on $\mathscr{A}$ with $P^*(S) = P_*(S) = 1$, $P^*(\varnothing) = P_*(\varnothing) = 0$, and $P^*(A) \geq P_*(A)$ for each A. If we let $\sigma(A) = P^*(A) - P_*(A)$, then

$$A \succ B \Leftrightarrow P_*(A) > P_*(B) + \sigma(B),$$

which is similar to the interval order model. Axioms for this case consist of the interval order conditions plus suitable nonnegativity and monotonicity assumptions. It requires a special additivity condition only if there is to be a probability measure P for which $P_* \leq P \leq P^*$.

Davidson and Suppes (1956) and Schmeidler (1984) axiomatize utility models in which P is monotonic but not necessarily additive. Davidson and Suppes adopt Ramsey's (1931) measurement approach. They first use an event with subjective probability ½ to scale outcome utilities and then use these to scale probabilities of other events. Their outcome utilities are evenly spaced and $P(A)$ is defined as

$$P(A) = \frac{u(d^*) - u(d)}{u(d^*) - u(d) + u(c) - u(c^*)}$$

when $u(c) \neq u(c^*)$, $u(d) \neq u(d^*)$, and the act that yields c if A and d if A^c is indifferent to the act that yields c^* if A and d^* if A^c. This gives $P(A) + P(A^c) = 1$, and their P is unique if $n \geq 5$.

Schmeidler (1984) weakens the lottery act axioms for the model at the end of the preceding section to obtain a modified expected utility representation in which P is monotonic and unique. Because P need not be additive, his definition of expectation (integration) is special. Schmeidler's model is designed to accommodate Ellsberg-type violations of additivity.

Another nonadditive model, axiomatized in both the intuitive and decision modes (Fishburn, 1983a, 1983b), allows intransitivities. Unlike preceding representations, it uses a two-argument function ρ on $\mathscr{A} \times \mathscr{A}$ with

$$A \succ B \Leftrightarrow \rho(A, B) > 0$$

along with $\rho(\Omega, \varnothing) = 1$, $\rho(B, A) = -\rho(A, B)$, $A \supseteq B \Rightarrow \rho(A, B) \geq 0$, and

$$\begin{aligned}&A \cap B = \varnothing\\ &\quad \Rightarrow \rho(A \cup B, C) + \rho(\varnothing, C) = \rho(A, C) + \rho(B, C).\end{aligned}$$

The last property is a nonseparable generalization of additivity. Necessary and sufficient conditions are nontriviality, monotonicity, and a generalization of strong additivity.

4. INFINITE ALGEBRAS

This and the next two sections assume that $\mathscr{A}$ is an infinite Boolean algebra of subsets of S. The present section introduces terminology and special axioms developed for this case. It also notes a partial agreement axiomatization that does not fit into the next two sections, which discuss agreement and almost agreement when $\succ$ is a weak order. The next section focuses on finite additivity; Section 6 looks at countably additive measures. We take $\succ$ as basic throughout.

The Boolean algebra $\mathscr{A}$ is a *σ-algebra* if A is in $\mathscr{A}$ whenever A is the union of a countable number of events in $\mathscr{A}$. A probability measure P is *countably additive* if

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

whenever the A_i are mutually disjoint events whose union is also in $\mathscr{A}$. Note that $\mathscr{A}$ does not have to be a σ-algebra for P to be countably additive.

Following Savage (1954), $\succ$ is *fine* if, whenever $A \succ \varnothing$, S can be partitioned into events $B_1, \cdots, B_m$ with $A \succsim B_j$ for all j. Events A and B are *almost equivalent* if, for all C and D in $\mathscr{A}$,

$$[C \succ \varnothing,\ D \succ \varnothing,\ A \cap C = \varnothing = B \cap D] \Rightarrow A \cup C \succsim B \quad \text{and} \quad B \cup D \succsim A;$$

and $\succ$ is *tight* if $A \sim B$ whenever A and B are almost equivalent. Savage (1954, page 37) subsequently strengthened his definition of fine by replacing $A \succsim B_j$ by $A \succ B_j$ for all j. We refer to $\succ$ as *superfine* in the latter case.

Following Villegas (1964), event A is an *atom* if $A \succ \varnothing$ and $A \succ B \succ \varnothing$ for no event $B \subseteq A$. The algebra $\mathscr{A}$ is *atomless* if it has no atom. It should be noted that these definitions depend on the behavior of $\succ$.

Three of the earliest axioms for infinite algebras are due to de Finetti (1931, 1937) and Savage (1954). The first of the following is from de Finetti, and also Bernstein (1917) and Koopman (1940a). The other two are Savage's. All three suggest the possibility of partitioning $\mathscr{A}$ into a very large number of events with very small probabilities.

Partition Axiom 1. For each $n \geq 2$, S can be partitioned into n equally likely ($\sim$) events.

Partition Axiom 2. For each $n \geq 2$, S can be partitioned into n events such that the union of no r events is more probable than the union of any $r + 1$.

Partition Axiom 3. For all A and B in $\mathscr{A}$, if $A \succ B$ then S can be partitioned into a finite number of events $C_1, \cdots, C_m$ such that $A \succ B \cup C_j$ for all j.

Partitions for Axiom 1 are *uniform* partitions, and those for Axiom 2 are *almost uniform* partitions. Given de Finetti's basic axioms (Section 2), Savage notes that Axiom 3 implies that $\succ$ is both superfine and tight and that it implies Axioms 1 and 2.

Partition Axiom 3 is used by Fishburn (1975) to axiomatize partial agreement with a finitely additive measure. Sufficient axioms for partial agreement are inclusion monotonicity, Axiom 5 with $=_0$ in place of $\geqq_0$, Partition Axiom 3, and the following part of additivity:

$$[A \cap C = \varnothing = B \cap C,\ A \cup C \succ B \cup C] \Rightarrow A \succ B.$$

Partial additivity is defended by the contention that if $A \cup C \succ B \cup C$ when $(A \cup B) \cap C = \varnothing$, then the difference between the two events will be even more apparent when the common part C is removed. The fourth example in the preceding section questions its general acceptability.

Because of the possibility of atoms, none of the partition axioms is necessary for agreement. We therefore note three more conditions that are necessary for the existence of a probability measure that agrees with $\succ$. These are referred to as Archimedean axioms because they lead to real representations with no infinitesimal nonstandard numbers or infinitely large numbers. A few preliminary definitions are needed.

A subset $\mathscr{C}$ of $\mathscr{A}$ is *order dense* in $\mathscr{A}$ if, whenever $A \succ B$ and neither A nor B is in $\mathscr{C}$, then $A \succsim C \succsim B$ for some C in $\mathscr{C}$. A countable (finite or denumerable) sequence of events $A_1, A_2, \cdots$ is a *standard sequence relative to event A* if for each A_i there are disjoint events B_i and C_i such that $A_1 = B_1$, $A_{i+1} = B_i \cup C_i$ (when $i + 1$ is present), $B_1 \sim A$, $C_i \sim A$, and $B_i \sim A_i$. The upshot of this is that if P agrees with $\succ$ then $P(A_i) = iP(A)$ for each i.

Archimedean Axiom 1. $\mathscr{A}$ includes a countable order dense subset.

Archimedean Axiom 2. For each event $A \succ \varnothing$, every standard sequence relative to A is finite.

Archimedean Axiom 3. For all $m \geq 2$ and all events $A_1, \cdots, A_m, B_1, \cdots, B_m$: if $A_1 \succ B_1$ then there is a positive integer N such that, whenever $(k\varnothing, nA_1, A_2, \cdots, A_m) =_0 (kS, nB_1, B_2, \cdots, B_m)$ with $k \geq 0$, $n \geq 1$, and $A_j \succsim B_j$ for all $j \geq 2$, then $k/n > 1/N$.

The first Archimedean axiom, which dates from Cantor's (1895) work on transfinite numbers, is necessary and sufficient in conjunction with weak order for the existence of a real valued function f on $\mathscr{A}$ such that, for all A and B in $\mathscr{A}$, $A \succ B \Leftrightarrow f(A) > f(B)$ (Birkhoff, 1967; Fishburn, 1970; Krantz et al., 1971).

Archimedean Axiom 2, from Luce (1967), is one of several standard sequence conditions used in Krantz et al. (1971). To be applicable, it requires structural conditions that allow the construction of such sequences. Neither of the first two axioms relates to additivity of probability, which must enter through another axiom.

Strong additivity is built into Archimedean Axiom 3 via $k = 0$ and $n = 1$. In its statement, due to Chateauneuf and Jaffray (1984), kA denotes A repeated k times. Its necessity for agreement follows from the fact that agreement and its hypotheses imply

$$k = n[P(A_1) - P(B_1)] + \sum_{j\geq 2} [P(A_j) - P(B_j)]$$
$$\geq n[P(A_1) - P(B_1)],$$

so $k/n \geq P(A_1) - P(B_1)$. Since $P(A_1) - P(B_1) > 0$ by $A_1 \succ B_1$, k/n cannot be arbitrarily small without violating agreement.

5. FINITE ADDITIVITY

The basic axioms of de Finetti, i.e., *weak order*, *nontriviality*, *nonnegativity*, and *additivity*, have become so standard that they are used more or less directly in all infinite $\mathscr{A}$ axiomatizations for agreement, and in most axiomatizations for almost agreement. Savage (1954) *defines* $\succsim$ (or $\succ$) as a *qualitative probability* when it satisfies de Finetti's axioms, and this name has persisted in more or less this form.

Since all axiomatizations in this section and the next use de Finetti's basic axioms, they will be taken for granted until the final section of the paper. The present section is concerned solely with the existence of finitely additive probability measures that agree, or almost agree, with $\succ$ when $\mathscr{A}$ is infinite. We consider almost agreement first.

Savage (1954) proves that there exists a P that almost agrees with $\succ$ when Partition Axiom 2 holds (almost uniform) and $\mathscr{A}$ is the σ-algebra of all subsets of S. Moreover, P is unique, and if $P(A) > 0$ and $0 < \lambda < 1$, then $P(B) = \lambda P(A)$ for some $B \subseteq A$. The same conclusions follow if $\succ$ is superfine and, in this case, $P(A) > 0$ if $A \succ \varnothing$ and $P(A) = P(B)$ if and only if A and B are almost equivalent. See also Wakker (1981, Lemmas 3–5).

Related results for almost agreement have been obtained by others when $\mathscr{A}$ is merely assumed to be a Boolean algebra. Niiniluoto (1972) and Wakker (1981) note that there is a unique almost agreeing P when $\succ$ is fine. Narens (1974) and Wakker (1981) show that almost agreement (not necessarily unique) follows from (4).

The earliest axiomatizations for agreement (Bernstein, 1917; de Finetti, 1931, 1937; Koopman, 1940a) used Partition Axiom 1 (uniform partitions). Savage (1954), again with $\mathscr{A}$ as the family of all subsets of S, obtains a unique agreeing P with Partition Axiom 3. Since the $P(B) = \lambda P(A)$ property holds here, $\mathscr{A}$ must be atomless. A related result is obtained in French (1982) by adjoining an auxiliary experiment (extraneous scaling probabilities) to S. This approach, of enriching a possibly finite S with events generated from a random device, was used earlier by DeGroot (1970), and indeed did not escape notice by Savage (1954, pages 33 and 38).

Other axiomatizations for agreement do not assume that $\mathscr{A}$ is a σ-algebra. Luce (1967), whose axioms apply also to finite S, shows that a unique agreeing P follows from (3) and Archimedean Axiom 2. Roberts (1973) obtains unique agreement from Partition Axiom 2 and Archimedean Axiom 1, after a similar result in Fine (1971). Wakker (1981) notes that, when $\succ$ is fine, unique agreement holds if and only if either $\succ$ is tight or $\mathscr{A}$ has an atom.

Necessary and sufficient conditions for agreement are discussed by Domotor (1969) and Chateauneuf and Jaffray (1984). Domotor's complex conditions will not be recalled here. Chateauneuf and Jaffray show that if $\mathscr{A}$ is countable, e.g., the family of finite subsets of $\{1, 2, \cdots\}$ and their complements, then some P agrees with $\succ$ if and only if Archimedean Axiom 3 holds. Because their conditions are necessary as well as sufficient, they do not imply uniqueness. Indeed, a price payed for uniqueness is nonnecessity since conditions that guarantee unique agreement cannot be wholly necessary.

Further references on mathematical details of finitely additive measures include Bochner (1939), Sobczyk and Hammer (1944), Yosida and Hewitt (1952), Hewitt and Savage (1955), Dubins (1974), Purves and Sudderth (1976), Bhashara Rao and Bhaskara Rao (1983), and Schervish, Seidenfeld, and Kadane (1984). Contributions of special interest in decision theory include Dubins and Savage (1965), de Finetti (1975), Heath and Sudderth (1978), Lane and Sudderth (1978), and Seidenfeld and Schervish (1983).

6. COUNTABLE ADDITIVITY

We continue to assume de Finetti's four basic axioms.

Villegas (1964) identifies the key assumption on $\succ$ needed for countable additivity. Modified slightly to accommodate Boolean algebras that are not also σ-algebras, his axiom is

Monotone continuity: For all $A, B, A_1, A_2, \cdots$ in $\mathscr{A}$, if $A_1 \subseteq A_2 \subseteq \cdots$, $A = \cup_i A_i$ and $B \succsim A_i$ for all i, then $B \succsim A$.

Thus, if the nondecreasing A_i converge to a limit event A, then the judgment that B is at least as probable as A_i for all i cannot be reversed in the limit by a jump to $A \succ B$. Monotone continuity is quite appealing and is necessary for agreement with a countably additive probability measure. In fact, Villegas (1964) proves that if $\mathscr{A}$ is a σ-algebra and if P is a finitely additive probability measure that agrees with $\succ$, then P is countably additive if and only if $\succ$ is monotonely continuous. Chateauneuf and Jaffray (1984) remark that this remains true even when $\mathscr{A}$ is not a σ-algebra.

Consequently, for every axiomatization in the preceding section that implies the existence of P that agrees with $\succ$, such a P is countably additive if and only if $\succ$ is monotonely continuous.

Other axiomatizations have been developed specifically for countable additivity when $\mathscr{A}$ is a σ-algebra. All assume monotone continuity. Villegas (1964) proves that if $\mathscr{A}$ is atomless then there is a unique countably additive P that agrees with $\succ$. DeGroot (1970) and French (1982) obtain the same conclusion with the explicit use of an auxiliary experiment that itself is atomless but allows atoms in the original state set. Similarly, Chuaqui and Malitz (1983) adopt strong additivity to obtain unique agreement whenever $\mathscr{A}$ has a nontrivial atomless subalgebra. They also consider the purely atomic case, where P is a discrete measure, and prove an almost agreement theorem under strong additivity. Finally, Chateauneuf and Jaffray (1984) show that Archimedean Axiom 3 is necessary and sufficient for agreement. Their complete set of axioms for countably additive agreement when $\mathscr{A}$ is a σ-algebra is equivalent to de Finetti's axioms, monotone continuity, and Archimedean Axiom 3. Since these are necessary as well as sufficient, their P is not necessarily unique.

7. CONDITIONAL PROBABILITY

A typical version of Bayes' formula for conditional probability when $S = \Theta \times Z$ and the underlying state or parameter space Θ and sample space Z are finite is

$$P_1(\theta \mid z) = \frac{P_2(z \mid \theta)P_0(\theta)}{\sum_{\theta'} P_2(z \mid \theta')P_0(\theta')}.$$

Here $P_1(\cdot \mid z)$ and $P_2(\cdot \mid \theta)$ are conditional probability mass functions, and P_0 is a (marginal) probability mass function on Θ. In the usual subjectivistic interpretation, P_0 is the prior distribution and P_1 is the posterior on Θ given observation z. Each of P_0, P_1, and P_2 may be based on a joint distribution over S as described in the next paragraph.

If there is a *unique* probability measure P on $\mathscr{A}$ that agrees or almost agrees with $\succ$ or $\succsim$, then subjective conditional probabilities can be defined unambiguously in the usual way as

$$(6) \quad P(A \mid B) = P(A \cap B)/P(B) \quad \text{when} \quad P(B) > 0,$$

thus putting Bayes' theorem and other mathematical machinery of conditional probability at our disposal. The theoretical foundations of Bayesian decision theory can be understood in this way when all uncertainties, including those associated with experiment's outcomes, are embedded in $\mathscr{A}$. At least in theory, if not in common practice, priors and other special probabilities can be extracted from the global P measure by the usual operations. For the setting of the preceding paragraph, $P_0(\theta) = P(\{\theta\} \times Z)$, $P_1(\theta \mid z) = P(\{(\theta, z)\})/P(\Theta \times \{z\})$, and $P_2(z \mid \theta) = P(\{(\theta, z)\})/P(\{\theta\} \times Z)$ when the denominators do not vanish.

A potential problem with (6) arises when P is not unique, as for example in Section 2, since conditional probabilities might not be uniquely ordered. For example, one agreeing P may give $P(A \mid B) > P(C \mid D)$ by (6), while another gives $P(C \mid D) > P(A \mid B)$.

Partly for this reason, and partly because some authors, including Koopman (1940a, 1940b), believe that conditional judgments are basic to subjective probability, there has been interest in axiomatizing conditional probability directly. This is usually done by means of a comparative conditional probability relation $\succsim_0$ on $\mathscr{A} \times \mathscr{A}_0$, where $\mathscr{A}_0$ is the set of nonnull events in $\mathscr{A}$ and $A \mid B \succsim_0 C \mid D$ is interpreted as "A given B is at least as probable as C given D." When $\succsim_0$ is complete and $\sim_0$ is defined by

$$A \mid B \sim_0 C \mid D \quad \text{if} \quad A \mid B \succsim_0 C \mid D \quad \text{and} \quad C \mid D \succsim_0 A \mid B,$$

a *null event* is an A in $\mathscr{A}$ for which $A \mid S \sim_0 \varnothing \mid S$. A typical representation for comparative conditional probability is, for all $A \mid B$ and $C \mid D$ in $\mathscr{A} \times \mathscr{A}_0$,

$$(7) \quad A \mid B \succsim_0 C \mid D \Leftrightarrow P(A \cap B)/P(B) \geq P(C \cap D)/P(D),$$

where P is a probability measure on $\mathscr{A}$.

Domotor (1969) includes necessary and sufficient conditions on $\succsim_0$ for (7) when $\mathscr{A}$ is finite. His axioms extend the approach of Kraft, Pratt, and Seidenberg (1959) for finite agreement. Suppes and Zanotti (1982) give necessary and sufficient conditions for a model like (7) for general $\mathscr{A}$ by using extensions of indicator functions in a manner similar to Suppes and Zanotti (1976). Other sufficient axiomatizations for (7) and related representations appear in Koopman (1940a, 1940b), Aczel (1961, 1966), Luce (1968), and Krantz et al. (1971). The last of these provides a nice description of previous work (pages 221–228). Their main axiomatization is based on Luce (1968) and consists of seven conditions necessary for (7) plus a nonnecessary solvability condition which says that if $A \mid B \succsim_0 C \mid D$ then there is a C' in $\mathscr{A}$ such that $C \cap D \subseteq C'$ and $A \mid B \sim_0 C' \mid D$.

Conditional probabilities also arise directly from the preference-based axiomatizations of Ramsey (1931), Pfanzagl (1967, 1968), Luce and Krantz (1971), Fishburn (1973), and Balch and Fishburn (1974). Reviews of their theories are included in Fishburn (1981).

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