Comment

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Professor Ferguson provides a nice introduction, both historical and mathematical, to the now "classical" secretary problem (CSP). He then refers to the game of googol as a striking example of the "true" secretary problem according to his definition, and tries to find the solution which he calls "a resolution" to the problem. In doing so he discusses a problem of information and sequential decision-making in a competitive two-person game with a very simple payoff structure. My purpose here is (a) to mention that Cayley's problem in 1875 already closely resembled the CSP which appeared 80 years later, (b) to notice that choosing the second best is more difficult than choosing the best, and (c) to note that the game of googol involves various classes of games according to the minimizing player's behavior in restricting his set of choices.

CAYLEY'S PROBLEM AND THE CSP

Cayley's problem in its first proposed form can also be resurrected from oblivion in the following way:

An urn contains n balls, each one labeled with one of the distinct positive integers $1, 2, \cdots, n$. One ball at a time is randomly drawn, the number on it read, and the ball removed. You may continue this procedure at most k times, each time one ball being removed. You are allowed to terminate the procedure at your convenience. If you observe the numbers on the balls $x_1, x_2, \cdots, x_{m-1}$ and stop at the mth drawing with the number x_m, then you win the game if the future draws at times $m+1, m+2, \cdots, k$ yield the result $x_m > \max\{x_1, \cdots, x_{m-1}, X_{m+1}, \cdots, X_k\}$. The problem is to find a stopping strategy that maximizes the probability of your winning.

This problem with $1 \le k \le n$ evidently (maybe) is a true secretary problem according to Ferguson's definition of it, but finding an optimal stopping strategy is far more complicated than in Cayley's original problem or in CSP. The reason I attached "maybe" above is that the absolute (as well as relative) ranks of the balls are known as they are drawn.

Define the state (m, y), for $m = 1, 2, \cdots, k$; $1 \le y \le n$, to mean that (1) you have not yet stopped the procedure and (2) you face the mth draw with the number $X_m = y$ which is a "candidate" (i.e., it is larger than any of the numbers which have been observed in the past $(m-1)$ draws). Then in state (m, y) the urn contains balls with numbers $(\{1, 2, \cdots, y\}\backslash\{x_1, \cdots, x_{m-1}, y\}) \cup \{y+1, \cdots, n\}$ and therefore the transition probability from state (m, y) is

$$\Pr\{(m', z) \mid (m, y)\} = \begin{cases} 0, & \text{if } z < y, \\ \dfrac{y-m}{n-m}\cdot\dfrac{y-m-1}{n-m-1}\cdots\dfrac{y-m'+2}{n-m'+2}\cdot\dfrac{1}{n-m'+1}, & \text{if } y+1 \le z \le n, \end{cases}$$

for $m < m' \le k$. Letting $V_{m,k}(y)$ denote the maximum expected probability of winning when the state of the process is (m, y), the principle of dynamic programming yields the equation

$$\begin{aligned} V_{m,k}(y) &= \max\Bigg[\frac{y-m}{n-m}\cdot\frac{y-m-1}{n-m-1}\cdots\frac{y-k+1}{n-k+1}, \\ &\qquad \sum_{m<m'\le k}\ \sum_{y+1\le z\le n} \Pr\{(m', z)\mid(m, y)\} V_{m',k}(z)\Bigg] \\ &= \max\Bigg[\frac{(y-m)_{k-m}}{(n-m)_{k-m}}, \sum_{m<m'\le k}\frac{(y-m)_{m'-m-1}}{(n-m)_{m'-m-1}} \\ &\qquad \cdot\frac{1}{n-m'+1}\sum_{y+1\le z\le n} V_{m',k}(z)\Bigg], \\ &\qquad 1 \le m \le k;\ V_{k,k}(y) \equiv 1, \end{aligned} \tag{1}$$

where $(j)_i$, with $1 \le i \le j$, means $j(j-1)\cdots(j-i+1)$.

For $n = 4$, as in Cayley's example, we find by direct computation without using the above Optimality Equation (1) that the probability of winning under an optimal stopping strategy is

$$W_k^{(4)} \equiv \frac{1}{4}\sum_{y=1}^{4} V_{1,k}(y) = 1, \frac{5}{6}, \frac{5}{6}, 1, \quad \text{for } k = 1, 2, 3, 4.$$

Going to the limit where $k, n \to \infty$, but with k/n tending to a fixed $\alpha \in (0, 1)$, dilutes the difference between sampling with and without replacement and therefore we conjecture that $W_k^{(n)}$ tends to the Gilbert-Mosteller constant 0.58106,

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independently of $\alpha \in (0, 1)$. When both the recall of past draws and a cost $c > 0$ per draw are considered, and the payoff for stopping after observing X_m is taken as $\max(X_1, \cdots, X_m) - mc$, the problem in this setting was treated by Chen and Starr (1980).

THE SECOND-BEST OBJECT IS DIFFICULT TO CATCH

The sixth feature possessed by CSP as listed by Ferguson can be modified in various ways. I will mention one possible modification. Suppose that a decision-maker facing CSP is a thoughtful and cautious person who prefers the second-best applicant to the best one for some appropriate reason of his or her own and will be satisfied with nothing but the second best. Payoff is unity if he chooses the second best of the n applicants and zero otherwise.

Rose (1982) derived the optimal strategy for this problem. It is: pass over the first $[(n+1)/2]$ applicants and choose the earliest 2-candidate (i.e., relatively second-best applicant among those which have already been interviewed) that appears thereafter. The expected payoff obtainable by adopting this strategy is ¼ for large n. Since $\frac{1}{4} < e^{-1}$, we observe that choosing the second best is more difficult than choosing the best. That is, to separate {the 2nd best} from {the best, 3rd best, $\cdots$, nth best} is more difficult than to separate {the best} from {the 2nd best, $\cdots$, nth best}. I would like to know what the resolution is in this situation by the Stewart–Ferguson approach to the "2nd best" choice problem along the line of the arguments made in Section 8 of the Ferguson paper.

THE MINIMIZING PLAYER IN GOOGOL

An intermediate problem of partial information lying between the cases of full information and no information occurs when observations are taken from a distribution of known form but containing one or more unknown parameters. A natural approach is to use a Bayes learning scheme to update knowledge about these parameters at the same time as deciding whether to stop or continue drawing the observations.

Professor Ferguson gives in Section 8 of the paper a resolution to solving the game of googol viewed as a zero-sum two-person game played by "someone" and "you" as described in Section 6. However the game itself can be variously interpreted according to someone's behavior of choosing the numbers on his "slips of paper." The game in the form as presented assumes that "someone" is an expert in mathematics and knows all kinds of non-negative real numbers including irrational and transcendental numbers such as $1 + \sqrt{2}$, $\pi/4$, $2e^{-1}$, log 100, etc. It may be just as natural to assume that "someone" is an outsider from mathematics and so non-negative integers are the only numbers he is familiar with. In such a case if "you" are a mathematical statistician, you will use a Bayes learning scheme based on the negative binomial/beta.

More precisely: Let $X_1, X_2, \cdots, X_n$ denote the values of the numbers on the slips. The X_j are chosen iid from a negative binomial distribution over the nonnegative integers $\{0, 1, 2, \cdots\}$, with the probability function

$$(2) \qquad f(x \mid k, \theta) \equiv \binom{x+k-1}{x}(1-\theta)^k \theta^x, \qquad x = 0, 1, 2, \cdots; k > 0, 0 < \theta < 1.$$

Assume that k is known but θ is unknown, and θ is chosen by "someone" from the beta (p, q) distribution over the unit interval $0 < \theta < 1$. Beta is a conjugate family of distributions for the negative binomial distributions, and the posterior distribution of θ given $X_1, \cdots, X_m$ is also beta $(p + X_1 + \cdots + X_m, q + mk)$.

Of course if "someone" is a very economy-minded person the underlying distribution may rather be taken as geometric, i.e.,

$$f(x \mid \theta) \equiv (1-\theta)\theta^x, \quad x = 0, 1, 2, \cdots; 0 < \theta < 1,$$

that is, $k = 1$ in (2). Also, in this case, the usual Poisson/Gamma learning scheme is possible as well, where

$$(3) \quad f(x \mid \mu) \equiv e^{-\mu}\mu^x / x!, \quad x = 0, 1, 2, \cdots; \mu > 0,$$

and μ is chosen by "someone" according to the gamma distribution. I would like to know whether "non-informative priors" exist for the partial information secretary problems with the underlying distributions (2) or (3).

ADDITIONAL REFERENCES

CHEN, W.-C. and STARR, N. (1980). Optimal stopping in an urn. *Ann. Probab.* **8** 451–464.

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