

# THE DEVELOPMENT OF MULTISSET THEORY

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## ABSTRACT

Multisets (sets with repeated elements) are of interest in mathematics, physics, philosophy, logic, linguistics and computer science. The development of multiset theory is surveyed from its earliest beginnings to its most recent applications in mathematics, logic and computational mathematics.

## INTRODUCTORY REMARKS

A multiset (or multiple membership set) is a collection of objects (called elements) in which elements are allowed to repeat. The multiset

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$[a,a,a,b,c,c]$  is said to contain the element  $a$  three times, the element  $b$  once, and the element  $c$  twice. The order of elements in a multiset is ignored, so that the multiset above is the same as  $[c,b,a,a,c,a]$  and  $[b,a,c,a,c,a]$ . We denote this multiset by  $[a,b,c]_{3,1,2}$  (although  $[c,a,b]_{2,3,1}$  et cetera are equally correct). Some authors denote this multiset by  $\{a^3, b^1, c^2\}$ ,  $\{3 \cdot a, 1 \cdot b, 2 \cdot c\}$  or  $[(3)a, (1)b, (2)c]$ . The number of times an element occurs in a multiset is called its multiplicity. The cardinality of a multiset is the sum of the multiplicities of its elements. Thus  $[a,b,c]_{3,1,2}$  has cardinality "six" - it contains a total of 6 elements. The multiset  $[a,b,c]_{3,1,2}$  is said to contain 3 distinct elements (namely,  $a$ ,  $b$  and  $c$ ), and repeated elements are sometimes called indistinguishables.

D. Knuth has observed, "Although multisets appear frequently in mathematics, they often must be treated rather clumsily because there is currently [1981] no standard way to treat sets with repeated elements. Several mathematicians have voiced their belief that the lack of adequate terminology and notation for this common concept has been a definite handicap to the development of mathematics." ([28], p. 636). Expanding on Knuth's remark, Meyer and McRobbie have suggested that the lack of adequate terminology and notation for multisets has also been a definite handicap to the development of logic and philosophy ([34], p. 107).

The word "multiset" was first suggested by N.G. de Bruijn in private correspondence with Knuth ([28], p. 636, [23], p. 170 and [21], p. 541) and is now the accepted name for this concept, replacing "bag", "heap", "bunch", "sample", "occurrence set", "weighted set" and "fireset" - finitely repeated element set. N.G. de Bruijn's interest in

multisets grew out of his investigations into the combinatorial properties of the set of divisors of a number – a number or any of its divisors is expressible as a multiset of prime factors (see, for example, [1], p. 27 and [28], p. 464).

The title of this paper uses the phrase "the development of" to imply a survey of the literature on multisets which extends through time and across disciplines. This account is certainly not a history of the subject, but it is more than simply an annotated bibliography. Annotated bibliographies are, for the most part, dull and boring, and it is most certainly too soon to write a comprehensive history of multisets. However, the already large amount of material on multisets scattered throughout the literature suggests that the time is right to consolidate the subject (as much as that is possible) into a single survey that permits access to the literature from a single source.

The development of multiset theory is, in fact, one small part of the remarkable proliferation of non-classical set theories over the last thirty years. These theories include Boolean- and Heyting-valued set theory, Zadeh's fuzzy set theory, Vopěnka's alternative (or semi-) set theory, Church's set theory with a universal set, intuitionistic and constructive set theory, quantum set theory, da Costa's paraconsistent set theory, and most recently, Pawlak's rough set theory, Tarski's set theory without variables, and Aczel's non-wellfounded set theory (or, what Barwise calls hyperset theory). In all cases (including multiset theory), either for purely aesthetic reasons or for hard-nosed practical application, classical set theory was found to be inadequate in some particular way. None of these theories realistically aspires to replace classical set theory, but only to supplement it or generalize it in some

respect. Classical Cantorian set theory as developed over the last one hundred years, and as formalized in its most popular form in Zermelo–Fraenkel set theory, is still our best candidate for a secure foundation for mathematics. Category theory is now a strong alternative candidate. It may very well come to pass that 'epsilons' are replaced by 'arrows', but that time is not yet here.

All of this recent activity in non-classical set theories may point the way to some very primitive theory of structure (a pre-set theory) from which all set theories, both classical and non-classical, can be derived. We now discuss one small part of this activity, the development of multiset theory. The division of this subject into "mathematics and logic" and "computational mathematics" is somewhat arbitrary and artificial, and will no doubt offend some readers.

### MATHEMATICS AND LOGIC

In his Grundlagen of 1883, Cantor defines a set as "... any multiplicity which can be thought of as one ... any totality of definite elements which can be bound up into a whole by means of a law." (Jourdain's translation, [11], p. 54). In 1895, the first sentence of his Beiträge further refines the definition of set to "... any collection into a whole  $M$  of definite and separate objects  $m$  of our intuition or our thought. These objects are called the 'elements' of  $M$ ." (Jourdain's translation, [11], p. 85). The last definition is considered the standard definition of a classical (Cantorian) set. The phrase "... definite and separate objects ..." is a clear restriction on

possible elements of  $M$ . Over the years, mathematicians and philosophers have quibbled about the exact meaning and translation of this phrase: for example, "... definite, distinguishable objects ..." (Wilder [50], p. 55), "... definite, well-distinguished objects ..." (Kamke [27], p. 1) and "... definite, distinct objects ..." (Fraenkel [19], p. 9). One unavoidable consequence of Cantor's definition is that an element may not occur more than once in a classical set. Whether one agrees or disagrees with the reasonableness of excluding repeated elements (see, for example, Singh [43], pp. 74-75 who disagrees), it is nevertheless a consequence of Cantor's definition. More precisely, the classical Cantorian notion of a set simply does not take account of repeated elements. Thus Kamke states, "... the same element shall not be allowed to appear more than once. The number complex 1, 2, 1, 2, 3, consequently, becomes a set only after deleting the repeated elements." ([27], p. 1). Fraenkel concludes that "... an object may belong, or not belong, to a set but cannot 'more than belong', for instance belong repeatedly ..." ([19], p. 10). Multisets are, therefore, non-Cantorian 'sets'. The classical axiom of extensionality forbids multiple occurrences of elements: the multisets  $[a,b]_{2,3}$  and  $[a,b]_{10,7}$  are forced equal to the set  $\{a,b\}$  since every element of one is also an element of the other two. Repetitions are simply ignored.

Despite official banishment from the kingdom (alias Cantor's paradise), multiset-like structures have arisen quite naturally in mathematics (and many other disciplines) since ancient times. There is a long tradition in mathematics of treating numbers as collections of units. As Hallett observes, "The most primitive idea of assessing the size of a collection of objects is to count through them replacing each

object by a tally mark ... for example a vertical stroke | ... There seems no doubt that this tally mark conception is actually the origin of our notion of number ... it is directly reflected in the ancient Babylonian (c. 2000 BC) and Egyptian (3500-1700 BC) symbolism for natural numbers ... And one finds much the same with the Greeks." ([24], p. 132). Thus, the number "seven", for example, has come to be identified with a collection of tally marks like |||||. (For a complete and detailed history of numbers, see Ifrah [26].) Hence, the very origin of numbers involves the use of collections of repeated units, or multisets of "ones".

There have been philosophical objections to such notions of number (most notably Leibniz, Frege and Wittgenstein) which can best be summarized as follows: the number (or plurality) of objects arises from their diversity (or their differences). Thus, without diversity, there is no plurality. In spite of these objections, such notions of number have a long tradition in mathematics and continue to this day. For example, the equation  $x^2 - 2x + 1$  is said to have two factors  $((x-1)$  and  $(x-1))$  and two roots (1 and 1).

Cantor himself makes use of the ancient notion of number in his first (1895) definition of the cardinality of a set. He defined the cardinality  $\bar{M}$  of a set  $M$  to be the result of replacing each element  $m$  in  $M$  by a "unit" ([11], p. 86). Thus  $\bar{M}$  is a collection of "units" - a record or tally of the exact number of elements in  $M$  (one "unit" for each element). Cantor insisted that collections like  $\bar{M}$  are sets themselves even though the repetition of "units" contradicts his restriction (in the same 1895 paper) on elementhood - repeated "units"

are certainly not distinct. For a detailed discussion of Cantor's "strange theory of ones", see Hallett [24], Sections 3.2 and 3.3, pp. 128-142.

In the final remark of his 1888 treatise [16] on number, Dedekind introduces the notion of a multiset. He observes that an image point in the range of a function can be said to occur with a multiplicity equal to the number of pre-images in the domain of the function that are mapped to it. He states, "In this way we reach the notion, very useful in many cases, of systems [sets] in which every element is endowed with a certain frequency-number which indicates how often it is to be reckoned as element of the system." ([16], p. 114). Although Dedekind does not explore this notion further, he does state that such deviations from the original meaning of a technical term (in this case, the number of elements in a system) occur frequently in mathematics.

In his lengthy introduction to the works of Cantor, Jourdain discusses Weierstrass's definition of real numbers - "With Weierstrass, a number was said to be "determined" if we know of what elements it is composed and how many times each element occurs in it. Considering numbers formed with the principal unit and an infinity of its aliquot parts, Weierstrass called any aggregate whose elements and the number (finite) of times each element occurs in it are known a (determined) "number quantity" (Zahlengrösse). An aggregate consisting of a finite number of elements was regarded as equal to the sum of its elements, and two aggregates of a finite number of elements were regarded as equal when the respective sums of their elements are equal." ([11], p. 18). Jourdain notes that the number of distinct elements in a "numerical quantity" need not be finite. Then, as Hallett observes, "...

Weierstrass regarded real numbers as certain collections of rational numbers in which finitely many repetitions are allowed. (One of these collections defines a real number if the sum of any finite number of its elements is less than some fixed rational bound ..." ([24], p. 134). Therefore, Weierstrass defined real numbers as certain multisets of rational numbers, thereby avoiding what both Jourdain and Cantor called the "logical error" of taking limits ([11], pp. 16-18).

According to Hailperin ([23], p. 170), the first recognition that multisets admit of mathematical treatment occurs in Whitney [49]. Whitney develops an algebra of characteristic functions of sets. He argues that the algebra of characteristic functions is preferable to (and more natural than) the algebra of sets because it more closely resembles the ordinary arithmetic of numbers. He then investigates "generalized sets" ("sets" whose characteristic functions may take any integer value - positive, negative or zero) which "... are useful in various mathematical theories." ([49], p. 405). He cites "chains in analysis situs" as one example in which "each element is counted any number of times." ([49], p. 412). Whitney investigates a variety of normal forms of such characteristic functions and establishes criteria which determine whether such functions represent "real" sets or "generalized" sets.

In Appendix B (entitled Ars Combinatoria) of his classic [48], Weyl explores the notion of multiple membership. His starting point is an "aggregate" containing red, white and green balls - "Generally speaking, in a given aggregate there may occur several individuals, or elements, of the same kind (e.g. several white balls) or, as we shall also say, the same entity (e.g. the entity white ball) may occur in several

copies. One has to distinguish between quale and quid, between equal (= of the same kind) and identical." ([48], p. 238). Weyl expresses this "equality of kind" using equivalence relations. An aggregate is defined as a set of elements and an equivalence relation on that set. Equivalent elements are said to be "in the same state". If an aggregate  $S$  contains  $n$  distinct elements, each of which may be in one of  $k$  distinct states, then an individual state of  $S$  is given if for each element of  $S$  it is known which state that element is in. An effective state of  $S$  is given if for each state of  $S$  it is known how many elements of  $S$  are in that state. Thus "... no artificial differences between elements are introduced by their labels ... and merely the intrinsic differences of state are made use of ..." ([48], p. 239). This is precisely the concept of multiset. In Weyl's words, "Balls may be white, red, or green; electrons may be in this or that position; animals in a zoo may be mammals or fish or birds or reptiles; atoms in a molecule may be H, He, Li, ... atoms." ([48], p. 239). Any two individual states of  $S$  are connected with the same effective state of  $S$  if and only if one may be carried into the other by a permutation of the labels. Weyl applies these concepts to a variety of sciences. In physics, for example, "Two individuals in the same 'complete state' [no further refinement is possible] are indiscernible by any intrinsic characters - although they may not be the same thing." ([48], p. 245).

A similar approach to multisets is taken in the first part of Monro [35]. Monro contends that "... the intuitive concept of multiset in fact contains two underlying ideas, and that these ideas should be separated. One of the resulting concepts is more set-like than the other, and the name 'multiset' has been appropriated for this concept;

the other concept is more numeric in character and has been named 'multinumber'." ([35], p. 171).

Like Weyl [48], Monro defines a multiset to be a set with an equivalence relation, where elements in the same equivalence class are said to be elements of the same sort. Defining a morphism between multisets as a function which respects sorts, Monro investigates the category Mul of multisets and multiset morphisms. Concepts like submultiset, union, intersection, complementation, and powerset arise naturally out of general category theory. Mul is then compared to Set, the category of ordinary sets.

However, Monro admits that the equivalence relation approach to multisets "... may perhaps be seen as not doing justice to the intuitive idea of multiset" (since objects in the underlying set are all distinct) ([35], p. 171). He introduces a second view "... arguably closer to the intuitive conception of multiset." ([35], p. 175). A multinumber is defined as a function from a collection of elements to the natural numbers (the multiplicity of element  $x$  in  $f$  is  $f(x)$ ). The partially-ordered algebra of all such multinumbers inherits much of its structure from the natural numbers. Monro argues that the concepts "multiset" and "multinumber" are indiscriminately mixed in the literature and should be distinguished. He finds that although the concept of "multiset" is more fundamental than that of "multinumber", the latter may be the more useful of the two ([35], pp. 171, 176).

Rado [38] uses multisets as a device to investigate the properties of families of sets. He observes, "The notion of a set takes no account of multiple occurrence of any one of its members, and yet it is just this kind of information which is frequently of importance. We need

only think of the set of roots of a polynomial  $f(x)$  or the spectrum of a linear operator." ([38], p. 135). Working within a set theory that admits classes, Rado defines a multiset to be any cardinal-valued function whose non-trivial domain (the collection of elements not mapped to zero) is a set. The class of all multisets is called the cardinal module since it "... possesses a rich structure which most resembles that of a module over the semigroup of all cardinals" ([38], p. 135). A multiset  $f$  represents a family of sets  $\alpha = (x_i)_{i \in I}$  just in case  $f(x_i)$  equals the number of times  $x_i$  occurs in  $\alpha$  (that is,  $f(x) = |\{i \in I \mid x_i = x\}|$  for all  $x$ ). Hence the multiset "... embodies everything concerning the structure of the family  $\alpha$  except for the notation for the indices of the members of the family." ([38], p. 137). The class of all sets is embedded into the cardinal module by identifying sets with their characteristic functions. Rado also makes use of signed multisets, functions which may take negative cardinal values ([38], Section 2.3, pp. 139-140).

Lake [30] was inspired by a lecture entitled "Multisets and multiscardinals" given by Rado to the London Mathematical Society on October 17, 1974. In that lecture, Rado mentioned that there was no axiomatization for multisets. Lake proposes a system of axioms similar to von Neumann's 1925 axiomatization of set theory in which the notion of function (rather than set membership) is taken as primitive. For a variety of reasons, von Neumann's system was eclipsed by the Zermelo-Fraenkel membership-primitive axioms for classical sets. Lake admits "... it might be thought desirable to have an axiomatization which does not go via functions. Such an axiomatization ... could be conveniently written out using  $x \in_z y$  (formally a three-place

predicate) to stand for 'x belongs to y precisely z times'." ([30], p. 325). This is the approach taken in Blizard [5].

In attempting to 'make sense' of Boole's algebra of logic, Hailperin [23] finds that "... correct interpretations or models are obtained if we consider, not classes, but multisets as the entities over which the variables range." ([23], pp. 3, 136). He adds "We are, to be sure, not attributing the idea of a calculus of multisets to Boole, only using it to explain his partially interpreted system." ([23], p. 170). He defines a multiset as "... a collection in which more than one example of an object can occur (indistinguishable balls of various kinds in an urn, roots of an equation with multiplicities counted, etc.)." ([23], pp. 3-4, 136). In the first edition (1976) of [23], Hailperin used the word "heap" for this concept, not realizing that "multiset" was the currently accepted name for this notion ([23], p. 170). In order to interpret Boole's unrestricted subtraction, Hailperin introduces signed multisets - multisets in which elements may occur a negative number of times ([23], pp. 4, 139). He notes, "While the notion of a signed multiset is not as intuitively simple as that of an unsigned multiset, a brief reflection on the history of the difficulties which were experienced until negative numbers were in good standing, should help one overcome resistance to the acceptance of signed multisets as a meaningful notion." ([23], p. 139). The interpretation of Boole's system using multiset algebras is discussed in detail in Chapter 2 (pp. 235-172) of [23].

In their [34], Meyer and McRobbie find that "... multisets have the right degree of abstraction needed for a number of logical purposes and in particular the right degree of abstraction needed in the study of

relevant implication." ([34], p. 107). In some theories of relevant implication, it matters how often a premiss is repeated in the course of an argument. Hence, Meyer and McRobbie make use of multisets — collections of premisses — and develop their own algebra of multisets for this purpose. We note, in this regard, that in multiple-conclusion logic, arrays of formulae rather than sets of formulae are sometimes needed, in order to take account of the multiplicity of premisses and conclusions in rules of inference (Shoesmith and Smiley [42], pp. 66–69, 113–114, 164, 211, 224).

In passing from a set to its cardinal number, Cantor invoked two levels of abstraction: first, ignore the order of elements, and second, ignore the nature of elements. Similarly, Meyer and McRobbie note that in passing from a sequence to a set, one first ignores the order of elements, obtaining a multiset. One then ignores the repetition of elements, obtaining a set ([34], diagram p. 125). Thus, "The level of abstraction that leads to multisets is halfway between that which leads to sequences, which give rise to ordinal numbers, and that which leads to sets, which give rise to cardinal numbers." They conclude, "It is surprising (to us) that multisets have not already attracted more logical, mathematical and philosophical interest." ([34], p. 125).

Commenting on the use of undefined multiset operations in Dershowitz and Manna [17], Hickman suggests that "... writers who wish to use multisets might present the reader with some fundamental definitions, or at least warn him that any analogy between multiset operations and set operations cannot be pushed too far." ([25], p. 212). He then demonstrates that there are significant differences between sets and multisets. For multisets, the classical axiom of extensionality

fails and the Schröder-Bernstein Theorem and the Cantor Powerset Theorem do not hold in general. Hickman also shows that the relative complement operation on multisets is not 'well-behaved' and that under certain conditions, de Morgan's laws fail. He concludes, "... if multisets are to play a role in mathematics, then their properties should not be taken lightly." ([25], p. 216).

In his 1925 paper "The Foundations of Mathematics", Ramsey [39] criticizes Russell and Whitehead's Principia Mathematica for defining 'identity' in such a way as to make indistinguishable things identical. In Principia, 'identity' means numerical identity; that is, identity in the sense of counting as one, not as two. Therefore, two things are identical if they have all their elementary properties in common ([39], p. 181). This definition, according to Ramsey, "... makes it self-contradictory for two things to have all their elementary properties in common. Yet this is really perfectly possible ... Hence, since this is logically possible, it is essential to have a symbolism which allows us to consider this possibility and does not exclude it by definition." ([39], p. 182). Ramsey concludes "... the treatment of identity in Principia Mathematica is a misrepresentation of mathematics ..." ([39], p. 183).

In this regard, we note an observation by Fraenkel, Bar-Hillel and Levy - "... the direct way to say, in the language of set theory ..., that a set  $z$  has exactly one member is to say that there is a member  $x$  of  $z$  such that every member  $y$  of  $z$  is equal to  $x$ ; if equality is not intended to be necessarily identity, then such a set  $z$  can contain two or more members equal to one another." ([20], p. 30).

With reference to Ramsey's remarks, Parker-Rhodes observes that "... no one has made any systematic attempt to open up the territory which lies behind these ideas ..." and "... there exists no branch of mathematics, in which a third parity-relation, besides equality and inequality, is admitted ..." ([36], p. xiii). In [36], Parker-Rhodes demonstrates that such a mathematical system is feasible and has useful applications. Indistinguishables behave as identicals when elements of different classes, but they behave as a plurality (they each contribute to cardinality) when they are elements of the same class ([36], p. 7). By such a definition, repeated elements in a multiset are indistinguishables. Parker-Rhodes develops a theory of sorts (collections of indistinguishables) that resembles multiset theory in some respects. However, sort theory (but not multiset theory) is a radical departure from classical mathematics because of its triparitous nature - objects may be identical, distinct or twins. The logical notation of sort theory is explained ([36], Chapter II, pp. 18-38) and the basic axioms and definitions of sort theory are introduced ([36], Chapter IV, pp. 56-74). The theory is then applied to fundamental problems in physics in Part II of [36]. In private correspondence, Parker-Rhodes indicated that he felt there was a one-to-one correspondence between sorts and multisets with the same structure, and that there was "a greater degree of constructiveness" on the multiset side. However, he also felt that it is not possible to develop multiset theory in a strictly "biparitous" mathematics.

In his treatment of the infinite arithmetic of well-ordered cardinals, Levy [31] introduces multisets as a conceptual aid. He observes that when one applies a mathematical operation (like addition,

multiplication, ...) to a collection of objects, one requires a collection in which elements are allowed to occur more than once ([31], pp. 100-101). He defines multiple-membership sets as functions whose domains are sets and whose values are non-zero cardinals. However, he notes "In spite of the theoretical advantage of dealing with multiple-membership sets we shall only keep them in mind and not deal with them directly." ([31], p. 101). Levy chooses to formally represent multisets using indexed families: "... whenever we speak about an indexed family we have in mind the corresponding multiple-membership set ..." ([31], p. 101). This approach is used to introduce addition, multiplication and exponentiation of cardinals ([31], pp. 102-111).

Corcoran [15] introduces multisets in an application of his main result on categoricity: any atom-complete set of sentences which includes induction is categorical (all models of the set of sentences are isomorphic) ([15], pp. 195 (footnote 9), 199-201). Let  $u$  be the set of all multisets with elements belonging to  $\{a,b\}$  (the set of all functions from  $\{a,b\}$  into  $\mathbb{N}$ ). The empty multiset is denoted by  $0$ . Two "successor functions"  $S_a$  and  $S_b$  are defined on  $u$  such that  $S_a$  increases the multiplicity of element  $a$  by 1 and  $S_b$  increases the multiplicity of element  $b$  by 1. For example,  $S_a[a,b]_{2,1} = S_b[a]_3 = [a,b]_{3,1}$ . Corcoran then shows that a simple set of axioms (the universal closures of  $S_a x \neq 0 \wedge S_b x \neq 0$ ,  $(S_a x = S_a y \rightarrow x = y) \wedge (S_b x = S_b y \rightarrow x = y)$ ,  $S_a S_b x = S_b S_a x$  and  $S_a x \neq S_b x$ , together with an induction formula) implies an atom-complete set of sentences including induction, which is categorical by the main theorem ([15], p. 201). For Corcoran's definitions of "induction" and "atom-completeness", see [15] pp. 195, 198.

Anderson [1] investigates properties of collections of subsets of a finite set and generalizations of these properties to collections of multisubsets of a finite multiset. This area of combinatorics is sometimes called "extremal set theory" ([47], p. 516). Of principal interest are the sizes, numbers and properties of chains (collections of pairwise  $\subseteq$  comparable subsets) and antichains (collections of pairwise  $\not\subseteq$  incomparable subsets) in the subset (or multisubset) lattice. For example, the unique maximum-sized antichain of subsets of an  $n$ -element set (where  $n$  is even) consists of all subsets of size  $n/2$ . Subsets of  $\{x_1, x_2, \dots, x_n\}$  can be identified with divisors of a square-free number  $m = p_1 p_2 \cdots p_n$  (a product of distinct primes). This correspondence between a subset lattice and the set of divisors of a number, when extended to arbitrary numbers  $m = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$ , leads naturally to the corresponding multiset lattice ([1], p. 18). One then investigates properties of chains and antichains of divisors (ordered by  $|$ ) of arbitrary numbers. Most notable in this regard is a series of papers by Clements (most recently [13] and [14], described fully in [1], Chapters 4, 9, 10) in which the multiset lattice of divisors of a number is generalized to that of an abstract multiset. Clements takes collections of billiard balls (in which there are  $k_i$  balls of colour  $i$ ) as his conceptual prototype of a multiset ([13], p. 153) similar to Weyl's [48] "aggregates" of red, white and green balls.

Also in combinatorics, we should take note of the papers by Bender ([2] and [3]) in which partitions of multisets are investigated and characterized. A partition of a multiset  $M$  is a collection of multisets (called blocks) whose multiset union (union with repetitions

counted) is  $M$ . There are four ways to count the number of partitions into a certain number of blocks (with or without repeated blocks and with or without repeated elements within blocks). For this purpose, multisets are classified into types: a multiset  $M$  is of type  $m = \langle m_1, m_2, \dots \rangle$  if exactly  $m_i$  distinct elements of  $M$  appear exactly  $i$  times in  $M$ . For example,  $[a, b, c, c]$  is of type  $\langle 2, 1 \rangle$ . As Bender observes, "A multiset of type  $m$  can be thought of as a set  $S$  of cardinality  $\sum m_i$  and a map  $u$  from  $S$  to the positive integers such that  $|u^{-1}(i)| = m_i$ ." ([2], p. 301).

Blizard [5] defines a first-order theory MST for multisets (in which elements belong any finite number of times) where the atomic formula  $x \epsilon^n y$  represents the multiple membership ( $n$  times) of element  $x$  in multiset  $y$ . The theory MST is a generalization of classical ZFC set theory: MST contains an exact copy ZFC' of ZFC; MST is a conservative extension of ZFC'; and MST is consistent relative to ZFC. Similar Zermelo-Fraenkel-like theories have been defined for multisets with negative-integer multiplicities in [7], with non-negative real multiplicities restricted to  $[0, 1] \subseteq \mathbb{R}$  in [6], and with arbitrary cardinal-number multiplicities in [8]. All of these theories contain an exact copy of ZFC and are relatively consistent <sup>1</sup>.

There are two possible approaches to the formalization of multiset theory. The first is to develop first-order theories using the classical predicate calculus (with or without equality) as in [5] and [30]. The second is to tamper with the underlying logic. Two approaches of the second kind are now surveyed.

Attempts to rescue Frege's set theory (set theory with extensionality and unrestricted comprehension using first-order

predicate calculus) from inconsistency have either imposed restrictions on the comprehension axiom, or weakened the underlying logic without restricting comprehension. The second approach based on logics developed by Curry, Meredith, Abbott, Thiele, et cetera, have led to weakened first-order logic called BCK-linear logic (see [9] for details). Using this weaker logic, Bunder [9] develops an elementary theory of multisets which is shown to be consistent. By representing population samples as multisets, possible applications of the theory to statistics are suggested. Bunder's set theory deals with finite multisets in which elements occur finitely often. The consistency proof relies inherently on this finiteness.

In Skolem [44] (Chapter 18), the possibility of set theory based on many-valued logic is discussed briefly. He states, "... it seems to be possible to obtain a consistent set theory with an unrestricted axiom of comprehension if all rational numbers  $\geq 0$  and  $\leq 1$  are allowed as truth values." ([44], p. 69). Skolem was able to prove that a rudimentary set theory, where the axiom of comprehension

$$\exists y \forall x (x \in y \leftrightarrow \Phi(x))$$

is restricted to  $\Phi(x)$  built up from atomic membership formulae and logical connectives  $\wedge$ ,  $\vee$  and  $\neg$  only (no quantifiers), is consistent ([44], p. 69). He concludes, "... research concerning set theories based on many-valued logic must be continued before we can say whether it is really promising or not." ([44], p. 70). It seems reasonable to expect that a set theory based on  $\omega$ -valued logic (in which the atomic formula  $x \in y$  can have a truth value  $n \in \omega$ ) can be

shown to be equivalent to multiset theory [5] based on classical logic (in which the atomic formula  $x \in^n y$  can have only two truth values).

### COMPUTATIONAL MATHEMATICS

The most frequently cited discussion of multisets has been Knuth [28]. It would appear that most individuals in computational mathematics first learned about multisets from Knuth's book. Knuth introduces multisets into algorithms that compute values of  $x^n$  where  $x$  is a real quantity and  $n$  is a large positive integer. His construction requires that collections respect the repetition of elements ([28], pp. 441-466). He develops a simple algebra of multisets, and introduces the representation of positive integers as finite multisets of prime factors. He also develops the natural correspondence between a monic polynomial over the complex numbers and the unique multiset containing its roots. He shows that generating functions with nonnegative integer coefficients correspond one-to-one with multisets of nonnegative integers ([28], pp. 464, 636-367). He adds, "Other common applications of multisets are zeros and poles of meromorphic functions, invariants of matrices in canonical form, invariants of finite Abelian groups, etc.; multisets can be useful in combinatorial counting arguments and in the development of measure theory. The terminal strings of a noncircular context-free grammar form a multiset that is a set if and only if the grammar is unambiguous." ([28], p. 636). Knuth's definition of a multiset has become the standard definition: a multiset is a mathematical entity that is like a

set, but it is allowed to contain repeated elements; an object may be an element of a multiset several times, and its multiplicity of occurrences is relevant ([28], p. 454).

On the other hand, very few people seem to be aware of the fact that multisets play a significant role in Knuth [29]. Here he explores permutations of multisets ([29], pp. 22-34) including a wealth of historical references on the subject. Multisets and permutations of multisets are then applied in a variety of search and sort procedures (see [29], p. 717 for page numbers).

We also note in passing the use of multisets in Reingold, Nievergelt and Deo [40] for their discussion of combinatorial algorithms (see [40] Section 2.4, "Sets and Multisets", p. 57).

In his [18], Eilenberg's objective is to give the theory of automata and formal languages a coherent mathematical presentation. A novel feature of [18] is its treatment of multiplicity - "When one has a set  $A$  defined by some explicit device (e.g., automaton, machine, grammar, or system of equations), the verification that an element  $a$  is in  $A$  involves a procedure which may be called a 'computation'. When for a given  $a$  two (or more) such essentially different computations exist, it is natural to assign to  $a$  a 'multiplicity' which is the number of ways by which the given algorithm leads to the conclusion ' $a \in A$ '." ([18], p. xv). In Chapter VI of [18], a general theory of multiplicity is developed and applied to automata (numerous other applications are given later in the book). The "behaviour" of an automaton is a collection of words, each of which has a "computation" (or "proof") associated with it. If the number of computations associated with a given word is  $n$ , then that word is said to belong to

the behaviour with multiplicity  $n$ . Therefore, the behaviour  $|\mathcal{A}|$  of an automaton  $\mathcal{A}$  is defined as a function from the collection of words  $\sum^*$  (a free monoid over a finite alphabet  $\Sigma$ ) to the natural numbers  $\mathbb{N}$  ([18], p. 120). The general algebraic device used for this purpose is called a K-subset (a multiset when  $K = \mathbb{N}$ ) which is defined as a function from a set  $X$  to a semiring  $K$ . Examples of semirings with addition and multiplication are: the trivial  $\{0,1\}$ , the non-negative integers (with or without the adjunct element  $\omega$ ), the non-negative rationals, and the non-negative reals (with or without the adjunct element  $\omega$ ) [18], pp. 123-124, 188. A K-subset  $A$  (a multiset of words over a finite alphabet) is recognizable if there exists an automaton  $\mathcal{A}$  such that  $|\mathcal{A}| = A$ . Extensions of these concepts to the cases  $K = \mathbb{Z}$  (the ring of all integers) and  $K = F$  (a field) are discussed as well ([18], pp. 158, 204, 207, 427).

A fuzzy set can be defined as a function from a classical set into the real unit interval  $(0, 1] \subseteq \mathbb{R}$ , whereas a multiset is often formulated as a function from a set into the positive natural numbers. Goguen [21] proposes a category-theoretic foundation for fuzzy set theory. As a by-product of this approach, he investigates the properties of "semiring sets" (functions from a set into a semiring as in Eilenberg [18]) with applications to the special case of multisets ([21], pp. 538-541). The usefulness of multisets in both combinatorics and the theory of formal languages is discussed ([21], pp. 541-543).

Manna and Waldinger [32] claim that mathematical logic plays the same fundamental role for computer science as does the calculus for physics and traditional engineering. After introducing the basics of

propositional and predicate logic, they present a series of first-order theories with induction for some of the most important structures in computer science: non-negative integers, strings, trees, lists, sets, multisets ('bags') and tuples. Chapter 11 of [32] (pp. 505-527) is dedicated entirely to a theory of multisets. The language employs a primitive binary function symbol  $\odot$  (the intended interpretation of the term  $u \odot x$  is "the multiset that results from the insertion of the atom  $u$  into the multiset  $x$ "). The induction principle ([32], p. 507) takes the form

$$[\rho(\phi) \wedge \forall u \forall x (\rho(x) \rightarrow \rho(u \odot x))] \rightarrow \forall x \rho(x)$$

for all sentences  $\rho$  of the language. The theory is limited to only finite collections of atoms. The different theories for sets, multisets and tuples are then combined into one theory by using a single unary predicate symbol atom and introducing three unary function symbols that map tuples into multisets, multisets into sets, and tuples into sets ([32], pp. 539-544).

Considerable use of multisets is also required in the formalization of Petri net theory. Net theory was introduced by C.A. Petri in 1962 to fill the need for a theory of asynchronous machine models. It is now a well established branch of theoretical computer science which models procedures, organizations and devices which involve regulated flows, in particular information flows. For example, in Peterson [37] the very definition of a Petri net requires the use of multisets ('bags') ([37], pp. 7-8) and an elementary theory of multisets is developed for this purpose ([37], pp. 237-240). Reisig [41] uses multisets to define

relation nets in which "... several individuals of some sort do not have to be distinguished" noting that "One should not be forced to distinguish individuals if one doesn't wish to. This would lead to overspecification." ([41], p. 126). As in Whitney [49], multisets are defined as generalized characteristic functions; that is, as integer-valued functions which allow for some elements to belong "negatively often." ([41], p. 126). Reisig also defines and makes use of multirelations - a multiset whose domain is the cartesian product of a set of sorts ([41], pp. 126-131).

Dershowitz and Manna [17] use multisets to prove the termination of certain programs. Given a well-founded set (a set ordered in such a way as to admit no infinite descending sequences of elements), one can induce a well-founded ordering onto the collection of all finite multisets whose elements belong to the set. The value of this construction is that "... the multiset ordering ... permits the use of relatively simple and intuitive termination functions in otherwise difficult termination proofs." ([17], p. 188). Multiset operations like union, relative complement and subset are used without definition ([17], p. 189).

Martin [33] defines a multiset on a set  $S$  as "... an unordered sequence of elements of  $S$ " ([33], p. 37). The primary concern of that paper is the construction and classification of various well-founded partial orderings of  $\mathcal{M}(S)$ , the set of all finite multisets on  $S$ . Martin is able to classify such orderings using the notion of a cone in  $\mathbb{R}^n$ . With this geometric interpretation, two new multiset orderings arise and interesting results about existing orderings are demonstrated.

Thistlewaite, McRobbie and Meyer [45] extend the results of Meyer and McRobbie [34] into the realm of automated theorem proving. The authors claim that [45] "... is the first sustained computational investigation of such [relevant] logics." Throughout [45] substantial use is made of multisets of formulas (within various Gentzen-style formulations of relevant logic). Most of the definitions involving multisets are taken from [34], although various alternative representations of multisets are needed ([45], pp. 113-115) for implementation using a theorem-proving program called KRIPKE.

Bundy [10] makes some use of multisets ('bags') in his quest for automated mathematicians. In particular, multisets of numbers are used to illustrate the definition- and conjecture-formation programs of Lenat ([10]; Chapter 13, pp. 225-240). The diagrammatic representation of the quartet - set, list, bag, and oset (an ordered list with no multiple elements) ([10], pp. 233-234) - is equivalent to that in Meyer and McRobbie - set, sequence, multiset and ordinal set ([34], p. 125).

In [22], Grzymala-Busse generalizes the notion of rough set (introduced in 1981 by Pawlak) to the concept of rough multiset (see [22], pp. 325-327 for details). Rough multisets are used to simplify the formalization of an information system (or data matrix, similar to a database) by using an information multisystem (which resembles a relational database) ([22], pp. 328-332).

Yager [51] develops an elementary algebra of multisets ('crisp bags') and suggests possible applications to relational databases. He then defines a rather curious object called a fuzzy bag: to each element  $x$  in a fuzzy bag  $A$  is associated a multiset containing elements  $\alpha$  (real numbers in the unit interval  $[0,1] \subseteq \mathbb{R}$ ) with

multiplicities  $n$  (non-negative integers). The number  $n$  indicates the number of times the element  $x$  appears with membership grade  $\alpha$  in the fuzzy bag  $A$  ([51], p. 33). Yager introduces an elementary algebra of fuzzy bags, and notes that classical sets are special cases of multisets, as are Zadeh fuzzy sets special cases of fuzzy bags ([51], p. 35). In [52], multisets are used in an attempt to define a meaningful notion of the cardinality of a fuzzy set.

Although the axiom system developed in Chapin [12] is intended to formalize fuzzy sets (introduced by Zadeh) and boolean-lattice-valued sets (developed by Brown) (see [12], pp. 619-623 for details), it is formulated in such a general way that it is also worth noting in relation to multisets. Chapin develops a formal theory of set-valued sets using the classical Zermelo-Fraenkel axioms as a guide. He states, "A theory having all of the objects involved of some uniform kind would seem profitable." ([12], p. 621). He, therefore, uses a single-sorted first-order language in which the intended interpretation of the atomic formula  $\epsilon(x,y,z)$  is "x is an element of y with degree of membership at least z" ([12], p. 624). Chapin does seem to have multisets in mind as a possible interpretation since he argues, "The necessity of both axioms, rather than just one or the other becomes apparent if the degrees are imagined to correspond to the natural numbers with their natural ordering." and "Again, if one imagines the degrees as natural numbers with their usual ordering, this becomes clear." ([12], pp. 627, 630). Unfortunately, Chapin's system does not succeed in its stated purpose - to formalize the naive set theories of Zadeh and Brown. For a detailed discussion of the problems associated with Chapin's proposed

axiom system and for a comparison with other systems see Blizard [4] pp. 173-176, Blizard [6] p. 95 and Weidner [46] pp. 61-62.

### CONCLUDING REMARKS

Given the variety of motivations for investigating multisets, it is surprising that the concept itself (collections of repeated elements) has remained so clearly intact. The basic idea that runs through all of the literature surveyed (with the possible exception of [36]) is that collections like  $[a,a,c,b,b,a]$  are either interesting in themselves, or useful in some specific circumstance. In most applications, it is preferable to work directly with such collections rather than with their formal equivalents (functions, permutations of sequences, collections of equivalence classes, families of sets, et cetera). Indeed, as Knuth points out, "... this formal equivalence is of little or no practical value for creative mathematical reasoning." ([28], p. 636).

It is probably fair to say that [2]-[9], [13]-[14], [25], [30], [35] and [49] approach multisets (or multiset-like objects) purely as mathematical objects (with little or no view to possible application), whereas [1], [10], [12], [15]-[18], [21]-[23], [28]-[29], [31]-[34], [36]-[38], [40]-[41], [45], [48] and [51]-[52] investigate multisets (bags, heaps, et cetera) with a very specific application in mind. In most applications, only finite multisets (which contain a finite number of distinct elements) with finite multiplicities (in which elements repeat a finite number of times) are needed. More theoretical approaches allow for an infinite number of distinct elements, while

others also permit infinite multiplicities of elements ([8], [18], [25], [30]–[31] and [38]). Certain investigations have dared to venture into the dark realm of negative multiplicities ([7], [18], [23], [38], [41] and [49]).

It should be emphasized that multisets and fuzzy sets are conceptually quite different: elements of multisets belong at least once, whereas elements of fuzzy sets belong at most once. However, multisets do arise in [12], [21], [46] and [51] as a side interest within fuzzy set theory (in [22], within rough set theory), whereas [6] and [30] make use of multiset theory to axiomatize fuzzy sets and [52] makes use of multisets to define the cardinality of fuzzy sets.

Multisets are thought of primarily as functions in [18], [21], [25], [30]–[31], the second part of [35], [38], [41] and [49]. An equivalence relation approach to multisets is taken in the first part of [35] and throughout [48]. Category theory is used throughout [35] and in parts of [18], [21] and [30]. Formal axiom systems for multisets are proposed in [5], [30] and [32], and somewhat indirectly in [9], [12], [21], [36] and [46]. The indistinguishability of repeated elements in a multiset-like structure is the main theme of [36], but it is also hinted at (the blurring of distinctions using permutations) in [23] p. 136 and [48] p. 240.

The overall flavour of [15], [23] and [34] is philosophical; of [5], [25] and [38] is set-theoretical; of [18], [21] and [35] is category-theoretical; of [1]–[3], [13]–[14] and [48] is combinatorial; and of [10], [17]–[18], [21]–[22], [28]–[29], [32]–[33], [37], [40]–[41], [45], [51] and [52] is computational. For the most part, multisets are formulated either against a classical set-theoretic

background, or within the classical first-order predicate calculus. The exceptions are [9] which uses BCK-linear logic, [36] which uses tripartitous logic, and [44] which suggests the possible use of many-valued logic. We have noted that multisets of formulae have proven useful in relevance logic ([34] and [45]) and multiple-conclusion logic ([42]). Applications of multiset-like structures in physics are given in [36] and [48].

#### NOTES

<sup>1</sup> My personal interest in multisets grew out of an earlier preoccupation with set annihilation. During the period 1970-72, I had read several popular accounts of matter-antimatter duality. Translating this phenomenon into set-theoretic language, I considered the situation in which to every classical set  $A^+$  there exists a unique anti-set  $A^-$  such that joining the two together (in some generalized union) results in the empty set  $\emptyset$ . It was not until 1982 that I hit upon the idea of raising the membership epsilon  $\epsilon$  to a power  $n$  (as in the atomic formula  $x \epsilon^n y$ ). I am told that Quine made use of a similar notation for a different purpose. The use of this new notation led naturally to multisets (where  $n \in \mathbb{N}$ ) and then to "shadow sets" (where  $n \in \mathbb{Z}$ ) with which set annihilation finds simple expression as  $x \bigoplus x^- = \emptyset$  in [7].

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