From Semantic Tableaux to Smullyan Trees: A History of the Development of the Falsifiability Tree Method

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Abstract

The tree method is one of several decision procedures available for classical propositional logic and first-order functional calculus and for nonclassical logics, including intuitionistic propositional logic and intuitionistic first-order logic, modal logics, and multiple-valued logics. The tree method provides exceptionally elegant proofs of the consistency and satisfiability of formulae. Falsifiability trees allow easy testing of the validity of proofs and are a canonization of proof by contradiction for natural deduction systems, while truth trees allow easy derivation of theorems in these systems. Tree proofs permit graphical-geometric representations of logical relations, and appear to be of greater intuitive accessibility than either the axiomatic method or the method of natural deduction. The proofs of the completeness and soundness of the tree method and its variants are also straightforward, and the method combines insights and results of model theory and proof theory in a fashion that clearly identifies the most basic concepts of proof involving such model-theoretic results as Craig's Interpolation Lemma, Beth's Definability Theorem, and Robinson's Consistency Theorem.

I trace the contemporary history of the tree method, from its its tentative origins in the Gentzen sequent-calculus and the method of natural deduction, through its evolution and comprehensive development as the Smullyan tree (1964-1968) from Beth tableaux (1955) and Hintikka's theory of model sets (1953-1955). Also considered is van Heijenoort's development of the falsifiability tree as a special case of the truth tree (1966-1974) and work in the 1970s on proving the soundness and completeness of the tree method.

AMS (MOS) 1980 subject classifications (1985 revision): primary: 03B35, 03F05, 03F07; secondary: 03-03, 01A60, 01A65, 03C95

§0. Introduction. In this work, we trace the development from Beth tableaux of the so-called *analytic tableaux* or *Smullyan trees* as a method of deduction for first-order quantification theory and as a test of the validity of proofs using natural deduction, and in particular of the recent evolution of truth trees and falsifiability trees for first-order quantification theory from Gentzen's natural deduction sequents (N-sequents) by Beth, Hintikka, Smullyan, and van Heijenoort. Deduction trees have been found in Hilbert and Bernays [1934] and in Kleene [1952]; Beth himself [1959] developed a version of the tree as a deduction scheme for his semantic tableaux. These trees were developed as partial orderings of natural deductions, to be distinguished from the linear or sequential deductions for the Gentzen sequent calculus. A *tree* is understood here, then, as a total ordering of natural deductions, and is derived as a one-sided Beth semantic tableau.

The standard history of the tree method as a deduction method for first-order classical quantification theory is the one given by Richard Jeffrey in the first edition of his book *Formal logic: its scope and limits* ([1967]). According to Jeffrey [1967, p. 227], "the tree method derives from Beth's method of *semantic tableaux* [1962, 1959] and equally from Hintikka's method of *model sets* [1955, 1955c]," that is, from Beth's *Formal methods* and *The foundations of mathematics*, and Hintikka's *Form and content in quantification theory* and *Notes on quantification theory.* Jeffrey [1967, p. ix] tells us that he borrowed from Smullyan the idea of one-sidedness that reduces tableaux to trees, and he hints that the roots of Beth's semantic tableaux and Hintikka's model sets (may) go back to Herbrand. The story given by J.A. Robinson in *Logic: form and function, the mechanization of deductive reasoning* [1979, p. 290] is essentially the same, although Robinson more carefully documents the roots of the work of Beth and Hintikka in Gentzen and Herbrand. In particular, the tree decomposition rules employed by Jeffrey in his *Formal logic* are precisely the inference rules which Gentzen introduced in his classical sequent-calculus **LK**. In fact, Jeffrey's rules are a subset of the rules of **LK**.

Gentzen [1934] showed that true sequents can always be given cut-free proofs, by providing an elaborate apparatus for converting any arbitrary proof with many cuts into a cut-free proof of the same sequent. (For an exposition of **LK** and a discussion of cut-elimination and its importance, see [Takeuti 1987].) Meanwhile, Herbrand, in his *Recherches sur la théorie de la démonstration* [1930], made the equivalent observation of the provability of any provable formula A by a proof that does not introduce any formulae that are not subformulae of the formula A (the subformula property). It is easy to show that **LK**$^-$ (i.e. **LK** without cut) is precisely the tree method. Indeed, a chain of Gentzen sequents would look very much like an inverted Smullyan tree, or like a Hintikka tree lying on its side. In fact, as Beth showed in a talk given at the International Congress of Mathematicians in Edinburgh in 1958 and published in [1960], his semantic tableaux, when closed, are equivalent to proofs in Gentzen's systems **LK** of classical sequents and **NK** of N-sequents. Moreover, Beth ([1960]) used this fact to prove the completeness of the semantic tableau method.

The remainder of the standard history states that a tree is a one-sided Beth tableau; and since the notion of one-sidedness is due to Smullyan, the tree is called a *Smullyan tree*. However, the first textbook to use the Smullyan tree is Jeffrey's *Formal logic*, essentially an undergraduate text, while the first graduate-level monograph is Smullyan's own *First-order logic*, which appeared in 1968. In the standard history, Smullyan's [1968] is (to borrow the phraseology of Robinson [1970, p. 290]), an investigation of the "nooks and crannies" of the semantic approach to cut-free proofs of sequents. This semantic approach was also independently and almost simultaneously developed by Beth and Hintikka. In this story, the real significance of Smullyan's book is that it gives the first unified and systematic exposition of semantic tableaux, which Smullyan called *analytic tableaux*.

The actual story of the development of the tree method is somewhat more complicated, but also much more interesting.

§1. Hintikka's model sets. There is disagreement as to who first created the tree method. Hintikka or Smullyan. Part of this disagreement hinges upon a question of the proper definition of the "tree method". What is agreed upon is that the Smullyan tree is an adaptation and unification of Hintikka's model set method and Beth's semantic tableaux. As Hintikka [1989] has noted, it is "completely arbitrary to drive any real wedge" between his and Beth's ideas on the one hand and Smullyan's on the other.

Hintikka's method of model sets intuitively defines a *model set* as a set of formulae that can be interpreted as a partial description of a model in which all formulae are true [Hintikka 1987, p. 11]. A proof of a formula F is a failed attempt to build a countermodel $\neg F$ to F. This work was carried out principally in Hintikka's two papers of 1955 ([1955] and [1955c]). The structure of the Hintikka tree. is based on the structure of Skolem's [1929] tree argument for creating models for the proof of the consistency of Hilbert's calculus. The Hintikka tree can serve the same purpose, applied to the theory D (of Hintikka sets) for natural deduction, as Skolem's tree argument serves for Hilbert's system.

If Δ is a theory in the first-order language L, then Δ is a *Hintikka set* if it satisfies all of the following properties:

1. $\perp$ is not in Δ
2. If φ is an atomic formula in Δ, then $\neg\varphi$ is not in Δ
3. If $\neg\neg\psi$ is in Δ, then ψ is in Δ
4. If $\psi \wedge \varphi$ is in Δ, then ψ and φ are both in Δ
5. If $\neg\ \psi \wedge \varphi)$ is in Δ, then either $\neg\psi$ is in Δ or $\neg\varphi$ is in Δ
6. If $\exists x\, \varphi$ is in Δ, then $\varphi[c\ /x]$ is in Δ for some individual constant *c*
7. If $\neg\ \exists x\, \varphi$ is in Δ, then $\neg\varphi[c\ /x]$ is in Δ for each individual constant *c*

If Δ is a Hintikka set, then an L-structure **D** constructed from Δ is a *model* of Δ. Formally, a *model set* is a model or set of sentences satisfying conditions (1) - (7). As Hintikka stated [1955, p. 26], "*model sets actually and literally constitute models in which all their members are true*, under a suitable interpretation". It turns out that every Hintikka set has a model. Extending language L to a language L^+ by adding infinitely many new individual constants to L, we can add new sentences $\mu_0, \mu_1, \ldots$ to Δ and obtain a new Hintikka set Δ^+, such that Δ^+ will have a model, which is also a model of Δ, since Δ is a proper subset of Δ^+. We may construct a hierarchy of such expansion models Δ_i of Δ. If Δ^+ is the union of the Δ_i within this hierarchy, then Δ^+ satisfies each of the properties (1) - (7), and is therefore a Hintikka set.

We construct the hierarchy of Δ_i according to the following rules, related to the properties of Hintikka sets listed above:

(3′) If μ_i is of the form $\neg\neg\psi$, then Δ_{i+1} is Δ_i together with ψ

(4′) If μ_i is of the form $\psi \wedge \varphi$, then Δ_{i+1} is Δ_i together with ψ and φ

(5′) If μ_i is of the form $\neg(\psi \wedge \varphi)$, then Δ_{i+1} is Δ_i together with at least one of $\neg\psi$ or $\neg\varphi$

(6′) If μ_i is of the form $\exists x\, \varphi$, then Δ_{i+1} is Δ_i together with $\varphi[c / x]$forsome individual constant *c* not already occurring in Δ_i

(7′) If μ_i is of the form $\neg\exists x\, \varphi$, then Δ_{i+1} is Δ_i together with $\neg\varphi[c / x]$ for the first individual constant *c* such that $\neg\varphi[c / x]$is not already in Δ_i

provided that, if μ_i is the i^{th} formula in the infinite list of formulae of L^+ and each μ_i occurs infinitely often in the list of formulae of L^+, then no opportunity for applying (3′) - (7′) may remain unused forever (for example, by requiring that μ_i is the oldest member of Δ_i to which a rule can be applied; see, e.g. [Hintikka 1955c, pp. 5-7] and [Hintikka 1989, p. 1]).

In the case of (5′), a branching occurs in the construction of the model for Δ_i, since it is not immediately certain without lengthy checking which of $\neg\psi$ or $\neg\varphi$ is consistent with Δ_i. The result is no longer a chain of theories, but a *Hintikka tree* which displays each of these possibilities. Those branches which fail to satisfy properties (1) and (2) in the definition of the Hintikka set may be trimmed off. König's tree lemma asserts that if a tree has a finite number of terms at its n^{th} node (for natural number *n*), then the tree has a branch containing each of the *n* nodes. It follows that the Hintikka tree has an infinite branch. If $\Delta_0, \Delta_1, \Delta_2, \ldots$ is an infinite branch and if Δ^+ is the set of sentences occurring in at least one of the Δ_i of that infinite branch, then Δ^+ satisfies all of properties (1) - (7) and is a Hintikka set. The *Henkin model* (named by [Leblanc & Wisdom 1972]; see [Leblanc 1982, p. 126f.] and [Leblanc 1983b, p. 197]) first developed by Henkin [1949] to study the completeness of first-order logic, provides the truth-theoretic interpretations for the Δ_i . More precisely, H is a *Henkin set* provided H is maximal consistent (that is, if H is consistent but not a proper subset of any consistent set of L) and property (7) for Hintikka sets holds for H. Clearly every Henkin set in our language L is a Hintikka set in L. Hintikka's model sets became one-sided Beth semantic tableaux when Smullyan transformed the Hintikka model sets by uniting with them the Henkin interpretation. But Hintikka [1989] claims that his [1955c] "in effect" already establishes a connection between Hintikka sets and the concept of the Henkin model and that this connection was "very much in my thinking right from the beginning." In fact, he had done so when he stated ([1955c], p. 2) that a consistent set of formulae ν can be embedded in a maximal model set μ. The simplification of the construction of expansions of Henkin models is due to the work of Henkin [1950], Hasenjaeger [1953], and Beth [1959], and was adopted by Smullyan [1968, pp. 93-97]. In either case, analytic tableaux are clearly Hintikka model sets which are also Henkin models (see [Smullyan 1968, pp. 91-96])

There is also a close relation between Hintikka model sets and the semi-valuations of Schütte presented in [Schütte 1960] for his system C^2_0 (of simple type theory). If in particular Σ is a set of (closed) signed formulae and each C^2_0-formula $A(x_1,\ldots,x_n)$ has an n-ary predicate parameter p^n associated with it, then Σ is a Hintikka set provided

1′′) for no formula X are both X and $\neg X$ in Σ

2′′) if $\alpha \in \Sigma$, then $\alpha_1 \in \Sigma$ and $\alpha_2 \in \Sigma$

3′′) if $\beta \in \Sigma$, then $\beta_1 \in \Sigma$ or $\beta_2 \in \Sigma$

4′′) if $\gamma^n \in \Sigma$, then $\gamma^n(c^n) \in \Sigma$ for each n-ary parameter $c^n (n \geq 0)$

5′′) if $\delta^n \in \Sigma$, then $\delta^n(c^n) \in \Sigma$ for some n-ary parameter c^n $(n \geq 0)$

6′′) if $\pi P^n c_1,\ldots,c_n \in \Sigma$ and there is a formula $A(x_1,\ldots,x_n)$ with which P^n is associated, then $\pi A(c_1,\ldots,c_n)$ is in Σ

Beth's semantic tableau is similar to Hintikka's theory model sets, except that, for the tableau method, a proof of a formula $F \to G$ is a failed attempt to construct a countermodel to $F \to G$ by describing a model in which F is true and G is not. Generally, Beth's tableaux are more complicated than Hintikka's model sets. Beth's tableaux list both true formulae and their consequences. There is an assumption colum on the left and an assumption column on the right. These columns are likewise divided. The true assumptions are are listed on the left-hand side of the assumption column, and the false assumptions are listed on the right-hand side of the assumption colum. True derivations are listed on the left-hand side of the derivation column, and flase derivations are listed on the right-hand side of derivation column. Thus, all formulae are presented in tabular form :

Assumptions		Derivations	
True	False	True	False

Beth deduction tableaux are somewhat simpler than these semantic tableaux of his, since there are only two columns; on the left, we list true formulae and their derivations, forming a truth tree, and on the right, we list false formulae and their derivations, forming a falsehood tree. In his *Remarks on natural deduction*, Beth [1955] gave an explanation for using semantic tableaux, turning a tableau into a formal derivation Nevertheless, tableaux are inelegant and difficult to use because of their complexity. As Smullyan ([Letter to Anellis, 1987]) writes: "When I saw Beth's tableaux I was immediately dissatisfied with the fact that he used two trees instead of one (I found it confusing to relate branches of one tree to the corresponding branches of the other) and so I (like Hintikka) use only one tree.... I present my tableaus in *two* versions— one using unsigned (ordinary) formulas and the other using *signed* formulas (expressions of the form TX or FX, where X is an unsigned formula). Whereas Beth would put X into either the left tree (the "truth" tree) or the right tree (the "falsehood" tree), I would respectively put TX, FX into my single tree." Aside from this the only other essential difference between Hintikka's method and that of Smullyan is that the nodes of Smullyan's tree contain single formulae rather than finite sets of formulae as in Hintikka's tree.

§2. Hintikka and Smullyan's analytic tableaux. Smullyan began work on his tree method, which he called the *analytic tableau method*, around 1959, and between 1962 and 1968 produced a series of papers, most of which were published between 1963 and 1968 (although publication of one was delayed until 1970, and one, which was supposed to have been published, was left unpublished (see [Smullyan 1963-1970]), which develop his method. That Smullyan had begun his work before 1963 is verified by Melvin Fitting, who arrived at Yeshiva University in New York City in 1963. Fitting recalled (for [Anellis 1989]) that, by the time he arrived at Yeshiva, Smullyan was already working on analytic tableaux and showed him some of that work. Smullyan's book *First-order logic* which appeared in 1968, is not simply the investigations of the "nooks and crannies" of the semantic approach to cut-free proofs of sequents that Robinson ([1979, p. 290]) calls it; rather, it is a complete and systematic exposition of the analytic tableaux that Smullyan developed during the previous decade of 1959-1968, and, as such, it is an original and important work.

Although, like Hintikka's model set method, Smullyan's analytic tableaux are left-sided Beth tableaux, that is, truth trees, Smullyan notes ([1987, Letter to Anellis]) that he was unaware of the work of Hintikka when he began his own work, and asserts that when he began his work on analytic tableaux, he had not yet even heard of Hintikka. According to Hintikka (Letter to Anellis, 10 March 1987, [1987a]), Smullyan had invited him to give a talk on the tree method at his [Smullyan's] institution [Yeshiva University] in 1962; but Smullyan [1987] reports that he cannot recall Hintikka's visit. In (Letter to Anellis, 16 October 1989, [1989]), Hintikka strongly recalls that he met Smullyan at that time, although he is no longer absolutely certain. Fitting told Anellis [1989] that he could not recall a visit by Hintikka to Yeshiva. This is compatible with Hintikka's recollection of the visit having occurred in 1962. Moreover, Hintikka [1989], does not now believe that his lecture took place at Yeshiva University, although he is not certain where else it may have been. He is clear that it was in the New York City area. He now also recalls that the invitation did not come from Smullyan, but from some other mathematician – he cannot recall who – interested in the tree method and seeking Hintikka as a co-author for an elementary logic textbook.* On the other hand, Hintikka [1987a] expressed his belief that Smullyan knew of his, as well as Beth's, work and surmised that Smullyan learned of his work through William Craig's [1957b] review for the *Journal of Symbolic Logic* of Hintikka's and Beth's work. Allowing for faulty memory, we can suppose that Smullyan learned of Hintikka's work before 1965; the earliest reference to any of Hintikka's work to be found in the published record in Smullyan's development of analytic tableaux being in Smullyan's [1965] paper, *Analytic natural deduction*. Since this paper was written in early 1963, since none of

Smullyan's earlier papers mention Hintikka, and since his later papers consistently cite Hintikka, we may give late 1962 or very early 1963 as the earliest that Smullyan learned of Hintikka's work, that is, probably in connection with a talk given in the New York City area in 1962 which Smullyan attended. It would appear, then, that Smullyan indeed became acquainted with Hintikka's work around 1962, as Hintikka had suggested, and that he began to produce his own work for publication precisely at that point. Moreover, Hintikka's work takes a prominent and explicit place in Smullyan's research on analytic tableaux from 1963 forward. While acknowledging that Smullyan did his own work independently before 1962, Hintikka [1989] expresses surprise that his and Beth's work was not better known in the late 1950s. Fitting has suggested (to [Anellis 1989]) that Smullyan would agree that Hintikka provided the apparatus for upward and downward satisfiability, that is, through Hintikka sets, and would agree too that he was influenced by Hintikka's model sets, but would likewise assert that he was mainly influenced by Beth's work. But however important Hintikka's work on model sets may have been for Smullyan after 1962-1963, we can take Smullyan at his word and assert that he arrived, independently of Hintikka, at the idea of reducing Beth's semantic tableau to a tree or one-sided tableau, since he began his own work several years before his first contacts with Hintikka, that is, around 1959, and since it is clear that, for Smullyan, as for Hintikka up until his publication of [1955], the major influence was Beth.

We must also take note of a [1960] paper by Zbigniew Lis on *Semantic deduction and formal deduction*, written in Polish, with Russian and English summaries, and completed before the third of March 1959. Lis, working in apparent independence from Hintikka and taking his inspiration largely from Beth and the work of Lukasiewicz and the Polish logicians employing natural deduction, developed a one-sided Beth tableau for first-order logic with identity and descriptions. Lis's "semantic tableaux" are one-sided Beth tableaux, truth trees in which formulae are signed as either +, −, or ±, thus combining in a single tree both the truth trees and the falsehood trees of Beth's semantic tableaux ; in a sense, they are also closely related to Beth's deduction tableaux. Lis's analytic tableaux are truth trees in which all formulae are signed by |−, rather than by truth-valuations as was done by Smullyan. Lis's tableaux, however, are not yet quite trees in the sense of either Hintikka, Smullyan, or Jeffrey, since Lis's deduction rules do not permit branching. Lis's work appears to have been altogether unknown to Smullyan.

Schütte's [1956] paper *Ein System des verknüpfenden Schliessens* and Stig Kanger's [1957] book *Provability in logic* provide alternative elaborations to Beth's tableaux. However, Smullyan's former student Sue A. Toledo [1975] has shown how to translate proofs back and forth between Schütte's system C^2_0, as these are presented in Schütte [1960] and developed in Schütte [1960a], and Smullyan's analytic tableau system.

The main import of work on completeness proofs in Henkin's [1949] development of the Henkin set for tableaux and arboreal systems was in the way it became a major tool for completeness proofs for the tableau method. Henkin [1954] and Orey [1956] showed that the ω-logic of the structure of natural numbers is complete, and Schütte [1960a] also proved the completeness of a cut-free system with ω-rule. It follows that the since the tree method is just a cut-free system, it too should be provably complete. Beth claimed to have proved the completeness semantic tableaux in various of his papers, although Kleene [1957] and Kreisel [1958] in print, and Hintikka, Kreisel and van Heijenoort in private communications pointed out an error in Beth's proof of the completeness of intuitionistic first-order tableaux (which shall be described momentarily). Beth's [1960] easily proved the completeness for first-order classical tableaux and simply brushes aside (see [Beth 1960, p. 285] the objections of Kleene and Kreisel. Kripke [1959], however, proved the completeness of the tableaux method for first-order modal logic with identity. In his [1971] book *Formal semantics and logic*, Bas van Fraassen also presented a proof of the completeness of the tableaux method for first-order classical logic with identity. Errors in the adequacy (soundness) proofs presented in these works of both Kripke and van Fraassen are corrected by Bencivenga in his [1981] paper *Semantic tableaux for a logic with identity.*

Hintikka gave a completeness proof of his theory of model sets in his [1955c], well before the completeness proofs given by Smullyan. Nevertheless, Smullyan's main contributions to the theory of trees, according to Fitting (see [Anellis 1989]), are his completeness and existence theorems. Smullyan's completeness proofs of [1968], similar to the ones later adopted by Bell and Machover [1977], made full use of Hintikka sets. Theodore Linden, in a doctoral thesis for Yeshiva University which was supervised by Smullyan [1968], extended Smullyan's own [1968] results by proving the completeness of the method of analytic tableaux for infinitary trees, that is, for a tree with length ω. Thus, the truth tree method could be shown to be applicable to propositional logics for the language $L\omega_\alpha$ and to models with first-order language $L\omega_\alpha\omega_\beta$. Soon thereafter, Fitting ([1971]) proved the soundness and completeness of Smullyan trees for first-order logic with empty domain. In the 1970s, van Heijenoort would prove the soundness and completeness of the tree method, and in particular of the falsifiability tree method for classical first-order logic, using techniques which were similar to those of Smullyan, but somewhat less elegant.

We must also note, however, that Hintikka has another claim to priority in the development of the tree method based upon Hintikka's work of 1955. As correctly understood by Hintikka, the tree method is already present (albeit merely as truth tree and falsehood tree taken as dual aspects of semantic tableaux) in Beth's paper *Semantic entailment and formal derivability* [1955a], and in his own two papers of 1955, *Form and content in quantification theory* [1955] and *Notes on quantification theory* [1955c]. Beth rushed to Hintikka an offprint of his paper *Semantic entailment and formal derivability* [1955a]; Hintikka received this a few days after the publication of his own paper *Form and content in quantification theory* [1955]. Possibly at the same time, but more likely somewhat later, Beth sent Hintikka a preprint copy of his paper *Semantic construction of intuitionistic logic* [1956], treating intuitionistic logic by the tableau method; and it is clear that Beth recognized the kinship of his method and that of Hintikka, a recognition which Beth made explicit in [1955a, pp. 340-341], and again in [1959, p. 201] in which a version of the trees already very clearly occurs. On the other hand. Hintikka [1989] does not believe that Beth understood the point of the well-known error in [Beth 1956] (and repeated thereafter, for example in [Beth 1957] where it was referred to by [Kreisel 1958] and by [van Heijenoort n.d.], and again in [Beth 1960] which Hintikka brought to his attention before the publication of the paper. (The error is in Beth's proof of the completeness of the first-order intuitionistic tableau; in particular, in order for his proof to work for intuitionistic first-order logic, Beth had to replace the classical idea of *decidability* with the idea of *security* (see, for example [Kleene 1957], [Kreisel 1958, p. 36], and [van Heijenoort n.d.]). For a sentence **S** to be *secure*, it is sufficient for **S** to be secure on some branch of the tree for **S**, that it occur as valid on some subbranch of its tree. In this case, a first-order universally quantified formula $S \vdash (\forall x)\Delta x$ is regarded as intuitionistically decidable or secure if for each of the (finitely many) branches of the tree for S, one of $\Delta x_1, \ldots, \Delta x_n$ occur on a subbranch. While Beth was prepared to allow that security looks like the decidability of $S \vdash (\exists x)\Delta x$ rather than of $S \vdash (\forall x)\Delta x$, he suggested that the occurrence of a subformula $\Delta x_1 \wedge \ldots \wedge \Delta x_n$ of the universally quantified formula on each tree was sufficient, given the finitary nature of intuitionistic logic.) Indeed, Beth no doubt sent the preprint of his [1956] paper to Hintikka in response to a letter from Quine, in which Quine, upon receipt of the manuscript of Beth's [1955], informed Beth of Hintikka's *Distributive normal forms in the calculus of predicates* [1953] and *Two papers on symbolic logic* [1955b] of 1955 (see the postscript, dated 11 July 1955, in Beth's [1955a, p. 340-341]). In the postscript of [1955], Beth pointed out in some detail the similarities and differences between his method and that of Hintikka.

The concept of model sets was first developed by Hintikka as an attempt to construct models for first-order formulae in distributive normal form, as these are defined in his 1953 work *Distributive normal forms in the calculus of predicates*, and he recalls explaining this idea to his colleagues in Finland as early as 1951-1952. In fact, Hintikka [1987a] claims that the "basic idea of the tree method" is mentioned as early as 1953 in his paper *A new approach to sentential logic* on the model set method in propositional logic. If, as Jeffrey ([Letter to Anellis, 6 May 1987]) has said, the tree method. as presented to Jeffrey by Smullyan, just presents one-sided tableaux which give graphical representations of Hintikka model sets, then the tree method is indeed already clearly present in Hintikka's paper *New approach...*, where ([1953, p. 8]) a graph of a tree, written from left to right rather than from top to bottom, is depicted. In this case, we can substantiate Hintikka's priority claim, noting in addition that a top-to-bottom tree occurs in Beth's *Semantic entailment...* of 1955. It was Smullyan, however, who brought together and developed these ideas systematically and first presented the tree method in the form in which we best know it today. It was Smullyan, too, who made explicit the connection between Beth's tableaux and Hintikka's model sets as graphical representations of one-sided Beth tableaux. For additional recollections by Hintikka of his work in logic, and especially concerning the connections of models sets with Beth tableaux. with Beth tableaux taken as proofs in Gentzen N-sequents "turned upside down", see Hintikka's *Self-profile* ([1987], especially pp. 10-14; quotation from p. 12).

The conception of the so-called Smullyan tree (analytic tableau) as a one-sided Beth tableau led to the first textbook using the tree method, Richard Jeffrey's *Formal logic* [1967]. As it is remembered by Jeffrey [1987]:

> One day (in 1965, I guess), I ran into him [Smullyan] on the street, in New York (I think). I told him that people had recommended Beth's method of semantic tableaux as the clearest and simplest way of dealing with first-order logic, and that I had tried to learn it from Beth's writings (e.g. the big fat yellow North-Holland book) – [the 1959 edition of *The foundations of mathematics*] – but without success. Somehow I couldn't follow Beth's examples, and he never seemed to describe the method in a general way -- not so that I could understand, anyway. I said I found it especially confusing to have to hop back and forth between the left and right sides of the tableaux, keeping track of what on the left matches what on the right. Ray said: "Just work with one side -- the positive one -- and whenever you'd put something on the other side [instead] just put its denial on the positive side." The scales dropped from my eyes; I rushed home. tried some examples, and was converted. I was an enthusiastic convert, and proceeded to convert my students at CUNY [City University of New York] by writing a mimeographed book on it for them, i.e. what became my Formal Logic.

Jeffrey also recalls attending Smullyan's lectures at Princeton, probably in 1964, where, he thinks, Smullyan probably used tableaux, and he surmises that these lectures were the basis of Smullyan's book *First-order logic*.

At the same time, Hintikka was convinced that everything necessary for a treatment of first-order logic by the tree method was present in his own work of 1953 and 1955, and that, as he expressed it ([1987, Letter to Anellis]), "all that was needed was to spell it out in textbook terms;" and that he was engaging the idea to write such a textbook when Jeffrey's book appeared in print. This was probably to have been the elementary textbook, to be co-authored with the person whose name Hintikka has forgotten, and for which Hintikka had already drafted several chapters. An application of the theory of model sets was given in Hintikka's paper *Reduction in the theory of types* [1955a].

There were, we noted, two versions of analytic tableaux worked out by Smullyan, one using signed formulae, the other using unsigned formulae. Beth used unsigned tableaux, putting an unsigned formula X into either the left tree of his tableau or into the right tree, as either true or false respectively. Smullyan created signed formulae

TX, FX from Beth's unsigned formula *X*, and instead of putting unsigned *X* into either the truth tree of the falsehood tree, as Beth did, Smullyan would place either *TX* or *FX*, as required, into his single tree. He showed both signed and unsigned versions of his tree to Jeffrey, who used the unsigned version in his textbook. Jeffrey's tree is a truth tree, and instead of placing an unsigned formula *P* into the falsehood tree of a Beth tableau, or of using the signed *FP* in the truth tree as Smullyan had done, placed its denial $\neg P$ in his truth tree. "I'm not absolutely sure," said Jeffrey [1987], "Ray told me to use denials; maybe what he said was ' Use t's and f 's to indicate which side of Beth's tableau the formulas would go on' and I turned t's into nothings and f 's into denials in my mind. But what I seem to remember is Ray suggesting denials... ." Jeffrey's *Formal logic*, an undergraduate textbook, was, then, the first book to present truth trees, and Smullyan's *First-order logic* was the first sophisticated treatment of the method. The approach using unsigned formulae taken by Jeffrey was also adopted by J.L. Bell and M. Machover in book *A course in mathematical logic* [1977], which currently serves as the standard graduate textbook on the tree method, especially since Smullyan's book has long since gone out of print.

New York City and its environs was an important center of activity in the development of the tree method in the mid- and late-1960s, where Richard Jeffrey, Raymond Smullyan, Jean van Heijenoort and other logicians would meet on occasion in an informal group to discuss logic. These meetings, which were organized by Hao Wang, took place primarily at Rockefeller University, where Wang arrived in 1966. It was also New York (although it is uncertain that it was at Yeshiva University) that Hintikka had visited in 1962 to discuss his theory of model sets. The allure of the tree method for Jeffrey and Fitting was its simplicity, its elegance – its beauty. Indeed, in his study comparing Gentzen's system with the falsifiability tree method, van Heijenoort [1973] pointed to the convenience, pedagogical and technical, of the truth trees and falsifiability trees as a critical factor in their favor. Moreover, we may surmise that it was the inherent clumsiness and difficulty of Beth's semantic tableaux applied to N-sequents that led Ono, in his paper *On a practical way of describing formal deductions* [1962], to seek a way of abandoning tree diagrams in favor of dealing with deduction trees as a series of propositions, while preserving the semantic aspects of tableaux by numbering the propositions and introducing so-called *nominating* *quantifiers* $\forall!$ and $\exists!$ which temporarily treat their bound variables as though they were free. Ono's method is therefore reminiscent of Quine's [1950] use in natural deduction of flagging. Likewise, Andrés Raggio, argued in his paper *Semi-formal Beth tableaux* [1977] that existentially quantified formulae on the right of a Beth tableau could be refuted, and universally quantified formula on the left of the Beth tableau could be verified, only in case every instantiation using terms of the variable already occurring in the tableau were considered, and this could not be done easily in light of the finitary nature of Beth's tableau rules. Raggio therefore proposed creating *semi-formal Beth tableaux* by dropping Beth's finitary restrictions. In Raggio's tableau, it is required that, for both existentially quantified formulae on the right, and universally quantified formulae on the left, the next line of the tableau must contain all instantiations uning the terms, both functions and variables, already occurring in the tableau. This would result in obtaining a new node in the tableau at which is listed a denumerable list of formulae containing one quantifier less than occurred in the original formula. Raggio's work, although it did not affect the creation or development of Smullyan trees, helps indicate the relationship between Beth tableaux and Smullyan trees and helps point out the greater simplicity and power of trees.

§3. Van Heijenoort and the falsifiability tree method. Jean van Heijenoort, who taught mathematics at New York University during the period when the tree method was developed, became one of the earliest and most enthusiastic advocates of the method as it was presented by Jeffrey. Van Heijenoort also made his own significant contributions to the development of the method. One of his contributions was to show that the tree method could be applied to quantified formulae whether or not they were in prenex form. By doing so, he showed that trees need not be finitary. This was done by modifying the character of the quantifier rules of the tree method by applying Herbrand quantification rules to the tree method [1968] and by proving König's infinitary tree lemma [1972a, p. 25]. This work was similar to work carried out by Thomas Patton [1963]. Patton developed a deductive system which he described as being related to the system in Quine's [1959] *Methods of logic*, but which might also be described as being between Herbrand quantification and the quantificational part of Smullyan tableaux. This system, for which Patton proved the soundness and completeness, was restricted to formulae in prenex normal form, and based upon proofs by contradiction. Smuulyan knew of Patton's [1963] work and refers to it in *First-order logic.* Richard Jeffrey (Letter to Anellis [1987]) knew Patton and his [1963] work, but was left untouched by it. Van Heijenoort's most significant and lasting contribution was to develop the falsifiability tree.

The concept of the *falsifiability tree* was developed in unpublished papers of van Heijenoort dating from 1966 to 1975. Many of the more pivotal of these papers have been described by Anellis [1988]. A detailed account of the contributions presented in these and related works by van Heijenoort is presented in [Anellis 1989]. In his manuscript *Interpretations, satisfiability, validity* [1966], van Heijenoort defined the concepts required for presentation of the falsifiability method, and in particular the concepts of *countermodel* or *falsifying interpretation*. In 1968, in his manuscript *On the relation between the falsifiability tree method and the Herbrand method in quantification theory*, he presented the falsifiability tree method as the "dual" of the truth tree presented by Jeffrey, while acknowledging that the falsifiability tree is studied, "in many variant forms", by Smullyan in *First-order logic* and codified in textbook form by Jeffrey's *Formal logic*. If so, then it was van Heijenoort who brought it to the fore.

The falsifiability tree can be seen as an application, or perhaps as a special case, of the right side, the falsehood tree, of the Beth tableau. Both truth trees and falsifiability trees make use of the truth-value semantic which was initially introduced by [Beth 1959], [Schütte 1962/63], and [Dunn & Belnap 1968], when they showed that any arbitrary statement S in the language L of first-order logic is logically true (i.e. valid) if and only if it is true on all truth-value assignments to the atomic statements of L, and when [Leblanc 1968] showed that any arbitrary statement S in the language L of first-order logic is logically true (valid) if and only if it is true on all truth-value assignments to the atomic substatements of L (see [Leblanc 1983b, p. 209] and [Leblanc 1982, pp. 127-129]). In [1976, §4.2, pp. 92-102] and [1977], Leblanc showed how Henkin models could be translated into truth-value assignments. In particular, he showed that if **M** is a Henkin model and *A* is a well-formed atomic formula in L, then the truth-value **T** is assigned to *A* if *A* is true in **M**; otherwise, the truth-value **F** is assigned to *A*. The translation is efficient and justifiable because (as Leblanc [1982, p 132] noted) truth-value assignments match all and only Henkin models, and because not every model is a Henkin model. In fact, Henkin models "translate straightforwardly into truth-value assignments" [Leblanc 1976, p. VI] This translatability is precisely the concept behind Smullyan's suggestion to Jeffrey in 1965 that Beth tableaux could be simplified by the use of signed formulae, and was presented by Smullyan [1963, p.828] in terms of "abstract consistency" and (in [Smullyan 1968, p. 87]) as *magic sets*, that is, as models having Boolean truth-functional valuations. Indeed, Smullyan's proposal to employ signed formulae depends upon Beth's [1959] truth-value semantic in *The foundations of mathematics*. The truth-value semantic interpretation of satisfiability and validity of first-order formulae was fully articulated by van Heijenoort in his paper *Interpretation. satisfiability, validity* [1966], was applied to truth trees and falsifiability trees in his paper *Notes on the tree method* [1971], and appeared in its final form in his paper *Falsifiability trees* [1972a]. Prawitz, meanwhile [1975], presented a systematic exposition of the correspondence in terms of truth valuations between the inference rules of the Gentzen sequent calculus and clauses of the semantic definintion of truth. Leblanc's [1976] book *Truth-value semantics* presents a systematic and unified non-historical exposition of published research in the subject; (see p. VII-VII); it also includes a brief ([1976, pp. 293-301]) history of published work on the subject. Since van Heijenoort's papers [1966]. [1971]. and [1972a] were unpublished. Leblanc's [1976] did not, unfortunately, consider them.

For the falsifiability tree, one takes the denial of the formula(e) at the initial node of a tree, assumes it / them to be true, and assumes at the same time that every (sub-)formula occurring in the tree obtained by applying tree decomposition rules to the formula(e) in this new tree is in fact true. The aim is to obtain a closed tree, that is, one in which every path of the tree contains both some formula F and its denial $\neg F$ In this way, the falsehood tree of Beth is substituted for a falsifiability tree, that is, a truth tree modified to present proof by contradiction of Gentzen N-sequents. (Discussions of the general concept, history, and technical workings of proofs by contradiction have been given by Goodstein [1948] and by Löwenheim [1946].) Van Heijenoort's manuscript *Comparison between the falsifiability-tree method and the Gentzen system* [1973] explores the connections between Gentzen's system **LK** and the falsifiability tree method.

To collect and simplify what has already been said about trees, we see that a *tree* is a one-sided (left-sided) Beth tableau in which all formulae are true. It has a small number of tree decomposition rules, which are precisely the inference rules which Gentzen introduced in his classical sequent-calculus **LK**, one for each of the truth-functional connectives which we choose for our base, and one each (at least) for the universal and existential quantifiers. Thus, for example, if we select as our base the set of connectives $\{\neg, \to\}$, then the following tree decomposition rules apply:

$$
\text{(DN)}\ \begin{array}{c} \neg\neg G \\ | \\ G \end{array}
\qquad
(\to\text{-rule})\ \begin{array}{c} P \to Q \\ / \quad \backslash \\ \neg P \quad Q \end{array}
\qquad
(\neg \to \text{-rule})\ \begin{array}{c} \neg (P \to Q) \\ | \\ P \\ | \\ Q \end{array}
$$

For quantifiers, we introduce the rules:

$$
(\forall\text{-rule})\ \begin{array}{c} \forall x\ Fx \\ | \\ F(x/\mu_0) \\ . \\ . \\ . \\ | \\ F(x/\mu_n). \end{array}
\qquad
(\neg\forall\text{-rule})\ \begin{array}{c} \neg\forall x\ Fx \\ | \\ \neg F(x/\mu), \end{array}
$$

where μ is a mutant (permissible substitution instance) of the bound variable, provided μ is new to the path in which it occurs.

where $\mu_0 \ldots\mu_n$ are mutants (permissible substitution instances) of the bound variable.

Let Φ be a sequence of formulae $F_0, F_1, F_2, \ldots, F_n, Q$. If $F_0, F_1, F_2, \ldots F_n, Q$ are subformulae of Φ, we say that $F_0, F_1, F_2, \ldots F_n$ of Φ a *proof of* Q (and write $\Phi \vdash Q$) provided $\Phi \vdash Q$ if and only if $\Phi \to Q$ and

$$(F_0 \wedge F_1 \wedge F_2 \wedge \ldots \wedge F_{n-1}) \to (F_n \to Q)$$

$$\vdots$$

$$(F_0 \to F_1 \to F_2 \to \ldots \to F_{n-1} \to F_n) \to Q$$

In that case, there is a truth tree for Φ such that the formula Q occurs at some node in a nonclosed branch of the tree for Φ, and that tree is a proof of Q. If a tree is a proof of and endformula Q, then Q is called a theorem. A branch of a tree for a set of formulae is closed if there is some formula Ψ of Φ such that Ψ and its negation $\neg\Psi$ occur in nodes of the same path of the tree for Φ. A *falsifiability tree*, or as Bell and Machover [1977] call it, a *confutation tableau*, is a tree (or sequence of trees) in which we attempt to find a falsifying assignment for a (set of) formula(e) Φ. We do so in either one of two equivalent ways. Either by constructing a corresponding truth tree for the negation $\neg\Phi$ of the entire sequence Φ or for the immediate successor sequent $\neg$Q of the initial sequent of Φ (that is for Φ'), and attempt to derive a contradiction for this new tree. If we construct a truth tree for the negation $\neg\Phi$ of the entire sequence Φ, we have created a truth tree for $\neg\Phi \vdash \neg Q$. Similarly, if we construct a corresponding truth tree for the immediate successor sequent $\neg$Q of the initial sequent of Φ (that is for Φ'), we have created a truth tree for $\Phi' \vdash \neg Q$. Thus, a *falsifying assignment* or *falsifying interpretation* of a formula Δ is an assignment α of truth-values to (the terms of) Δ such that Δ is false, that is, $v[\alpha:\Delta] = f$. A path or (sub-)branch of a tree closes, then, provided that there is an assignment α of truth-values to the subformulae of Δ such that $v[\alpha:\Delta] = t$ and there is a related assignment α' to subformulae of Δ such that $v[\alpha':\neg\Delta] = t$. If each path of such a falsifiability tree closes, and there is a subvaluation α' such that $v[\alpha'':F_i] = t$ for each F_i of Φ, then the falsified sequence Φ' for the sequence Φ is inconsistent, and the original sequence Φ is *valid*. Thus, the falsifiability tree, as first proposed by van Heijenoort [1968] as the dual of the Smullyan tree or truth tree, announced by van Heijenoort in [1970], is a canonization, or algorithmization and codification, of proof by contradiction for **LK** and for axiomatic

Van Heijenoort developed the falsifiability tree method for both propositional calculus and the classical first-order functional calculus in 1971 in his unpublished papers *Notes on the tree method* and *Falsifiability trees* [1972a], the latter revised in [1974]. These three works [1971; 1972a; 1974] also present the falsifiability tree method as a technique which the author employs to present his first proof of the completeness and soundness of Jeffrey's [1967] version of the tree method. Van Heijenoort's unpublished paper *Soundness and completeness of the falsifiability tree method for sentential logic* [1973a] is a revised and improved proof; likewise, his unpublished paper *Falsifiability trees* [1974] provides an improved presentation of the falsifiability tree for classical first-order logic, and presents a simplified version of a proof of the soundness and completeness of the method for classical first-order logic.

Using the deduction theorem as a schema for rewriting sequences of formulae of $\Phi \vdash Q$ as a single expanded formula, van Heijenoort was able to treat proofs as formulae. This permitted him to treat both proofs and formulae truth-functionally, and to prepare model- theoretic proofs of the soundness and completeness of the tree method. Formulae as well as proofs taken as extended formulae have truth-theoretic interpretations according to which they may be valid, satisfiable, or invalid. Hence, no distinction needs to be made between deductive validity, satisfiability, or invalidity and the validity, satisfiability, or invalidity of formulae, and hence no difference between the validity of formulae and deductive validity or the validity of proofs. According to this conception, a tree for Φ is thus a *proof* of each formula at a terminal node of an open path in the tree for Φ, and the set of all formulae in the open paths of the tree for Φ is a *satisfiability model* for Φ. A formula/proof Θ is *valid* if there is a truth tree for Θ such that every path in the tree for Θ is open and there is a falsifiability tree for Θ' or for $\neg\Theta$ such that every path in the tree for Θ' or for $\neg\Theta$ is closed. A formula/proof Θ is *satisfiable (or nonvalid)* if there is a truth tree for Θ such that at least one path in the tree for Θ is open and there is a falsifiability tree for Θ' or for $\neg\Theta$ such that at least one path in the tree for Θ' or for $\neg\Theta$ is closed. A formula / proof Θ is *invalid* if there is a truth tree for Θ such that every path in the tree for Θ is closed and there is a falsifiability tree for Θ' or for $\neg\Theta$ such that every path in the tree for Θ' or for $\neg\Theta$ is open. Alternatively, we can say that *satisfiability* is validity with respect to a specific model, or that *validity* is satisfiability invariant with respect to a model. This means that $\Phi \mid\!- Q$ (Φ *is a model* of Q) only in case $\Phi \models Q$ (Φ *proves* Q) for every interpretation of Φ The falsifiability tree method is *sound* if each provable formula in the system can be proven to be valid by the falsifiability tree method; and the falsifiability tree method is *complete* if each valid formula of the system is provable by the method. As noted, van Heijenoort's proofs, unlike those given by Smullyan [1968] and by Bell and Machover [1977], do not make explicit use of Hintikka sets. As a result, even though they employ the same concepts and follow the same patterns, they are somewhat longer and require more bookkeeping. (For an example of van Heijenoort's proofs, see the sketch by Anellis [1989] of his proof for the soundness and completeness of the falsifiability tree method for propositional logic.) Van Heijenoort's result's are therefore closely related to Smullyan's proofs in *A unifying principle in quantification theory* [1963] that consistency implies satisfiability and that denumerable satisfiability implies sentential satisfiability. Of course, much of the work on the completeness of the tree method had already been done by Beth, who ([1960]) proved the completeness of the semantic tableau method.

Van Heijenoort also gave proofs of the completeness and soundness of the falsifiability tree method for the simple theory of types with extensionality [1972d]. In 1975, he used the signed formulae version of Smullyan's analytic tableaux to provide proofs of the soundness and completeness of the tree method for intuitionistic propositional logic [1975] and for intuitionistic first-order logic [1975]. The way for van Heijenoort's proofs was prepared by the work of Fitting. In a doctoral thesis for Yeshiva University supervised by Smullyan, Fitting [1968] employed Kripke models (as these were presented by Kripke in [1965] for his semantic analysis of intuitionistic logic) to prove the completeness of intuitionistic Beth tableaux. Fitting's thesis was published, with revisions, in [1969]. In his 1975 proofs of the soundness and completeness of the tree method for intuitionistic logic, van Heijenoort followed the concepts set forth in

Fitting's proofs, making full use of Kripke models (see [Anellis 1989]). In his book *Introduction à la sémantique des logiques non-classiques* [1979], van Heijenoort extended these results to propositional and first-order modal logic as well as to propositional and first-order intuitionistic logic, meanwhile improving his proofs, and he suggested ways to extend the tree method to versions of trivalent logics as well. In this respect, van Heijenoort lagged behind logicians such as W. Suchon, whose *La méthode de Smullyan de construire le calcul n-valent de Lukasiewicz avec implication et négation* [1974] and S.J. Surma, whose paper *A method of construction of finite Lukasiewiczian algebras and its application to a Gentzen-stile* (sic) *characterization of finite logics* [1974] applied Smullyan trees to multiple-valued algebra. He was, however, contemporary with, and sometimes even ahead of, Copeland [1983], Davidson [1982], Davidson, Jackson, and Pargetter [1977], and Burrieza and León [1987], all of whom applied Jeffrey's version of trees to modal logics. Kanger [1957] applies a method similar to, but independent of, the tableau method, to modal logic. M. Guillaume [1958; 1958a] extends semantic tableaux to modal logic, and Kripke [1959] mentions an unpublished result in which he extended semantic tableaux to modal logic. The pioneer in the application of Kripke models to modal Smullyan trees, however, was again Fitting, whose paper *Tableau methods of proof for modal logics* [1972] borrowed Kripke's [1963; 1963a] model- theoretic approach to modal logics, as adapted by Fitch [1966], and applied it to Smullyan trees. In [1989], Fitting discussed the application of Smullyan trees to a four-valued logic developed by Nuel Belnap. Meanwhile, van Heijenoort continued to explore applications of Beth tableaux to classical and nonclassical logics, including his unpublished proof of the completeness of Beth tableaux for classical propositional logic [1972], his unpublished proof of the completeness of Beth tableaux for intuitionistic first-order logic [1972b], and his application of the tableau method to Grzegorczyk first-order logic [1972c].

§4. Some uses of the tree method. One of the more interesting and important uses of the tree method, in addition to its use in obtaining completeness results, is its use in proving Craig's Interpolation Theorem (Interpolation Lemma). Beth [1958] had already employed semantic tableau in proving Craig's lemma. Craig first presented this result for classical first-order logic in [1957], and Schütte extended it to the intuitionistic case in [1962]. The Interpolation Lemma says that, if $\phi \models \psi$, then we can show that one of

(i) ϕ is refutable
(ii) ψ is valid
(iii) there exists a sentence Γ such that $\phi \models \Gamma$, $\Gamma \models \psi$, and every sentence symbol occurring in Γ also occurs in both ϕ and ψ

holds. Γ is called the *interpolation formula*. There is a clear connection between a tree proof of $\phi \vdash \psi$ and the structure of Γ for which we have $\phi \vdash \Gamma$ and $\Gamma \vdash \psi$. If in particular Φ and Ψ are finite sets of formulae of the first-order language L and β is an L- formula (or any empty string), then a formula α is an *interpolant for* $<\Phi, \Psi, \beta>$ provided each of

(i′) $\Phi \nabla\alpha$ (where $\Phi\nabla\alpha$ means that we can construct a closed first-order tree for $\Phi, \neg\alpha$)
(ii′) $\Psi, \alpha \nabla\beta$
(iii′) every free variable of α is free both in Φ and $\Psi \cup \{\beta\}$ and every extralogical symbol (constant or predicate symbol) occurring in α occurs in both Φ and $\Psi \cup \{\beta\}$.

Thus Craig's lemma says that, given a proof of $\Phi\nabla\alpha$, we can interpolate in $<\Phi, \alpha>$. Replacing '∇' by '$\vdash_\Gamma$', Craig's lemma says that if we have $\Phi \vdash_\Gamma \alpha$, we can either

(i″) prove that $\Phi \vdash_\Gamma$
(ii″) prove that $\vdash_\Gamma \alpha$
(iii″) find a formula β common to both Φ and α and prove that $\Phi \vdash_\Gamma \gamma$ and $\gamma \vdash_\Gamma \alpha$ (so that γ is an interpolant for $<\Phi, \alpha>$).

Hintikka, who recalls ([1989]) having been aware of connections between the complexity of proofs in tableaux and arboreal proofs and interpolation, without. however, writing on the subject, has suggested ([1989]) that, with proper caution, one could consider the interpolation formula Γ to be an index of the complexity of the tree proof of $\phi \mid\!- \psi$. His suggestion is correct, as we now know, since Daniele Mundici has since the early 1980s (see, for example, [Mundici 1982]) studied the complexity of Craig's Interpolation Lemma, that is, how fast the length $\|\Gamma\|$ of the interpolant Γ grows as a function of the length of ϕ and ψ, and has shown that a knowledge of the depth of a Turing machine computation of $\|\Gamma\|$ is related to the length of the proof of $\phi \mid\!- \psi$ while the complexity of the proof of $\phi \mid\!- \psi$ determines the length of the interpolant (although this work is not specifically concerned with tableaux or arboreal proofs).

Craig's lemma may in turn be used to give tree proofs of Beth's Definability Theorem and Robinson's Consistency Theorem. These connections have been presented by Craig [1957a]. If we replace ∇ with $\mathbb{V}$ (constructibility of closed first-order intuitionistic trees), then we obtain the intuitionistic version of Craig's lemma.

Use began to be made of Smullyan's work within a few years of its appearance. A favorable review was given *First-order logic* by Craig [1975]. That same year, Toledo's book *Tableau systems for first order number theory and certain higher order theories* ([1975]) appeared; in it, she applied analytic tableaux to proofs of formal arithmetic theories, proving the soundness and completeness of the infinitary system of first-order number theory and the soundness of full second-order number theory and the completeness of the cut-free part of the second-order theory. In his review of Toledo's book, Zucker [1980] declaimed on the elegance of the method, but suggested that it might not be appropriate for arithmetic theories which include an induction rule. In this respect any of Zucker's worries should be dispelled since we have both upward and downward induction on the trees (in particular in the slightly simplified version of the tree method as presented by Jeffrey and van Heijenoort). Some of this work on trees naturally duplicated ideas which were also being simultaneously but independently carried out by van Heijenoort but left unpublished, while others developed ideas similar to his, but carried out in different directions or with somewhat different techniques; and some of this work appeared only years after van Heijenoort had completed his work. Thus, for example, while van Heijenoort was developing the falsifiability tree method and proving its soundness and completeness for various theories, this concept of the falsifiability tree was also developed in late 1969 or early 1970 by Hugues Leblanc and D. Paul Snyder and published in 1972 in their paper *Duals of Smullyan trees.*

Some interesting results on Smullyan trees were presented by Hugues Leblanc at the 1982 annual meeting of the Society for Exact Philosophy held at Arizona State University, the 1982 meeting of the Philadelphia Logic Colloquium held at Haverford College, and the special session on proof theory during the American Mathematical Society (AMS) meeting in Denver in January 1983. These results were announced in a published abstract on *Smullyan trees: their model-theoretic uses* [1983] prepared for the AMS meeting, and were first presented in detail in the paper *Of consistency trees and model sets* [1983a] which was prepared for publication for the proceedings of the AMS special session. Leblanc pointed out that Smullyan trees can test the validity of finite sets of first-order formulae, and that, when closed, they can serve as proofs, while, when open, they can provide information about models in which these sets of formulae are satisfiable. In his paper, Leblanc presents a new routine for Smullyan trees which – unlike Jeffrey's – shows the consistency of any set with a finite model and shows the least possible cardinality of such a model. This expands van Heijenoort's unpublished result ([1974(?)]) showing that the law of lesser universes holds for falsifiability trees. Leblanc then is able to prove that a finite set of first-order formulae has a finite model if and only if it extends to a finite Hintikka model set. Leblanc's routine bears a strong resemblance in its intent to the structure trees presented by van Heijenoort ([1973a, p.3]) for falsifiability trees. For each falsifiability tree with branch β, a *structure tree* $S(\beta)$ is a tree for which there is an isomorphic mapping ψ between nodes of β to corresponding nodes of $S(\beta)$, and the formula at the node n of β is also at the corresponding node $\psi(n)$ of $S(\beta)$, provided, if a tree decomposition rule is applied to the formula at node n of β, then the successor node n′ (and, if necessary, also the successor node n′′) created by that application has a corresponding node

$\psi(n')$ in $S(\beta)$ (and, if necessary, also the successor node $\psi(n'')$ in $S(\beta)$), and the formula(e) at the successor node(s) n (and n'') in β are also the formula(e) at the corresponding successor node(s) $\psi(n')$ (and $\psi(n'')$) in $S(\beta)$. That is, a *structure tree* is a fully articulated truth tree for some branch β of a falsifiability tree, and its purpose, like that of Leblanc's routine, is to find the set of all models satisfying formulae at the initial node of the truth tree for which the falsifiability tree was constructed.

It appears that in recent years, a number of logicians, among them George Boolos (for example in his unpublished papers *Don't eliminate cut!* [1982], and *Keep cut* [1983], which were early versions of his published paper *Don't eliminate cut* [1984], and especially his paper *Trees and finite satisfiability: proof of a conjecture of Burgess* [1984a]), have produced and published results on trees that were first dealt with by van Heijenoort in his unpublished papers. In particular, Boolos [1984a] proved Burgess' conjecture of 1982 that the Smullyan tree method proves the finite satisfiability of any finitely satisfiable first-order formula to which it is applied; he thereby proved the "weak" soundness (and completeness) of the tree method. His result was extended by Kapetanovic and Krapez [1987] to languages with function symbols. In addition, van Heijenoort gave rather cursory proofs of the principles of converging and diverging induction (upward and downward induction) for trees in his paper *Falsifiability trees* (see, e.g. [1974, p. 24]), and N. L Wilson published his 1981 proof of upward and downward induction on trees in [1983] in *The transitivity of implication in tree logic*.

The popularity of the tree method has grown rather slowly, although both the first and second editions of Jeffrey's *Formal logic* were reviewed enthusiastically, the first edition by Sibajiban [1973], the second by McCarthy [1984]. This growth has largely been a recent phenomenon. As early as 1967, computer scientists interested in automated theorem proving began to explore the theoretical use of Beth tableaux ([Poppleston 1967]) and of semantic trees for natural deductions [Kowalski and Hayes 1969]). In [1988] Austen Clark developed a popular program, Twootie, for the exploration of truth trees for teaching undergraduate symbolic logic courses in philosophy departments; this computer-assisted logic instruction is based on the second edition ([1981]) of Jeffrey's *Formal logic*.

Thus, the tree method developed from Beth tableau which even professional logicians such as Jeffrey had found confusing, into a method which has become increasingly popular even with undergraduate philosophy students. It began with Hintikka, who first presented some of the basic ideas of model sets as graphical representations of truth trees of Beth tableaux, and was systematically developed and given its present form by Smullyan, as a left-sided Beth tableau, was first presented to logicians by Smullyan in its full development in 1968, and to logic students in 1967 by Jeffrey in his textbook, while van Heijenoort developed the falsifiability tree as the dual of the truth tree presented by Jeffrey and Smullyan, as a modification of the right-sided falsehood tree of the Beth tableau. A study of the history the tree method strongly suggests that Hintikka is owed much more credit than is generally given to him by the standard history. Like most mathematics, the method has a history, and is not the sole work of one person. Indeed, as Abeles [1989] shows, there are syllogistic versions of the trees of Beth tableaux to be found in the work from the last century of Charles Dodgson. Perhaps this valuable contribution to proof theory ought to be called the *Hintikka-Smullyan tree method*, or even the *Dodgson-Hintikka-Smullyan tree*, rather than the Smullyan tree. Likewise, and for similar reasons, perhaps the falsifiability tree could be called the *van Heijenoort confutation tree*.

Note.

*Inqueries of the logicians who might have been in the New York City area at the time in question failed to clarify the situation. Elliott Mendelson, who was then, as now, at Queens College, wrote [1990] that he has "no recollection of a talk on model sets (or any other topic) by Jaakko Hintikka in New York in 1962", and he suggested that Martin Davis might be of some help. Martin Davis was at Yeshiva University in 1962, along with Smullyan; Davis [1990] wrote that he recalls a visit by Hintikka to Princeton some time during the academic years 1952-54 (since Hintikka was a guest in his home at the time), but he has no recollection of a visit by Hintikka to New York City in 1962. He suggested contacting Mendelson. The only other logician of note in New York at the time was Jean van Heijenoort. A search of the van Heijenoort papers at the Archives of American Mathematics at the University of Texas at Austin by archivist Frederic F. Burchsted failed to find any documentation (lecture notes, etc.) that could confirm or disconfirm Hintikka's presence in New York at the time or that he gave a lecture [Burchsted 1990]. Mitsuru and Ann Yasuhara did not arrive in New York until 1965 [Yasuhara 1989]. Hao Wang, who organized the occasional logic seminars at Rockefeller University in the second half of the 1960s that were attended by the logicians of the New York City area, wrote [1989] that he did not arrive in New York until 1966.

Hintikka has not yet responded to the inquiry of whether he might not have given the talk on model sets in the New York City area a decade earlier than he initially remembered. that is, some time during his visit to Princeton in 1952-1954.

Acknowledgements.

I am most grateful to George S. Boolos. Melvin C. Fitting. Jaakko Hintikka. Richard C. Jeffrey, Hugues Leblanc, and Raymond M. Smullyan for their assistance, and for sharing with me their recollections, ideas, and copies of various documents relating to this study. Boolos and Leblanc presented their latest work on Smullyan trees in the AMS special session on proof theory, Denver, 1983, organized by the author. I am also grateful to Frederic F. Burchsted, archivist of the Archives of American Mathematics at the University of Texas at Austin, for making available to me materials in the van Heijenoort collection of the Archives. Thomas Drucker kindly wielded his expert editorial blue pencil.

This paper is dedicated to Michael Marlies, who first taught me the tree method, and especially to the memory of my Doktorvater Jean van Heijenoort (1912-1986), in whose classes I received not only many of his writings mentioned here, but also a knowledge and love of the tree method.

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