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LOGIC OF ACTUAL INFINITY AND G. CANTOR'S DIAGONAL PROOF OF THE UNCOUNTABILITY OF THE CONTINUUM

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ABSTRACT. The idea of actual infinity is central to our contemporary understanding of the foundations of mathematics. Notwithstanding this fact the notion has resisted precise and universally accepted definition. The well-known methodological principle enunciated by Wittgenstein that "mathematical statements get any meaning they may have from rule-governed activities that involve them" here receives application in elucidating the logical and mathematical nature of actual infinity within the framework of Cantor's diagonal proof of the uncountability of the continuum.

Since Cantor first constructed his set theory, two *independent* approaches to infinity in mathematics have persisted: the Aristotle approach, based on the *axiom* that "*all infinite sets are potential*," and Cantor's approach, based on the *axiom* that "*all infinite sets are actual*." A detailed analysis of the "rule-governed" usage of 'actual infinity' reveals that Cantor's diagonal proof is based on two hidden, but nonetheless *necessary* conditions never explicitly mentioned but in fact algorithmically used both in Cantor's so called "naive" set theory as well as modern "nonnaive" axiomatic set theories. An examination of "rule-governed" usage of the first necessary condition opens the way for a rigorous proof that in reality Cantor's diagonal procedure proves nothing, and merely reduces *one* problem, that associated with the uncountability of real numbers (the continuum), to *three new and additional problems*. The second *necessary* condition is simply a *teleological* one possessing no real relation to mathematics.

Further analysis reveals that Cantor's Diagonal Method (CDM), being the only procedure for distinguishing infinite

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sets on the basis of their cardinalities, does not distinguish *infinite* from *finite* sets just on the basis of the number of their elements (cardinality); the results of CDM depend fatally upon the order of real numbers in the sequences to which it is applied. Cantor's diagonal proof itself is formally a "*half*" of the well-known "Liar" paradox but which can be used to produce a *new* set-theoretical paradox of the "Liar" type.

The "rule-governed" usage of 'actual infinity' within the framework of Cantor's theory of *transfinite ordinal* "integers" shows that the mathematical sense of his transfinite ordinals is expressible using the Aristotle-Peano axiom that "if a 'thing' is an integer then 'thing' + 1 is an integer as well," defined only for *finite* natural numbers regardless of the referent of 'thing.' It will follow that the "ordinal" part of Cantor's "Study on Transfinitum" is consistent and irrefutable; a 1-1–correspondence, indeed, exists between the series of the common *finite* natural numbers and Cantor's series of *transfinite* ordinals, *preserving the natural order* of the finite natural numbers; i.e., the series of common *finite* natural numbers and the series of Cantor's *transfinite* ordinals up to the famous Cantor's transfinite ε_0 are homothetic (similar) and therefore their differences are inessential mathematically speaking. The mathematical and logical legitimacy of the operation '+1' applied to Cantor's 'omega' is neither more nor less than the mathematical and logical legitimacy of the operation '/2' applied to a common point of Euclidean geometry.

The fact to be demonstrated is that ultimately Cantor's diagonal proof engages us in an endless, *potentially* infinite, and quite senseless paradoxical "game of two honest tricksters" (a new set-theoretical paradox) which, as Wittgenstein alleged, "has no relation to what is called a deduction in logic and mathematics."

Here it is argued that Cantor's proof does not in fact prove the uncountability of the continuum, but rather proves something else entirely, viz. Aristotle's Thesis (stated in its later canonized Latin form): "*Infinitum Actu Non Datur.*" In other words, it proves that an actual infinity "*is never permitted in mathematics*" (Gauß), or alternatively speaking, that in the words of Poincaré "*there is no actual infinity; Cantorians forgot that and fell into contradictions. [...] Later generations will regard set theory as a disease from which one has recovered!*" Stated in the more compelling language of Brouwer: what we conclude is that *Cantor's theory as a* *whole is "a pathological incident in the history of mathematics from which future generations will be horrified."*

Contents

1. *Introduction.* ACTUAL INFINITY: THE QUESTION IS NOT CLOSED.

Infinity is one of the most enigmatic as well as one of the most important conceptions in mathematics—which is, by right, called the "science on infinity" since the ancient Greeks' time—as well as logic, philosophy, cognitive science, and the intellectual history of humankind more broadly construed.

One of the most outstanding mathematicians of the twentieth century, D. Hilbert, characterizes the important epistemological aspect of the problem of infinity in the following oracular terms: "... the final elucidation of the essence of infinity oversteps the limits of narrow interests of special sciences and, moreover, that became necessary for the honour of the human mind itself" [Hilbert 1948]. Unfortunately, the problem of "the final elucidation of the essence of infinity" is hitherto far from its final solution. I shall try to ground this statement in this Introduction.

Two sorts of infinity, opposed in nature, have been distinguished: the *potentially infinite* and the *actually infinite.* This distinction was explicitly introduced for the first time by Aristotle more than two millenniums ago [Aristotle]. In regard to the Aristotelian definition of *potential infinity*, it will be shown in what follows that it was in fact quite rigorous, even from a contemporary point of view. A potential infinity, understood as a *process* which can in principle be continued forever, can never produce "a complete and existing entity" *as its final result* [Aristotle]. Such a conclusion was generally accepted in mathematics and from its inception posed few, if any, problems.

The earliest example of potential infinity was the series of common finite natural numbers:

$$\text{(NN)} \qquad 1, 2, 3, \ldots, n, \ldots$$

How to extend the series is clear enough: for any natural number, n, there exists another, $n+1$, that is greater than all previous numbers, $1, 2, 3, \ldots, n$. Consequently there is no maximal, last element of the series (**NN**). Imagining the series (**NN**) completed would entail a flagrant contradiction: indeed, a step-by-step process of constructing the series (**NN**) may, allegedly, be completed, but a 'halting' state of the process cannot be shown. Such a situation conflicts not only with our primitive intuitions, but with the millennial practice of the humankind interaction with and cognition of the outer world, since according to Hilbert (and to modern science) "infinity is realized nowhere in physical world" [Hilbert 1948].

Since the time of Aristotle there have been on going debates over matters such as whether an actual infinity exists, what such an actual infinity might be, and the mathematical treatment of the actually infinite, generally. Since the notion of an actual infinity was never strictly defined, all such debates were largely in vain.

Cantor was the first to explicitly introduce the concept of an actual infinity in mathematics. According to him, "the actual infinity is such the quantity that, on the one hand, is not changeable, but is definite and invariable in all its parts and represents a veritable constant magnitude, however, on the other hand it surpasses by its magnitude any finite magnitude of the same kind" [Cantor 1914]. There are a number of other definitions of the 'actual infinity' to be found in Cantor, but all are based on vague or speculative intuitions, just as are the more contemporary arguments in favor of the actual infinity. However vague the definitions may have been, Cantor and his followers declared the series (**NN**) an actually infinite object along with any other infinite sets.

By the end of the nineteenth century and into the early twentieth century, Cantor's actualization of infinite sets generated a number of set-theoretical paradoxes and led to the Third Great Crisis in foundations of mathematics. Much work of a remarkably high quality describing the Crisis, its reasons and possible ways of resolving it has been written—much of it by key players in the ensuing debates (see, e.g., historical reviews in [Fraenkel and Bar-Hillel 1958], [Bourbaki 1965], [Kleene 1957], and [Kleene 1967]). We can here address only a few points specifically relevant to what is to follow.

1. For about two millenniums, the idea of the actually infinite was criticized and rejected based for the most part on Aristotle's dictum "*Infinitum Actu Non Datur*"—his main statement in its canonized later Latin formulation. Critics included, among other, Leibniz, Berkeley, Locke, Descartes, d'Espinosa, Gauß ("*I must protest most vehemently against [the] use of the infinite as something consummated, as this is never permitted in mathematics*"), Kant, Cauchy, Kronecker ("*I don't know what predominates in Cantor's theory—philosophy or theology, but I am sure that there is not any mathematics here*"), Hermite, Poincaré ("*There is no actual infinity; Cantorians forgot that and fell into contradictions*" [Poincaré 1983]), Bair, Borel, Brouwer (*Cantor's theory as a whole is "a pathological incident in history of mathematics from which future generations will be horrified"* [Fraenkel and Bar-Hillel 1958]), Quine, Wittgenstein, Weyl, Luzin, etc., many of whom were *creators* of what S. Feferman, aptly describes as "*"really working"* mathematics [Feferman 1998].

On the other hand, after Cantor, a pleiad of outstanding twentieth century metamathematicians and set theorists including Hilbert ("*No one shall expel us from the paradise which Cantor has created for us!*" [Hilbert 1948]), Gödel, Turing, Church, Tarski, etc., with their famous "negative" theorems and paradigmatic epistemological conceptions constructed the great edifice of modern *meta*mathematics and axiomatic set theory basing it on Cantor's transfinite conceptions and, most importantly, on his idea that "all infinite sets are *actual*."

So, we happen upon an almost unique situation in the history of mathematics where one large group of outstanding mathematicians and logicians would have nothing to do with the very idea of an actual infinity, while another group, equally distinguished, stood resolutely opposed.

One at first might be inclined to say that only modern Cantorians know the truth about infinity, simply because they alone enjoy possession of the greatest accumulation of all the humankind wisdom. However, infinity is a very special 'thing' in the sense that no amount technical progress will likely ever provide us with an understanding equal by comparison to what we have come to know of what Kant described as the "starry sky over a head and the moral law in our soul"; this notwithstanding the fact that Aristotle (not to mention Gauß, Kronecker, Poincaré, Weyl, Brouwer, etc.) knew more about the mathematical nature of the infinity than many modern day apologists for Cantor's paradigm of the actual infinity (see below).

In a word, as regards the ancient problem of the true (mathematical) nature of the actual infinity, the question remains unresolved.

In this connection it seems quite doubtful a widespread metamathematical opinion that "Cantor's view [as to the actualization of all infinite sets—AZ] prevailed and modern mathematics accepts actual infinity" [Wikipedia]. Indeed, a number of well-known mathematicians and logicians remain in a state of disagreement. For example, one outstanding logician and expert on the foundations of mathematics, S. Feferman writes in his recent (and remarkable) book [Feferman 1998]:

> But there are still a number of thinkers on the subject who in continuation of Kronecker's attack, object to the panoply of transfinite set theory in mathematics [...] In particular, these opposing points of view reject the assumption of an actual infinity (at least in its nondenumerable forms). Following this up, alternative schemes for the foundations of mathematics have been pursued in a direct and straightforward way on philosophically acceptable non-Cantorian grounds.
>
> Furthermore, a case can be made that *higher set theory is dispensable in scientifically applicable mathematics*, i.e., in that part of everyday mathematics which finds its applications in the other sciences. Put in other terms: *the actual infinite is not required for the mathematics of the physical world.*

J. Peregrin ("*There is not an actual infinity*" [Peregrin]), V. F. Turchin ("*For actual infinity we have no place in [...] the global cybernetic theory of evolution and in the constructivist foundation of mathematics*" [Turchin]), and many other modern experts in foundations of science today adhere to a rejection of the actual infinity conception. So we see that the acceptance of actual infinity is by no means unanimous in contemporary mathematics. This is worth examining in greater detail.

2. In the mid-twentieth century Cantor's original set theory was declared "naive" (see, e.g., [Kleene 1967]). Following Kleene, modern

axiomatic set theory assumed that "set theoretical paradoxes are connected with the introduction of sets that are "too large," e.g., sets like the set T of all sets" [Kleene 1967]. But this is almost certainly incorrect. Indeed, the cardinality of the set T must be greater than the cardinality of any other set, but, according to Cantor's theorem, the cardinality of all subsets of T is greater than the cardinality of the initial set T of *all* sets (Cantor's paradox). However, the paradoxical conclusion obtains if and only if the set T is an *actually* infinite object, since if T were *potentially infinite* then, in accordance with Aristotle's view, the question itself as to a differentiation of infinite sets by their cardinality simply never arises. The same holds in the case of the known Burali-Forti paradox with respect to an ordinal number of all ordinal numbers. The real basis of the set-theoretical paradoxes is just the *actuality* of *infinite* sets, and not that *any* predicate $P(x)$ generates a definite set of all x, for which $P(x)$ is true ([Cantor 1914], [Bourbaki 1965], [Kleene 1967], [Zenkin 2001], [Zenkin 1997a], [Zenkin 1997b], [Zenkin 1997c]).

However that may be, following the declaration that Cantor's set theory is "naive," the notion itself of actual infinity disappeared from logical analysis, metamathematical and set-theoretical textbooks and tractates, regarded as speculative, vague, and fuzzy. Consider the following which is the key axiom of any modern "nonnaive" axiomatic set theory.

Axiom of Infinity ([Fraenkel and Bar-Hillel 1958], [Kleene 1967])**.** There exists at least one infinite set—the set $\{1, 2, 3, \dots\}$ of natural numbers.

N. Bourbaki has an even shorter formulation:

Axiom of Infinity ([Bourbaki 1965])**.** There exists an infinite set.

If one attempts to get at whether these axioms require an actual or potential infinity one will remain at a loss. For it is only in the historical literature that the alternative is profitably discussed. On the other hand, every modern metamathematician and set-theorist knows all too well that if the "infinite set" of these axioms is potential, then the set-theoretical theories of transfinite ordinals and cardinals can be deleted without any prejudice for "really working" mathematics.

3. Any modern metamathematical and set-theoretical tract contains reference to Cantor's famous theorem that for any set M the cardinality of the set of all subsets of M is greater than the cardinality of the initial set M, and "the most important partial case of this theorem—the theorem on the uncountability of the continuum" ([Cantor 1914],

[Alexandrov 1948]). This theorem is the sole logical basis allowing modern set theorists to distinguish infinite sets on the basis of their cardinality. On the other hand, the theorem is the "Trojan horse" of a "naive" Cantorian set theory, one contained within the framework of the modern "nonnaive" axiomatic set theory in the sense that the theorem is not provable if the corresponding infinite sets are potentially infinite. In other words, accepting the theorem requires that axiomatic set theory itself include Cantor's conception of the *actualization* of *all* infinite sets. Thus, the problem of the logical legitimacy of actual infinity in mathematics is real as well as unavoidable—independently of whether modern *meta*mathematics and axiomatic set theory include discussion of the problem.

However, the axiomatic set theorists are absolutely right in stating that all existing definitions of 'actual infinity' are unsatisfactory and have no role to play in serious logical and mathematical analysis ([Fraenkel and Bar-Hillel 1958], [Bourbaki 1965], [Kleene 1957]). In such a circumstance, in order to reveal the true sense of the actually infinite and to render the notion accessible to nonspeculative analysis, there is only one way, which, strictly speaking, has been well known for a long time, but which was probably best formulated by L. Wittgenstein [Wittgenstein 1956] (cited by [Hodges 1998]): "Mathematical statements get any meaning they may have from rule-governed activities that involve them."

In the paper we shall consider the central notion of modern "nonnaive" axiomatic set theory, viz. Cantor's famous theorem regarding the uncountability of the continuum and his diagonal procedure for proving this theorem. In addition there will follow a detailed logical analysis of the nature of actual infinity based on its real, algorithmical, "rule-governed" usage *within the framework of Cantor's diagonal argument*, rather than the traditional "fuzzy" conception.

2. Aristotelian and non-Aristotelian (Cantorian) mathematics.

Consider, then, the main problem of mathematics—the problem of infinity. As noted earlier, it was Aristotle who first explicitly distinguished two opposing types of infinity—*potential* infinity and *actual* infinity. The logical essence of potential infinity was defined by Aristotle as follows: "...the infinite exists through one thing being taken after another, what is taken being always finite, but ever other and other" [Aristotle] (cited by [Moore 1993]). With little effort it can be shown that Peano's now famous axiomatic system is a literal but natural formalization of the Aristotelian definition of the potential infinite.

Indeed, from Aristotle's definition of the potentially infinite we have:

(1) there is a first 'thing,' say, '0' (since any ordered *finite* sequence of 'things' has a first 'thing');
(2) if n is a 'thing' then $n+1$ is a 'thing' too (since "one thing $[n+1]$ being taken after [>] another $[n]$ ...");
(3) there are no other 'things' aside from those derived by means of (1) and (2).

(1)–(3) represent the *inductive definition* of the series (**NN**) of the common *finite* natural numbers in its modern form [Kleene 1957].

Adding to (1)–(3) the later axioms

(4) for any 'things' n and m, if $n = m$ then $n+1 = m+1$;
(5) for any 'thing' n, $n+1 \neq 0$;
(6) $[P(0)\ \&\ [\forall n\,[P(n) \to P(n+1)]]] \to [\forall n\, P(n)]$,

we arrive at Peano's axiomatic system for arithmetic from which, according to Poincaré, "*almost all mathematics can be deduced*" [Poincaré 1983]. In fact, one cornerstone of the arithmetic just is Aristotle's inductive definition of the potential infinite series (**NN**) of the common *finite* natural numbers; and this is a fact that must be given considerable emphasis.

As for the actually infinite, Aristotle postulated his famous, and well argued for, thesis: "there will not be an actual infinite. ...the infinite has a potential existence" [Aristotle], or, briefly, "*Infinitum Actu Non Datur.*" So, according to Aristotle, actual infinity is simply the logical negation of potential infinity and, as such, is a scientific impossibility in addition to being logically flawed. In a word, the actually infinite is self-contradictory, i.e., it is something like a "finitized infinity," see below.

However, from the mathematical standpoint, Aristotle's statement that "the infinite has a potential existence" was never proved mathematically, amounting to a highly intuitive guess by a great genius, nothing more or less. Statements which are accepted and used in mathematics without a proof are called "axioms"; therefore, Aristotle's guess must be regarded as an "axiomatic" statement and must be written in the language of modern mathematics as follows.

Aristotle's Axiom (third century B.C.). *All infinite sets are potentially infinite.*

In the second half of the nineteenth century, Georg Cantor "rejected the [scientific] authority of Aristotle, Leibniz, Gauß, Cauchy, and others" who objected categorically against the actual infinity and declared the *contradictory* thesis: "all infinite sets are actual" ([Cantor 1914], [Cantor 1985]).

However that may be, since Cantor's thesis is so far unproven (but is implicitly used in all modern "nonnaive" axiomatic set theories without a proof), it, too, must be rewritten in *modern mathematical* language as follows.

Cantor's Axiom (nineteenth century A.D.). *All infinite sets are actually infinite.*

Thus, in reality we now have not merely one, but two mathematical systems: first, the classical, Aristotle-Peano mathematics of infinite sets, based upon Aristotle's Axiom, and, secondly, non-Aristotelian (Cantorian) mathematics based upon Cantor's Axiom, that is, the logical negation of the Aristotle's own Axiom.

Just the axiomatic character both of the statements explains, in particular, why during roughly two millenniums nobody has been able to prove the actuality of infinity, nor have they been able to disprove its potential character.

From the very invention of Cantorian set theory in the 70s of the nineteenth century, these two mathematical conceptions lead an independent existence: indeed, the really working Aristotle-Peano mathematics used never, even to this day, the "paradigmatic achievements" of transfinite Cantorian mathematics (see [Feferman 1998], [Turchin]).

This situation resembles the dramatic history of Euclidian and non-Euclidian geometries based as they were on acceptance or rejection of Euclid's Fifth Postulate regarding parallels. However, there remains an important difference: the consistency of non-Euclidian geometries was strictly proved, but the consistency of modern "nonnaive" axiomatic set theory based upon Cantor's Axiom has so far gone unproven. This means that there is a nonzero possibility that Cantorian 'transfinite mathematics' is inconsistent and will therefore ultimately be rejected.

The fact that statements as to either the actuality or potentiality of infinity have the character of axioms also explains exactly why Aristotle's adherents (first of all Kronecker and Poincaré) failed to exclude Cantorian actual infinities from mathematics.

3. The rule-governed usage of the 'actual infinity' conception within the framework of Cantor's diagonal proof.

As it was said above, Cantor's epochal achievement was his now famous theorem showing the uncountability of real numbers or, what amounts to the same thing, the uncountability of the continuum. It must be underscored once again that Cantor's theorem is a cornerstone of all current metamathematics and all current "nonnaive" axiomatic set theories in the sense that without this theorem any talk itself of distinguishing infinite sets by the number of their elements (i.e., by their cardinality) is reduced to nonsense [Kleene 1967].

The value of Cantor's theorem for "naive," as well as "nonnaive," axiomatic set theory is defined also by the other fact. Usually, when mathematicians talk about set theory, they mean something whole. But in reality, set theory consists of two absolutely independent parts: the theory of transfinite *ordinal numbers*, and the theory of transfinite *cardinal numbers*. There is only a bridge between these two theories, and that is just Cantor's theorem regarding the uncountability of the continuum and its direct consequence—the famous Cantorian continuum hypothesis, subject matter of Hilbert's First Problem ([Zenkin 1997a], [Cohen 1966], [Zenkin 1999], [Zenkin 1997d], [Zenkin 1998]). As Kleene observes, the idea itself of a comparison of infinite sets on the basis of a number of their elements would be of little interest were it not for Cantor's theorem on the quantitative nonequivalence of the sets of natural and real numbers [Kleene 1967]. This theorem is based on Cantor's axiom inasmuch as without it the theorem is simply unprovable. It is well worth considering here that in modern "nonnaive" axiomatic set theory this fact is hardly, if ever, explicitly mentioned.

Now we shall examine in some detail how "rule-governed" usage of the *actually* infinite is realized within the framework of this proof of Cantor's theorem.

To begin with, consider Cantor's traditional diagonal proof ([Cantor 1914], [Alexandrov 1948], [Hodges 1998], [Capiński and Kopp 1999]). Below $N = \{1, 2, 3, \dots\}$, X is the set of all real numbers (points) belonging to the segment $[0, 1]$ and, for simplicity, we shall use the binary system to represent real numbers. Note that the further conclusions of this paper hold for any radix greater than 2 (for details see [Zenkin 2000a], [Zenkin 2001], [Zenkin 1997b]). In the curly braces $\{\mathbf{L}:\}$ we place a label $\mathbf{L}$ for further references to the sentence positioned immediately following the label.

Cantor's Theorem (1890). {**A**:} The set X is *uncountable.*

Proof (by the Reductio ad Absurdum method).
Assume that {¬**A**:} the set X is *countable.*
Then there is an enumeration of *all* real numbers of X.
Let {**B**:}

(**E**) $$x_1, x_2, x_3, \ldots$$

be an *arbitrary* enumeration of *all* real numbers of X.

Consider a so-called 'diagonal' of the enumeration (**E**),

(**D**) $$0.x_{11}x_{22}x_{33}\ldots x_{nn}\ldots,$$

i.e., consider the *infinite* sequence of binary digits, where the digit x_{11} is the first digit of the first real number x_1 in the *given* enumeration (**E**), the digit x_{22} is the second digit of the second real number x_2 in the enumeration (**E**), the digit x_{33} is the third digit of the third real number x_3 in the enumeration (**E**), and so on.

Next, Cantor constructs (creates, defines, etc.) a *new infinite* sequence of binary digits,

(**anti-D**) $$y_1 = 0.y_{11}y_{12}y_{13}\ldots y_{1n}\ldots,$$

by application of the *constructive algorithm* of his famous diagonal method in the form of the following "diagonal rule" to the 'diagonal' (**D**):

(**R**) $$\textit{for any } i > 1:$$
$$[[\text{if } x_{ii} = 0, \text{ then } y_{1i} := 1] \text{ and } [\text{if } x_{ii} = 1, \text{ then } y_{1i} := 0]].$$

where the notation '$a := b$' means "to give the value 'b' to the variable (place) 'a'" as is usually done in any modern algorithmical language.

It is evident that an *infinite* sequence of binary digits, such as y_1, will by definition be a real number belonging to the set X and will by its construction differ from the first real number x_1 of the enumeration (**E**) in at least its first digit y_{11}, from the second real number x_2 of the enumeration (**E**) in at least its second digit y_{12}, and so on. So, the (*anti*)diagonal real number y_1 will differ *from each* (or *from any*) real number within the enumeration (**E**), and—*as a consequence*(!)—*from all* real numbers of the enumeration (**E**). This necessitates the conclusion that:

{¬**B**:} the *given* enumeration (**E**) is **not**
an enumeration of *all* real numbers of X.

The contradiction between $\neg\mathbf{B}$ and $\mathbf{B}$ proves—according to the metamathematical version of the Reductio ad Absurdum (further—RAA) method [Kleene 1957]—that the assumption $\neg\mathbf{A}$ is false. $\square$

So, the key points of Cantor's proof are as follows.

(1) The ultimate aim of the diagonal method is *to create a new individual* mathematical object which a) is a real number of X and b) differs *from all* real numbers of a given *infinite* sequence $(\mathbf{E})$;

(2) The contradiction in the proof is between a formal consequence (here: $\mathbf{B}$) of the assumption, i.e., of a *nonauthentic* premise (here: $\neg\mathbf{A}$), and its *formal* consequence (here: $\neg\mathbf{B}$), i.e., a contradiction of the following quite specific form:

$$[\mathbf{B} \to \neg\mathbf{B}].$$

It should be remarked here that, all other contradictions of the form $[y_1 \in (\mathbf{E})$ (due to assumption)$]$ & $[y_1 \notin (\mathbf{E})$ (due to construction)$]$ or $[y_1 \neq y_1]$ (by virtue of the same reason) that are usually used in traditional metamathematical tracts and set theoretical textbooks, are *secondary* consequences just of the *initial* contradiction, $[\mathbf{B} \to \neg\mathbf{B}]$.

(3) The falsity of the *nonauthentic*(!) consequence, $\mathbf{B}$, of the *nonauthentic*(!) assumption $\neg\mathbf{A}$ is proved not by RAA, but by means of the law of contradiction: Cantor's (*anti*diagonal) real number y_1 is a counterexample to the *common* statement $\mathbf{B}$.

(4) Strictly speaking, the 'diagonal rule' $(\mathbf{R})$ says that the real number y_1 differs from *every* (or from *any*) *given* real number x_i of the *given* sequence $(\mathbf{E})$. The statement that the number y_1 differs from *all* real numbers of the *infinite* sequence $(\mathbf{E})$ is a *consequence* based on the quite specific "metalogical" rule:

(R1) If $P(m)$ holds 'for *every*' ('for *any*') *given* $m \in M$ then $P(m)$ holds 'for *all*' $m \in M$.

It should be remarked here that, of course, the statement "$P(m)$ holds 'for *every*' ('for *any*') *given* $m \in M$" is a formal consequence of the statement "$P(m)$ holds 'for *all*' $m \in M$"; however, the inverse statement $(\mathbf{R1})$ implies that the set M is either *finite* or *actually* infinite (see below). It is common knowledge that the universal quantifier in $\forall m\, P(m)$ tells us nothing about how many elements are in the domain M of the interpretation of the variable m, i.e., whether M is finite or infinite. It's also evident that the universal quantifier fails to reflect the obvious semantical distinction between the two (contradictory!) cases, viz. where the domain M is an *actually* infinite set and, secondly, where the domain M is *potentially* infinite. Within the framework of Cantor's proof, the set X and the sequence (**E**) are *actually* infinite and only therefore the "metalogical" rule (**R1**) is applicable. However if the domain, M, is considered as a *potentially* infinite set, then the logical legitimacy of the rule (**R1**) becomes doubtful and requires serious clarification or emendation (concerning the details of the problem see [Zenkin 1997b], [Brouwer 1954]).

4. Three odd peculiarities of Cantor's Diagonal Method (CDM).

4.1. **The CDM does not distinguish finite and infinite sets by their cardinalities.** [Zenkin 2000a].

It is obvious that if the enumeration (**E**) is finite, then its diagonal (**D**), and the corresponding "antidiagonal" sequence (**anti-D**) will also be finite, and in such a case neither (**D**) nor (**anti-D**) defines any real number at all in the accepted mathematical sense of 'real number' as an infinite sequence of (binary, decimal, or any other) digits. The inapplicability of CDM to finite enumerations has in the past always appeared so obvious that nobody ever made the effort to examine the issue. This is, however, merely symptomatic of an ingrained *psychological stereotype* attributable to traditional *metamathematical* thinking.

Indeed, as is generally conceded, the final result of an algorithm's work can not depend upon a concrete place where intermediate as well as the final results are produced and saved (in memory, on paper, in a computer memory, and the like); otherwise, one and the same algorithm would produce different final results, depending on facts extraneous to mathematics such as the mental health of a human being, the size of the paper or its quality, or more significantly on preceding states of the computer's memory and the current "mood" of a computer, etc. Keeping this in mind, let's take an arbitrary *infinite* binary sequence as an initial "half-finished product" for CDM-building the Cantorian (antidiagonal) real number y_1. The simplest of such sequences is obviously the identical (real) zero:

(**D0**) $$0.0_1 0_2 0_3 \ldots 0_n \ldots,$$

Now apply Cantor's 'diagonal rule' (**R**) to the enumeration (**E**) taking the following equivalent form:

(**R2**) $$\text{for } any\ i \geq 1:$$
$$[[\text{if } x_{ii} = 0, \text{ then } 0_i := 1] \text{ and } [\text{if } x_{ii} = 1, \text{ then } 0_i := 0]].$$

where 0_i is the ith digit of the "half-finished product" (**D0**); that is, we shall use the (**D0**) as a *place* for the storage of the digits of Cantor's (*anti*)diagonal real number constructed by means of applying CDM to (**E**). It is easy to see that now, literally repeating Cantor's diagonal reasoning, we shall get a real number which identical to Cantor's real number y_1.

But what will happen just in case the enumeration (**E**) is *finite*? It is obvious that while applying Cantor's diagonal method in the form (**R2**) to any *finite* enumeration of some real numbers from X, say, to

$$\textbf{(E1)} \qquad x_1, x_2, x_3, \ldots, x_n,$$

we shall obtain a *rational number* (even with the zero "tail"):

$$z_1 = 0.z_{11}z_{12}z_{13}\ldots z_{1n}000\ldots$$

This number is an element of X, and differs *from each (any)* real number of the *finite* enumeration (**E1**), and, according to the rule (**R1**), *as a consequence* ... it differs *from all* elements of the *finite* enumeration (**E1**).

Thus, Cantor's diagonal method is precisely applicable to both *infinite* and *finite* enumerations *without alteration.* Consequently, the method does not distinguish and nor does it take into account *quantitative* characteristics of those sets and enumerations which the method is applied to. We arrive at the following conclusion, strange as it may be from the classical logic point of view: the *only* method, which hitherto allows metamathematicians "successfully" to differentiate *infinite* sets according to their *transfinite* "powers/cardinalities," i.e., according to the number of their elements, does not differentiate *finite* from *infinite* sets just according to their "powers/cardinalities"!

If so, then which properties of the *infinite* enumeration (**E**) does Cantor's diagonal method take into account and explicitly, algorithmically employ? It is obvious that it can be only a property which is *the same both for finite and infinite sets*, viz. the property of 'being actual.' The nature of this property from the algorithmical point of view will be examined (see §§5 and 6), but for now we may conclude that the condition that application of the 'diagonal rule' (**R2**) to *every (any)* element of a given enumeration (**E**) generates an *individual* mathematical object y_1 *as a final result*, i.e., the requirement of the *actuality* of the infinite set, X, and the infinite enumeration (**E**), allows Cantor to assert that his *new* real y_1 differs *from each (any)* element of the given enumeration (**E**). Then, according to the quite doubtful rule (**R1**), it follows that Cantor's real y_1 differs *from all* elements of the given enumeration (**E**), and therefore, the enumeration (**E**) does not contain *all* elements of the set X.

Thus, it is only this property of the *actuality* that enables Cantor to achieve the desired contradiction his proof requires, taking the form, as above indicated, of that quite strange deductive "implication" $[\mathbf{B} \to \neg\mathbf{B}]$.

It is appropriate, nonetheless, to emphasize that there are very different contradictions in logic, and every contradiction has its own reason, origin, and structure. Analyzing carefully such reasons, origins and structure is much to be desired, before some *may* claim: "Oh! A contradiction is obtained, *consequently* ...!"—This is particularly true in connection with such a power technique as the classical RAA-method applied to such fine 'things' as the *actually* infinite sequences of real numbers.

4.1.1. *Remark. Some detailed points concerning the logic of Cantor's RAA-proof.* Indeed, within the framework of Cantor's RAA-proof, the desired falsehood of the assumption $\neg\mathbf{A}$ follows not from the obtained contradiction $[\mathbf{B} \to \neg\mathbf{B}]$. As has already been said, from that contradiction and the *independently proven truth* of the formal consequent $\neg\mathbf{B}$ it follows that the antecedent $\mathbf{B}$ is false by the contradictory law, but not by way of the traditional propositional calculus with its tabular "if consequent is true then antecedent is false" which is here simply irrelevant and is empty talk. And only then the desired falsehood of the assumption $\neg\mathbf{A}$ follows from the (now *proven*) *falsity* of the formal consequence $\mathbf{B}$ arrived at from the premise $\neg\mathbf{A}$ by the classical *modus tollens* rule. Thus, we see that the logic of Cantor's RAA-proof is not trivial and has some quite subtle distinctions when compared with Aristotle's classical Reductio ad Absurdum ([Zenkin 2001], [Zenkin 1997a], [Zenkin 1997b]).

In this regard, the logical explication of the real "rule-governed" usage of Cantor's famous diagonal method shows that the method makes *explicit* (but tacit) use of the *actuality* property of *infinite* sets and enumerations, while making no reference to their *quantitative* properties. This fact necessitates a deeper consideration of the issues.

4.2. **The results associated with applying CDM depend crucially upon the order of real numbers in the sequences to which CDM is applied.** [Zenkin 1997a].

As we have seen above, application of CDM to the countable sequence (**E**) generates an infinite sequence of binary digits (**anti-D**) which

(1) by definition, is a *new* real number $y_1 \in X$ and

(2) by its construction, differs *from all* real numbers of the sequence (**E**).

These two conditions, (1) and (2), by themselves entail the conclusion that the real number y_1 is not an element of the given enumeration (**E**), i.e., $y_1 \notin$ (**E**).

Now, based on the well-known clauses of Cantor's theory of transfinite ordinals, we transform the given *countable* sequence (**E**) into the following new *countable* sequence of *the same* elements:

$$\textbf{(E2)} \qquad x_2, x_3, \ldots, x_1,$$

that is, we simply remove the first real number x_1 after all other real numbers in the initial actually infinite enumeration (**E**).

It is obvious that the countable enumeration (**E2**) comprises all elements of the initial enumeration (**E**) and differs from (**E**) only in *the order* of only one of its elements.

Now we apply CDM to the *countable* enumeration (**E2**). It is easy to see, in such a case that we will obtain the following infinite sequence of digits:

$$y_1^* = 0.y_{12}y_{13}y_{14}\ldots y_{11},$$

This is a countable sequence consisting in *the same binary digits* as Cantor's real number y_1, but now the ordering is of Cantor's ordinal $(\omega + 1)$-type. Obviously such a sequence is not a common real number. By definition, it is a *nonstandard real* number ([Davis 1980], [Uspensky 1987]), one which is not an element of the set X. Consequently, though the infinite binary sequence y_1^* differs in construction from any of the real numbers in the enumeration (**E2**) as well as any in Cantor's (**E**), it now provides no reason to reject the assumption $\neg$**A** that the set X is countable.

In this instance Cantor's proof is not valid.

So, the result of applying CDM to a countable sequence of real numbers essentially *depends upon the order of elements* in such a sequence. It is a quite strange peculiarity of the method which was specially invented by Cantor in order "to prove" just the *quantitative* nonequivalence of *infinite* sets. But, according to Cantor's well-known definition of the cardinality notion, the *quantitative* equivalence or nonequivalence of *infinite* sets can't depend upon the ordering of elements of the sets.

4.3. **Cantor's proof is 'half' the "Liar" paradox.** ([Kleene 1957], [Fraenkel and Bar-Hillel 1958], [Zenkin 2000a], [Zenkin 2000b], [Zenkin 1997e], [Zenkin and Zenkin 2002].)

Consider Epimenides' charming case of the "Liar": someone asserts "I am a liar—who am I?"

If I am a liar then I am lying in stating that "I am a liar." Consequently, I am not a liar.

But if I am not a liar then I tell truth when I say "I am a liar." Consequently, I am a liar.

Where $\mathbf{A}$ = "I am a liar," we have the following canonical schema of the "Liar" paradox (as well as the Russell's paradox concerning the "the set of all sets which are not elements of themselves," etc.):

$$(\mathbf{L}) \qquad [\mathbf{A} \rightarrow \neg\mathbf{A}] \;\&\; [\neg\mathbf{A} \rightarrow \mathbf{A}]$$

Cantor's proof, as we see above, contains the following deductive fragment (here: $\mathbf{B}$ = "an enumeration contains *all* real numbers of X"):

$$(\mathbf{L1}) \qquad [\mathbf{B} \rightarrow \neg\mathbf{B}],$$

Here we have, formally speaking, the left half of the "Liar" paradox ($\mathbf{L}$).

Of course, the "deduction" ($\mathbf{L1}$) is not a quark of theoretical physics; but, as is well known, the like halves were not ever revealed in Nature either. It's natural, therefore, to conjecture that Cantor's formal "implication" ($\mathbf{L1}$) is in reality one half of a new set-theoretical paradox and can be completed to form an entire canonical paradox of the "Liar"-type ($\mathbf{L}$).

Indeed, this can be shown formally: to begin with we substitute $\mathbf{B}$ in ($\mathbf{L1}$) for $\neg\mathbf{B}$, producing the formula $[\neg\mathbf{B} \rightarrow \neg[\neg\mathbf{B}]]$, and then, by double negation (here: $\neg[\neg\mathbf{B}]$ is $\mathbf{B}$), we arrive at the inverse "implication" $[\neg\mathbf{B} \rightarrow \mathbf{B}]$. If within the framework of a system it is possible to deduce the "implication" $[\mathbf{B} \rightarrow \neg\mathbf{B}]$ ($\mathbf{L1}$), then within the framework of *the same* system it is possible to deduce the inverse "implication" $[\neg\mathbf{B} \rightarrow \mathbf{B}]$ and consequently within the framework of such a system it is possible to deduce a complete paradox of the "Liar"-type:

$$(\mathbf{L2}) \qquad [\mathbf{B} \rightarrow \neg\mathbf{B}] \;\&\; [\neg\mathbf{B} \rightarrow \mathbf{B}],$$

Later on it will be shown how such a new set-theoretical paradox ($\mathbf{L2}$) can actually be deduced within the framework of Cantor's original proof (see §9) while conforming to Wittgenstein's "rule-governed" criterion.

5. Algorithmical definition of *actual infinity.*

Now we shall give an interpretation of Cantor's diagonal proof in terms of what we will call a "Cantor-Turing machine" or a CT-machine for short.

We define the CT-machine as follows.

Suppose we have

(1) two boxes: a left box (L-Box) and a right box (R-Box);
(2) a limitless tape T divided into cells.
(3) an arbitrary set X.

In such a case the CT-machine works as follows.

(1) The initial state of the CT-machine: all elements of the set X are placed in the L-Box; the R-Box is empty, and the tape T is positioned in the first cell with the number '1.'
(2) At *every (any)* ith step, $i \geq 1$, the CT-machine:
    (a) removes an element $x \in X$ from the L-Box;
    (b) indexes the element x using the number i of the current step thereby producing the indexed element x_i;
    (c) prints, in the ith cell of the tape T, a pair $\{i, d_i\}$ of natural numbers, where i is the number of a current step and d_i is defined by some specific, individual properties of the element x_i and is explicitly specified by the rule $d_i = f(x_i)$ for *every (any)* ith step; the rule itself is, of course, explicitly defined before the CT-machine begins its work (in Cantor's case, such a rule amounts to being his 'diagonal rule' (**R**) or (**R2**));
    (d) places the element x_i into the R-Box.
(3) The 'halting' condition:

IF the L-Box is *empty* THEN *stop* ELSE go to the step (2a).

As for the 'halting' problem in relation to the CT-machine, there are the following cases.

(1) If the set X is *finite*, then the CT-machine will always eventually reach its STOP-state.
(2) If the set X is *infinite*, then we have the two cases:
    (a) If the *infinite* set X is *potential*, then (by definition) the CT-machine *will never reach* its STOP-state, i.e., the L-Box will never be empty.
    (b) If the infinite set X is *actual*, then the L-Box will eventually be empty and the CT-machine will eventually reach its STOP-state.

So, if the *infinite* set X is *actual*, then the CT-machine reaches its STOP-state and generates as a final result the *actually infinite* sequence of pairs,

(P) $$\{1, d_1\}, \{2, d_2\}, \{3, d_3\}, \dots, \{n, d_n\}, \dots,$$

or, equivalently, two *actually infinite* sequences:

(NN) $$1, 2, 3, \dots, n, \dots$$

and

(D1) $$d_1, d_2, d_3, \dots, d_n, \dots.$$

It is worth mentioning that the sequence (**D1**) contains in a fixed form *information* about every element of the set X in the sequence (**E**) and also about every step of the CT-machine's action.

We are now able to give, for the first time, *algorithmical* definitions of the property 'to be actual' *for finite and countably infinite sets*, i.e., we are now able to define the fundamental notions of modern metamathematics and axiomatic set theory, based on their real, "rule-governed" (Wittgenstein) usage *within the framework of Cantor's diagonal proof.*

Definition 1 of 'actual infinity'. An *infinite* set X is *actual* if and only if the corresponding CT-machine, applied to X, reaches its STOP-state.

The same definition in terms of Cantor's diagonal proof can be formulated as follows.

Definition 2. The *infinite* set X of real numbers is *actual* if and only if the application of a step-by-step algorithm (here—Cantor's Diagonal Rules (**R**) or (**R2**)) to *every (any)* element of the set produces *as a final result* of the algorithm's work *an individual* object (here—a new real number as a *completed, invariable* infinite binary sequence) which is different "*from every*" ("*from any*"), and, due to the rule (**R1**), *as a consequence*, "*from all*" elements of the infinite set X.

6. Algorithmical definition of the *real number* notion.

6.1. **Definition of the real number notion based on *actual* infinity.** From what has been said above we can now give a rigorous definition of the real number concept based on how it is used *within the framework of Cantor's diagonal proof.*

Definition 1. A real number (of the segment $[0, 1]$) is an *infinite* sequence of digits generated by a CT-machine,

(D2) $$d = 0.d_1d_2d_3, \dots,$$

where:

(1) the *infinity* of the sequence (**D2**) is *actual*;
(2) the *actual infinity* of the sequence (**D2**) is *countable*;
(3) the *countable actual infinite* sequence (**D2**) is a well-ordered one of Cantor's ω-type.

6.1.1. *Remark. A hidden vicious circle within Cantor's diagonal procedure.* It is obvious that if even one of the conditions (1)–(3) fails to hold, then Cantor's proof becomes invalid. Now assume an impossible, that is, assume that the set N and the set X are both uncountable. Then the enumeration (**E**) will really contain *all* real numbers of the uncountable set, X. In such a case, the application of Cantor's diagonal procedure to such an enumeration as (**E**) will produce an *uncountable* sequence of digits like (**D2**) which, however, fails to satisfy point (3) of our Definition 1 of 'real number.' In other words, the sequence of digits will not be an element of the set, X. If the set, X, were really uncountable, and if the sequence (**E**) really were to contain all elements of X, then application of Cantor's diagonal procedure to such the sequence (**E**) would simply generate an "object" that is not an element of X, something that does not provide the desired final result. Thus, Cantor's RAA-proof includes, at least implicitly, the condition that the diagonal procedure itself is simply not applicable to sequences of reals having a cardinality that is greater than that of the countable. That is a very fine point, one which requires further consideration—especially in connexion with the quite significant isomorphism of logical structures in Cantor's *diagonal* proof of the continuum *uncountability* as well as Pythagoras' *diagonal* proof of the *potential* infinity of the set of common *finite* natural numbers (see Table 1).

6.2. **Definition of the real number notion based on *potential* infinity.** What are the real numbers in 'really working' mathematics?

One possible definition can be formulated in term of a CT-machine as follows.

Naturally, we must now change the last step of the CT-machine algorithm to the form:

(3) Go to (2a), i.e., the 'halting' condition is simply absent in this case.

We get the following

Definition 1. A real number (**D2**) is a 'thing' that exists through [its every nth approximation] being taken one after another, what is taken being always finite, but ever other and other" (almost by Aristotle).

We shall now provide a different definition of 'real number' more akin to that used in "really working" mathematics. Suppose, first, that we

Notations: $X = [0, 1]$, {**L**:} is a label '**L**' for references.	*Notations:* $N = \{1, 2, 3, \dots\}$, {**L**:} is a label '**L**' for references.
Cantor's Theorem (1890). {**A**:} The set X is *uncountable.* *Proof* by the Reductio ad Absurdum method. Assume that $\{\neg\mathbf{A}$:} the set X is *countable.* Then there is a list of *all* reals of X. Let { **B**:} the list (1) $x_1, x_2, x_3, \dots$ comprise *all* reals of X. Applying to list (1) the *diagonal* rule, $\forall i \geq 1\, [[[x_{ii} = 0] \to [y_{1i} := 1]]$ $\&\ [[x_{ii} = 1] \to [y_{1i} := 0]]]$, Cantor creates a *new* real of X, say, $y_1 = 0.y_{11}y_{12}y_{13}\cdots$, such that: $\forall i \geq 1\, [y_1 \neq x_i]$, i.e., $y_1 \notin (1)$. Consequently, $\{\neg\mathbf{B}$:} the list (1) does **NOT** comprise *all* real numbers of X. So, the contradiction of a special kind, $[\mathbf{B} \to \neg\mathbf{B}]$, is obtained. From the contradiction, it follows that the assumption $\neg\mathbf{A}$ is false. Consequently, it is proved that {**A**:} the set X is *uncountable.* □	**Pythagoras' Theorem** (sixth century B.C.). {**A**:} The set N is *infinite.* *Proof* by the Reductio ad Absurdum method. Assume that $\{\neg\mathbf{A}$:} the set N is *finite.* Then there is a list of *all* integers of N. Let { **B**:} the list (1) $1, 2, 3, \dots, n$, comprise *all* integers of N. Applying to list (1) the *diagonal* rule, $C := 1;\ \forall i > 1\, [C := C + 1],\ 1 < i \leq n$, Pythagoras creates a *new* integer of N, say, $C = n + 1$, such that: $\forall i \geq 1\, [C > i]$, i.e., $C \notin (1)$. Consequently, $\{\neg\mathbf{B}$:} the list (1) does **NOT** comprise *all* natural numbers of N. So, the contradiction of a special kind, $[\mathbf{B} \to \neg\mathbf{B}]$, is obtained. From the contradiction, it follows that the assumption $\neg\mathbf{A}$ is false. Consequently, it is proved that {**A**:} the set N is *infinite.* □
NO COMMENT	
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TABLE 1. Who in fact is the originator of the Diagonal Procedure?

ask: What are the well-known real numbers: π, e, $\sqrt{2}$, $\ln 10$, $\sin 13°$, and so on? To answer the question consider, e.g., the number π.

'π' is:

(1) a *proper name* of the corresponding algorithm which allows to calculate any *finite* nth digit of the decimal representation of the sequence:

$$\pi = 3.14159265358979323846264338327950288419716939937510 58\ldots$$

(2) a *proper name* of an *ideal 'final' result* of the *potentially infinite process of* this algorithm working, which by definition *will never reach* its final STOP-state.

(3) a *proper name* of a special type of a *variable* which takes its value from the *potentially infinite* set of *finite* approximations, generated by the corresponding algorithm.

With this example in mind we now proceed to make more accurate our definition of 'real number.'

Definition 2. Any real number (including any rational one), say, d of the segment $[0, 1]$ is a *symbolic proper name* possessing the following properties:

(1) It is a *name* of a concrete mathematical *algorithm* which allows us to calculate and present a *finite* sequence of, say, decimal digits of some length, n, for any *given finite* n,

$$0.d_1 d_2 d_3, \ldots, d_n. \tag{D3}$$

(2) It is a *name* of an *imaginary, ideal (not in Plato's "objective" sense) final result* of the *potentially infinite* process, that is, of an "*upshot*" of the CT- machine working not having a STOP-state by definition.

(3) It is a *name* of a special type of a *variable* which takes its value from the *potentially infinite* set of *finite* approximations of the (**D3**)-type generated by the corresponding algorithm mentioned in §1.

(4) It is a *name* of a *variable* which can be used in any mathematical expressions and may be substituted for any *finite* nth approximation of the (**D3**)-type.

It is easy enough to see, the problem as to an actual infinity of the set of 'all' approximations of the (**D3**)-type simply doesn't arise within the framework of 'really working' mathematics. The problem exists only in our imagination, not in actually working mathematics.

6.2.1. *Remark. Cantor's continuum "is mostly made of gaping holes."* As is known, the modern noted metamathematician, Gregory Chaitin, at IBM's T. J. Watson Research Center in Yorktown Heights, New York, "shattered mathematics with a single number" [Chaitin 2001]. Chaitin named this number Omega ['Ω'] (not to be confused with Cantor's patent transfinite ordinal 'omega' ['ω']!). Chaitin's Omega is "an unending, random string of 0s and 1s" where the value of nth binary digit is defined by the condition of whether a certain Diophantine equation ("that was 200 pages long and had 17,000 variables") has a finite or infinite number of solutions for each value of some parameter $n = 1, 2, 3, \ldots$. Chaitin states that each digit of such a binary sequence "is as unrelated to its predecessor as one coin toss is from the next," and therefore the number Omega is "the outstanding example of something which is unknowable in mathematics."

According to Chaitin, his "Omega infects the whole of mathematics, placing fundamental limits on what we can know." He has found that "the core of mathematics is riddled with holes." Chaitin has also shown that "there are an infinite number of mathematical facts but, for the most part, they are unrelated to each other and impossible to tie together with unifying theorems. If mathematicians find any connections between these facts, they do so by luck. Most of mathematics is true for no particular reason. Randomness is the true foundation of mathematics. Maths is true by accident."

"The fact that randomness is everywhere has deep consequences," says John Casti, a mathematician at the Santa Fe Institute in New Mexico and the Vienna University of Technology. "Chaitin's work shows that solvable problems are like a small island in a vast sea of undecidable propositions," Casti says [Chaitin 2001].

"The Omega strain of incalculable numbers," Chaitin concludes his metamathematical sentence upon mathematics, "reveals that mathematics is not simply moth-eaten, it is mostly made of gaping holes. Anarchy, not order, is at the heart of the Universe" [Chaitin 2001].

Of course, no really working mathematician will ever take seriously Chaitin's apocalyptical 'diagnosis.' Even some metamathematicians are shocked by such scandalous claims: "It has been shown conclusively [...] that Chaitin's philosophical interpretations of his work are unfounded and false; they are based on various fatal confusions," Panu Raatikainen states in his review of Chaitin's last books in *Notices of the AMS* in 2001 [Raatikainen 2001], "And thus we have all the more reason for doubting the claim that his approach can explain the true source of the incompleteness and unsolvability theorems. As his philosophical interpretations fall, so does this claim."

However, Chaitin's claims above are quite reasonable if to attribute them to the problem of limits "on what we can know" as to not mathematics but the nature of Cantor's continuum. Indeed, as it has been shown above any *mathematical* definition of real number is based on an infinite binary sequence of the (**D2**)-type generated by a definite *algorithm* that allows to construct *any* nth digit of the sequence (now we do not touch the question whether the infinite sequence (**D2**) is actual or potential). It is reasonable to state that if such an algorithm is given then the corresponding real number is given as well and is existent and *cognizable* in the obvious mathematical sense defined by the definition above. It is also quite reasonable to state that if for some real number the corresponding algorithm generating a sequence of the (**D2**)-type for this real number will not be ever defined, then such a real number will not be existent ever and therefore such a real number will be an *unknowable* 'thing' in the mathematical sense.

As is known, any algorithm is a finite string in some finite alphabet, say in Chaitin's version of the LISP language, and therefore, according to Cantor and Chaitin, the set of all possible algorithms is countable. It means that the set of all real numbers that can be ever defined by all these algorithms, i.e., a set of all *cognizable* real numbers, is countable as well. Even if the humankind will live for ever and will be engaged solely in writing new LISP algorithms and running them on all computers to produce new and new real numbers in the form of the (**D2**)-type sequences.

On the other hand, the set of all real numbers is, according to Cantor, uncountable. The last means that a subset of all real numbers that will not be ever presented as a sequence of the (**D2**)-type, i.e., a set of all *unknowable* 'things,' is uncountable. Since, according to known 'algorithmic information theory,' a knowledge that is contained in a *countable* set of *cognizable* real numbers in comparison with a 'knowledge' that is contained in an *uncountable* set of *unknowable* real numbers is a value that is much less than, say Leibniz's infinitesimal, then we can state that Cantor's continuum is, using Chaitin's expression, "not simply moth-eaten, it is mostly made of gaping holes." Shortly, Cantor's continuum as a whole is simply an *unknowable* 'thing.' Of course, if and only if the continuum is actual and uncountable in Cantor's sense.

Other aspects of this problem connected with a rigorous *mathematical* proof of the *theorem* that there is an infinite set of 'real' gaps between any two different *cognizable* real numbers are presented and discussed in [Zenkin and Zenkin 2000].

7. Two hidden *necessary* conditions of Cantor's diagonal proof.

There are no deductive rules, or algorithms, nor any 'really working' mathematical proofs that are always applicable and valid: the applicability of a rule, the effectiveness of any algorithm, as well as a validity of any mathematical proof are strictly defined by *necessary* (and sufficient) conditions for their realization. For example, no mathematician will ever try to prove, say, the Pythagorean theorem $a^2 = b^2 + c^2$ without the explicit showing of the condition that the corresponding triangle ABC is right-angled since it is a *necessary* condition of such a proof without which the proof simply lacks sense.

It will be quite useful to analyse those very conditions required for the realization of Cantor's diagonal method; it will turn out that such an analysis will generate some new, unexpected "rule-governed" meanings of "Cantor's diagonal argument" and its ultimate conclusions ([Zenkin 2001], [Zenkin 2002a], [Zenkin 2003], [Zenkin 2000c]).

7.1. **First hidden necessary condition of Cantor's proof.** Both Cantor's "naive" set theory, as well as modern "nonnaive" axiomatic set theories, require the *actuality* of the infinite set X and the enumeration (**E**) to be a *necessary* condition of Cantor's diagonal proof, since if the enumeration (**E**)—and as a consequence the X itself and all its infinite subsets—is *potentially* infinite then, as shown above, Cantor's diagonalization process will never reach a STOP-state, and therefore will never permit complete construction of Cantor's (anti)diagonal real y_1 as an *individual* mathematical object. So, from the point of view of 'really working' classical mathematics, and taking into account the fact that Cantor's axiom (above) was never formulated explicitly within the framework of modern axiomatic set theory, the traditional metamathematical formulation of Cantor's theorem is incorrect and must be set right as the following *conditional* statement:

Cantor's Theorem (correction 1). IF the set X is *actually* infinite, THEN X is *uncountable.*

It should be mentioned that the explication of this necessary condition of Cantor's proof, and the corrected formulation of the theorem itself, violates no restriction either of classical, or metamathematical logic. However, this necessary condition of the proof might be said to be a hidden meaning of Cantor's proof since it has never been explicitly mentioned.

Now we shall show that a number of unexpected and fatal consequences follow from such a natural and legitimate specification of Cantor's theorem.

To begin with, recall some familiar assertions of classical logic.

If we have two nonauthentic, i.e., so far unproven, premises (assumptions), say A and B, in any deductive inference and if we have deduced from these A and B "something known to be false" (according to Aristotle) or "known to be a contradiction" (according to the modern metamathematical version of the RAA-method [Kleene 1957], [Fraenkel and Bar-Hillel 1958], [Hodges 1998]), then we can know only that *at least one* of these *two* premises must be false. But as has been well known since Aristotle, that "*at least one*," here means that we fail to arrive at a single conclusion, instead we are faced with the following *three*, equal in their logic rights, conclusions (here T = Truth, F = False):

(1) $[A = \mathrm{F}]$ & $[B = \mathrm{T}]$,
(2) $[A = \mathrm{T}]$ & $[B = \mathrm{F}]$,
(3) $[A = \mathrm{F}]$ & $[B = \mathrm{F}]$.

Thus, that version of the RAA-method which uses two assumptions states nothing as to which alternative from among these three is correct. Clearly, the RAA-method with, say, $n = 10$ assumptions is possible, but it reduces n initial problems to $2^n - 1$, i.e., to 1023, new ones, *including all the initial ones*(!). Since, in general, to solve even 10 initial problems is usually easier than the same 10 plus 1013 new ones, the usual RAA-method has never more than a single assumption, at least in classical logical and mathematical proofs.

Thus, the RAA-method with two assumptions proves nothing and is from the standpoint of classical Aristotelian logic and "really working" mathematics without value.

Now consider three nontrivial consequences of the "hidden" fact that in Cantor's proof the set X is *actually* infinite.

7.1.1. *Relativity of the continuum uncountability.* "Naive" Cantorian as well as all modern "nonnaive" axiomatic set theories try to convince us that the number of reals in the *infinite* set X is considerably greater than the number of natural numbers in the *infinite* set N. P. J. Cohen for one writes in his famous monograph [Cohen 1966]: The Continuum hypothesis "is a rather dramatic example of what can be called (from our today's point of view) an absolutely undecidable assertion ..." (p. 13), and further: "Thus, [the cardinality of the set X—AZ] is greater than $\aleph_0$, $\aleph_n$, $\aleph_\omega$, $\aleph_\alpha$, where $\alpha = \aleph_\omega$, and so on.

Therefore, we shall even not try to imagine visually a set of integers of a cardinality succeeding $\aleph_0$ ..." (p. 282).

Moreover, the notorious uncountability of the continuum is given by modern metamathematics as its inner, 'genetic,' absolute property. However, as correction 1 of Cantor's formulation above shows, uncountability is not an absolute property of the continuum, but rather a *conditional* one, valid (if any, see below) only within the framework of Cantor's (axiomatic, i.e., not proven, i.e., taken on trust) paradigm of the actualization of all infinite sets.

7.1.2. *'Deductive' pearl I. From logic to belief.* Since no metamathematician so far has disproved the Aristotle's potentiality of the infinite sets, in general, nor of the set X, in particular, the actuality of X is an unwarranted presupposition, i.e., in reality, it is a *second* assumption of Cantor's proof and so we arrive at a RAA-method with ... *two* assumptions: "X is *actual*" and "X is *countable*." As shown above, we have in such a case not the only Cantor's famous conclusion, but following *three*, alternative conclusions all equally correct:

(1) X is *actual* and X is NOT-*countable* (Cantor).
(2) X is NOT-actual (i.e., *potential*) and X is *countable* (Aristotle).
(3) X is NOT-actual (i.e., *potential*) and X is NOT-*countable* (nameless, so far).

From the point of view of classical logic, Cantor's proof is itself unable to answer the question as to which alternative among these three, *including* Cantor's famous statement (1), is in fact true.

Thus, Cantor's "proof" of the uncountability of the set X of all real numbers proves nothing and reduces the sacramental metamathematical question as to the distinguishability of infinite sets on the basis of their cardinalities to a question of a *belief*: if your preference is for the *actual*, then you may freely choose the first alternative (1) after Cantor, if you together with "really working" classical mathematics put your trust in Aristotle's *potentially* infinite, then the second alternative (2) may well suit your taste; and finally, if you are someone who trusts in nothing, then alternative (3) can be adopted without so much as a single contradiction with classical logic.

Of course, some "ideological" logicians and metamathematicians may shrug, saying: "no wonder, simply, by analogy with Euclidean/non-Euclidean geometries, we have here not one, but three, equivalent, but absolutely different, mathematics: first, Cantor's mathematics, based on his axiom (above), second, a non-Cantorian mathematics, based on Aristotle's axiom (above), and (so far) nobody's mathematics based, most likely, on accepting both of the two incompatible axioms."

Fortunately, as we shall soon discover, one need not go so far.

7.1.3. *'Deductive' pearl II. From meta- to para-mathematics.* Every generation of school children proves the Pythagorean Theorem for roughly the last 2600 years. And every (good) pupil arrives at the same result: $a^2 = b^2 + c^2$. This means that in 'really working' mathematics any once proven theorem remains valid regardless of how often its proof is constructed. Apparently, some metamathematical theorems do not enjoy similar "onerous" peculiarity of classical mathematics.

Indeed, from the once proven Cantor's theorem (correction 1), by virtue of the contraposition law ("if '$A \to B$' is proved then '$\neg B \to \neg A$' is valid too") of elementary logic which, by the way, holds even in metamathematical logic, ([Bourbaki 1965], [Kleene 1957]) we obtain the following quite surprising and epistemologically quite deep conclusion.

Corollary 1. If X is *countable*, then X is NOT-actually, i.e., it is *potentially*, infinite.

Now let us repeat once more the traditional proof (above) of Cantor's theorem.

Proof 2. Assume that X is a *countable* set. Then the sequence (**E**) of all real numbers of X is countable as well. According to Corollary 1, the sequence (**E**) will be a *potentially* infinite set. However, Cantor's diagonal method is not applicable to such a *potentially* infinite set. Consequently, Cantor's theorem becomes ... *unprovable.* □

Thus, Cantor's theorem, in contrast to all other theorems in mathematics, is not provable twice and, thus, is an expendable metamathematical theorem. It's, perhaps, more aptly described as a "*para*"- rather than a "*meta*"-mathematical phenomenon.

Taking into account that the set N of finite natural numbers is *countable by definition*, we deduce from Corollary 1 the following quite unexpected but rigorous consequence.

Corollary 2. The *countable* set $N = \{1, 2, 3, \dots\}$ of finite natural numbers is *potentially* infinite.

Thus, from the main Cantor's theorem itself it follows that Cantor's Axiom as to "*all sets are actual*" (above) which is the only basis for all his transfinite ordinal and cardinal constructions is wrong, i.e., according to Poincaré, all Cantor's set theory as well as all modern "nonnaive" axiomatic set theories are really "built on a sand" [Poincaré 1983].

7.2. **Second hidden necessary condition of Cantor's proof.** The first hidden necessary condition of Cantor's proof was quite obvious, but it was simply never formulated explicitly and so never considered as a necessary condition of the proof. The second necessary condition of Cantor's proof which will now be formulated is much more subtle and is well masked. It will be of some assist to make to appeal to metamathematical experts. One highly regarded logician in particular, W. Hodges, has analyzed numerous (naturally, unpublished) papers devoted to criticism of Cantor's diagonal proof [Hodges 1998]. Of course, most of Hodges' arguments against Cantor's opponents are certainly both correct and valid. However at one point his conclusion appears somewhat doubtful. Namely, Hodges writes:

"It was surprising how many of our authors failed to realise that *to attack an argument; you must find something wrong in it.*" The most common among the objections against Cantor's proof was "that Cantor had chosen the *wrong enumeration of the positive integers.* His argument only works because the positive integers are listed in such a way that each integer has just finitely many predecessors. If he had reordered them so that some of them came after infinitely many others, then he would have been able to use these late comers to enumerate some more reals, for example [the nonindexed antidiagonal real y_1—AZ]." W. Hodges objects, "...The existence of a different argument that fails to reach Cantor's conclusion tells us nothing about Cantor's argument." He goes on to restate his objection another way: "How does anybody get into a state of mind where they persuade themselves that you can criticise an argument by suggesting a different argument which doesn't reach the same conclusion?"

I believe that this argumentation of W. Hodges is not correct due to the following reasons ([Zenkin 2002a], [Zenkin 2003]).

1. In classical mathematics, if we have two deductive inferences (proofs), say, $A \to B$ and $A \to C$, the second one, usually, indeed "tells us nothing" [to be bad—AZ] about the first one, since the mathematics is consistent. But if we have two deductive inferences, say, $A \to B$ and $[A \,\&\, B] \to \neg B$, then the second one tells us a great deal not only about the first inference, but also about the doubtful logical merit of the premises set, A, itself. Unfortunately, Hodges passes over in silence the fact that the unnamed authors' conclusion of "a different argument that fails to reach Cantor's conclusion" *contradicts* and, indeed, *invalidates* the famous conclusion of "Cantor's argument."

2. On Cantor's RAA-proof assumption "X is *countable*," it follows that the set X is *equivalent* to ***any*** other *countable* set. In other words, according to Cantor's (and modern "nonnaive" axiomatic set theory's) definition of the equivalence relation itself, there is *always* a 1-1–correspondence between the set X and *any other countable* set as well as *any proper infinite* subset of the set N. E.g., there is a 1-1–mapping of the set X to any of such *countable* sets as $N = \{1, 2, 3, \dots\}$, $N_1 = \{2, 3, 4, \dots\}$, $N_2 = \{3, 4, 5, \dots\}$, and so on, and, say, $N_{\text{even}} = \{2, 4, 6, \dots\}$, $N_{\text{odd}} = \{1, 3, 5, \dots\}$, $N_{\text{prime}} = \{2, 3, 5, 7, 11, 13, \dots\}$, and so on. This means that, according to Cantor's RAA-proof assumption, *all* elements of the *countable* set X can be enumerated or indexed by the elements of *any* of above *countable* sets. The prohibition of such indexings and reindexings is a breach of the transitivity law for the equivalence relation, and simply contradicts straightforwardly to the known strict metamathematical definition of the equivalence relation.

3. Modern set theorists place special emphasis on the fact that according to Cantor's RAA-proof the enumeration (**E**) is *arbitrary*. However, this is a half of truth. Indeed, the *final result* of Cantor's diagonal method (CDM) is defined only by the sequence of real numbers and does not depend on the indices ascribed to them. Therefore, *any* reindexing of reals in the initial enumeration (**E**) by means of elements of any of the countable sets shown above, which (this reindexing) does not change the *number* or *order* of the real numbers in the sequence (**E**), will produce the same (anti)diagonal real number, y_1; that is, any such reindexing is admissible from the mathematical, algorithmical and set theoretical points of view. Thus, in Cantor's 'arbitrary' enumeration (**E**), not only the sequence of real numbers itself is arbitrary, but so too is the indexing of these real numbers.

4. According to W. Hodges (and all modern axiomatic set theories after Cantor himself), only the set $N = \{1, 2, 3, 4, 5, \dots\}$ is allowed in indexing the real numbers of the countable set X, i.e., in the mapping $X \to N$. However, any reindexing of real numbers in (**E**) is forbidden. Why?—Because only the indexing using *all* elements of $N = \{1, 2, 3, \dots\}$ leads to the desired Cantorian conclusion, whereas other indexings which make use of elements of any *proper infinite* (i.e., *countable*) subsets of N prohibit reaching Cantor's conclusion, and therefore are strongly excluded. In other words, according to W. Hodges, there are 'good' indexings which allow to prove Cantor's theorem, and there are 'bad' indexings which don't.

It is obvious that in such a circumstance modern set theory as well as symbolic logic demands that a strict logical criterion be formulated in order to distinguish these two sorts of enumerations. Unfortunately, Hodges provides no such criterion. However, if we translate his quite emotional and purely intuitive as well as dubious metamathematical "veto" on the "bad" enumerations that "fail to reach Cantor's conclusion" into the common mathematical language, we arrive at what follows ([Zenkin 2002a], [Zenkin 2003]).

The hidden Cantor-Hodges Postulate. Within the framework of Cantor's diagonal proof, from the RAA-assumption "X is *countable*" it follows that only such indexings of real numbers in the sequence ($\mathbf{E}$) are admissible which use *all* elements of the set $N = \{1, 2, 3, 4, 5, \dots\}$. Any other indexing which employs fewer than *all* natural numbers (e.g., any *proper infinite* subsets of the countable set N) is categorically forbidden.

I would like to emphasize here that the *hidden Cantor-Hodges postulate* is the *second necessary* condition of "Cantor's diagonal argument," since all 'bad' indexings of real numbers of the set X, according to W. Hodges, really "fail to reach Cantor's conclusion."

Thus, a completed and explicit formulation of the statement which Cantor proved in 1890 and which is used hitherto in modern metamathematical and set-theoretical educational courses and textbooks is actually as follows.

Cantor's Theorem (correction 2). The set X is *uncountable* if and only if:

(1) the set X is *actually* infinite;
(2) the *hidden Cantor-Hodges postulate is valid.*

Now I agree absolutely with the ultimate conclusion Hodges reaches: "there is nothing wrong with Cantor's argument" [Hodges 1998, p. 16]. Of course, this is so if and only if Cantor's theorem includes explicitly both of the *necessary* conditions of its proof acceptable to "really working" classical mathematics.

This is very important, so it merits repeating: both conditions, (1) and (2), are *necessary*, i.e., unremovable, conditions of Cantor's proof in the sense that if either is jettisoned, Cantor's theorem becomes simply unprovable. Moreover, according to Wittgenstein, the true, "rule-governed" sense (meaning) of Cantor's proof must be supplemented with the two necessary conditions (1) and (2) which essentially transform the traditional understanding and the final results of Cantor's proof itself.

In a word, the first necessary condition (1) of Cantor's theorem renders it invalid. In regard to the second necessary condition (2), the

Cantor-Hodges hidden postulate, it is simply a *teleological* assertion (only 'good' indexings are permitted, since only these allow us "to reach a desired Cantor's conclusion") which has no relation to mathematics and is an issue for what has been called "*para*mathematics."

7.2.1. *Why neither Kronecker nor Poincaré could disprove* mathematically *the ten* philosophical *strings of Cantor's proof?* Now we have an opportunity to answer a historical (and psychological) question of considerable importance, viz. why, for example, the greatest mathematicians of the nineteenth and twentieth centuries, such as Kronecker, Poincaré and others, who knew well Cantor's "Study on Transfinitum" and categorically rejected the actual infinity usage in mathematics (see the Introduction), could not *mathematically* prove the inconsistency of the main statement of the "Study"—Cantor's theorem on the uncountability of the continuum based upon the actual infinity usage? [Zenkin 2002a].

The answer would appear to be as follows. In mathematics, the "suspicious" theorem may be disproved in *at least two* ways: 1) if it is possible to detect an error in the proof itself of the theorem (as W. Hodges demands, see above), or 2) if it is possible to prove that even one necessary condition of the proof is logically incorrect (false, contradictory or unrealizable). In view of the second possibility we would do well to reconsider Hodges' statement that "It was surprising how many of our authors [criticizing Cantor—AZ] failed to realise that *to attack an argument, you must find something wrong in it.*" Indeed, it sounds very strange, since really working mathematicians know full well that in mathematics any proof is to be considered absurd not only when it contains "something wrong," but also when *even one necessary* condition of the "proof" is absurd.

Apparently, Cantor's diagonal procedure itself contains no 'deductive,' formal mistakes—otherwise really working mathematicians would have detected such errors a long time ago. That is why the first, common way to disprove Cantor's proof turned out not too convincing and effective. So, only the second way remained—to prove that "even one necessary condition of the proof is absurd." But here we come across a rather strange situation in all visible history of mathematics: one of the most influential metamathematical and set-theoretical theorems, Cantor's theorem on the uncountability of the continuum, never made explicit the necessary conditions of its own proof (see above). The last allows us to explain why neither Kronecker nor Poincaré, nor any other of Cantor's opponents was able to give a *mathematical* refutation of the ten *philosophical* strings of Cantor's diagonal proof: really, it's impossible to prove the absurdity of what does not exist at all! And what is more, no mathematician, even most ingenious one, simply could ever come to mind even a thought itself that it's possible during hundred years "to teach other people" (Hodges) how to formulate and prove Cantor's, ostensibly, mathematical theorem, while omitting ... the necessary conditions of its proof. From the 'really working' mathematics as well as Aristotle's logic point of view, it is a scandalous nonsense and, to borrow from the wisdom of Brouwer, really represents a "pathological incident in history of mathematics" [Fraenkel and Bar-Hillel 1958]. Such a situation has exerted a deleterious influence on progress in our understanding of mathematical foundations and is in fact "very harmful and has grave scientific, pedagogical, and social consequences," according to the opinion of the outstanding modern mathematician and educator Prof. V. I. Arnold ([Arnold 1999], [Zenkin 2000d], [Zenkin 2002b]).

8. Four odd peculiarities associated with Cantor's transfinite ordinal numbers.

We next consider several aspects of the second half of Cantor's "Study on Transfinitum"—his famous theory of the transfinite *ordinal* numbers.

8.1. **Arithmetization of names or a child game "draw, snip off, and glue together."** Gödel invented a way of numbering formal statements, i.e., he replaced formulas by natural numbers, arithmetized all "quite rich" formal systems ([Kleene 1957], [Zenkin 2002c]), and then produced his famous "negative" metamathematical results (incompleteness, unprovability of consistency, etc.). Now we shall show that Cantor, anticipating Gödel, successfully used a similar idea in creating his famous theory of transfinite ordinal numbers ([Zenkin 1997a], [Zenkin 1997b].

Cantor begins with an argument for the legitimacy of the actualization of the infinite series (**NN**) of finite natural numbers. Here we stick close to his original procedure ([Cantor 1914], [Cantor 1985]) (almost verbatim): it is well known that the number of *finite* natural numbers in the series (**NN**) is infinite, and therefore [according to Aristotle—AZ] there is no last maximal number in (**NN**) ...; however contradictory it may seem (it is really very much contradictory, as Cantor himself well understood!), there is in fact no absurdity ("*The essence of ... mathematics is its freedom*"!—So any fantasy is admissible!) in *denoting* the *series* (**NN**) as a whole using a *symbol* (or *name*), say, 'omega' (further: ω, for short), *calling* the *name* 'ω' an *integer* and then going on to count:

$$\textbf{(C1)} \qquad \omega, \omega+1, \omega+2, \omega+3, \ldots,$$

in the complete conformity with ... the Aristotle-Peano axiom: "if a 'thing' is [called—AZ] integer then the 'thing' + 1 is integer too" (*for any* 'thing' independently of a "real nature" of the 'thing' and what we think of the 'thing').

From a formal point of view, Cantor's "transfinite" series (**C1**) is the same series (**NN**) with the new *name* 'ω' instead of the traditional *name* '0' for zero:

$$\textbf{(NN)} \qquad 0, 0+1, 0+2, 0+3, \ldots, 0+n, \ldots$$

$$\textbf{(C1)} \qquad \omega, \omega+1, \omega+2, \omega+3, \ldots, \omega+n, \ldots$$

It is reasonably obvious that if we prefer common *names* in common Latin alphabet, say 'tankard' (according to Hilbert, if say in Euclidian system of axioms to replace the words 'point,' 'line,' and 'plane' by the words 'table,' 'chair,' and 'tankard' then nothing will be changed in Euclid's geometry, see [Bourbaki 1965]), call this name 'tankard' an integer, then, according to Peano's axiom in the form "if a 'tankard' is integer then the 'tankard' + 1 is integer too," we can generate the series,

$$\text{tankard}, \text{tankard}+1, \text{tankard}+2, \text{tankard}+3, \ldots, \text{tankard}+n, \ldots,$$

the set-theoretical semantics of which will be identical to the number-theoretical semantics of the series (**NN**) and (**C1**). However, since Cantor creates a "*transfinite* stairway to Heaven" [Cantor 1914], a countable set of sensible (as well as meaningless) finite *names* which can be produced in the finite Latin alphabet will be exhausted quite soon owing to the mere countability of the *names* set. To avoid the last difficulty Cantor makes use of an ingenious innovation, viz., in order to generate an admittedly unlimited number of new *names* he uses as *names* the notation for the separate components (summands) of the common algebraic polynomial,

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_\alpha x^\alpha + \ldots,$$

with integer coefficients and exponents written in a *nonpositional* notational system, i.e., the exponents α in $P(x)$ are presented in the same polynomial form if for a fixed integer x the exponent α becomes greater than x.

Now, replacing the name 'x' of the traditional mathematical variable by the Greek name 'ω,' we get the following familiar, "well-ordered"

$$
\begin{array}{lllll}
0, & 0+1, & 0+2, & 0+3, & \dots \\
\omega, & \omega+1, & \omega+2, & \omega+3, & \dots \\
2\omega, & 2\omega+1, & 2\omega+2, & 2\omega+3, & \dots \\
3\omega, & 3\omega+1, & 3\omega+2, & 3\omega+3, & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
n\omega, & n\omega+1, & n\omega+2, & n\omega+3, & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\omega^2, & \omega^2+1, & \omega^2+2, & \omega^2+3, & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\omega^n, & \omega^n+1, & \omega^n+2, & \omega^n+3, & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\omega^\omega, & \omega^\omega+1, & \omega^\omega+2, & \omega^\omega+3, & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

and so on

TABLE 2. The only series (**NN**) of *finite* natural numbers rewritten with different names for zero.

series of "quasi-arithmetical" *names*:

(Ord) $\omega, 2\omega, 3\omega, \dots, n\omega, \dots, \omega^2, \dots, \omega^n, \dots, \omega^\omega, \dots,$ and so on,

which consists in the *names* of Cantor's "transfinite ordinals of a second kind" [Cantor 1914]. Using this names, we may continue the process of the "transfinite" recursion (see Table 2).

Now, if we "snip off" the strings of the Table 2 and then successively "glue" them together into one string, we arrive at the main object of Cantor's theory of the transfinite ordinals up to Cantor's famous countable *name* 'ε_0' ([Hilbert 1948], [Cantor 1914], [Cantor 1985], [Kleene 1957]):

(C2)
$$
\begin{gathered}
1, 2, 3, \dots, \omega, \omega+1, \omega+2, \omega+3, \dots, \\
2\omega, 2\omega+1, 2\omega+2, 2\omega+3, \dots, \\
3\omega, 3\omega+1, 3\omega+2, 3\omega+3, \dots, \\
\omega^2, \dots, \omega^\omega, \dots, \omega^{\omega^\omega}, \dots, \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}} \quad (= \varepsilon_0)
\end{gathered}
$$

It is obvious that now the same senseless procedure can be carried out with this *new name* ε_0 for zero in (**NN**). Then the same child's game can be repeated with the obvious new names ε_1, ε_2, ε_3, $\dots$, ε_ω,

$\ldots, \varepsilon_\varepsilon, \ldots$, and so on, generated by the same 'genetic algorithm' of the series (**Ord**).

Thus, if we will believe not in the speculative Cantor's statements about a high transfinite nature of the name 'ω' and symbolic constructions of the (**C2**) type, but, according to Wittgenstein, in the real, mathematical, "rule-governed" *usage* of the name 'ω' in such symbolic constructions, then we shall see that the true mathematical sense of Cantor's series (**C2**) of "transfinite" ordinals is exhausted by the mathematical sense of the Aristotle-Peano axiom "if a 'thing' is integer then the 'thing' + 1 is also an integer." As has been remarked above, this axiom tells nothing as to a real nature of these 'things.' Therefore nothing we can say as to the true nature of these transfinite 'things' (whether they are "in reality" finite/infinite, actual/potential, *consistent/inconsistent*, big/small, white/black, etc.) has any relation to mathematics. As for Cantor's series (**C2**) of 'transfinite ordinals,' it is, unfortunately, a senseless unlimited reiteration of the same series (**NN**) of the common *finite* natural numbers with new symbolic *names* for common zero '0' at each step of this cloning process, having a vacuous mathematical semantics and ontology. In a word, Cantor's "*transfinite stairway to Heaven*" is in reality a version of a well-known old-fashioned child's game: "Draw, Snip off, and Glue together."

8.2. **Whether Cantor was really in need to actualize infinity in order to tell about 'transfinity'?** According to Cantor, 'omega' means that "the sequence (**NN**) contains *all* its elements, i.e., it is a *completed, constant, invariable, actually* infinite set" ([Cantor 1914], [Cantor 1985]) otherwise it would be inadmissible at all to call the 'omega' a number. As is known Cantor put a lot of work in order to convince mathematicians of a legitimacy of such a doubtful step. However what would happen if Cantor's 'omega' would, 'in reality,' denote the sequence (**NN**) as something *potential*, i.e., as an "incomplete, variable," growing sequence whose elements are, according to Aristotle, "always finite, but ever other and other"? I believe nothing, since, as was shown above, in order to work Cantor's "transfinite" series (**C2**) it is *sufficient to call any symbol* an integer—and further in reality Peano begins "to work," not Cantor. Following Wittgenstein, the real, "rule-governed" *usage* of all Cantor's 'omegas,' 'epsilons,' etc. shows that in order to overstep the *potential* infinity of the classical series (**NN**) of the common natural numbers and produce Cantor's "transfinite" construction (**C2**) it is, contrary to Cantor, *absolutely irrelevant* to whether the *infinite* series (**NN**), (**C1**), (**C2**), and so on are *actual.*

Thus, Cantor wasted words arguing in favor of the actualization of the infinite series (**NN**), but what Cantor was really in need in order to construct his transfinite ordinals in (**C2**) is only the immutability (i.e., indeed the real actuality) of these *names* and *symbols* themselves. However the last is a problem of a syntax of the symbolic 'game' above, but not of a mathematics.

8.3. **When finite and transfinite are a matter of taste.** There is another interesting aspect of Cantor's series (**C2**) of the transfinite ordinals ([Cantor 1914], [Cantor 1985], [Zenkin 1997a], [Zenkin 1997f], [Zenkin 1997g]).

Let us rewrite the common series (**NN**) of the *finite* natural numbers which, according to Kronecker, were created by the Lord, in the following quite ancient and *primitive*, but natural enough, *additive*, for the sake of simplicity *decimal*, but *nonpositional* notation:

$$(\mathbf{C3})\qquad \begin{gathered} 1, 2, 3, \ldots, 10, 10+1, 10+2, 10+3, \ldots, \\ 10+10, 10+10+1, 10+10+2, 10+10+3, \ldots, \\ 10+10+10, 10+10+10+1, 10+10+10+2, 10+10+10+3, \ldots, \end{gathered}$$

In order to reduce the length of (**C3**), we shall use common symbols for multiplication and the raising to a power as follows: $10+10 \to 2\cdot 10$, $10+10+10 \to 3\cdot 10$, etc., and, for example, $10+10+10+10+10+10+10+10+10+10 \to 10^2$, etc., and so on. The series (**C3**) then becomes:

$$(\mathbf{C4})\qquad \begin{gathered} 1, 2, 3, \ldots, 10, 10+1, 10+2, 10+3, \ldots, \\ 2\cdot 10, 2\cdot 10+1, 2\cdot 10+2, 2\cdot 10+3, \ldots, \\ 3\cdot 10, 3\cdot 10+1, 3\cdot 10+2, 3\cdot 10+3, \ldots, \\ 10^2, \ldots, 10^{10}, \ldots, 10^{10^{10}}, \ldots, 10^{10^{10^{10^{\cdot^{\cdot^{\cdot}}}}}} \end{gathered}$$

But it is obvious that the series of the *finite* natural numbers—in the form (**NN**) or in the new form (**C4**), which is the same thing—can be written in any other number system, and, as is commonly understood, neither the quantity of its elements, nor the order of these elements will not be changed. Therefore, we *shall choose* an arbitrary *symbol*, for example, the Latin symbol 'w,' then we *shall call* this symbol 'w' an *integer*, and lastly we shall rewrite the series (**C4**) in the *arbitrary* w-*adic number system notation* as follows:

$$\text{(C5)}\qquad \begin{gathered} 1, 2, 3, \ldots, \mathrm{w}, \mathrm{w}+1, \mathrm{w}+2, \mathrm{w}+3, \ldots, \\ 2\mathrm{w}, 2\mathrm{w}+1, 2\mathrm{w}+2, 2\mathrm{w}+3, \ldots, \\ 3\mathrm{w}, 3\mathrm{w}+1, 3\mathrm{w}+2, 3\mathrm{w}+3, \ldots, \\ \mathrm{w}^2, \ldots, \mathrm{w}^{\mathrm{w}}, \ldots, \mathrm{w}^{\mathrm{w}^{\mathrm{w}}}, \ldots, \mathrm{w}^{\mathrm{w}^{\mathrm{w}^{\mathrm{w}^{\cdot^{\cdot^{\cdot}}}}}} \end{gathered}$$

Underscore once again that since until now we have nowhere changed either the quantity or the order of natural numbers in the initial series (**NN**), it can be asserted that the series (**C5**) is the same initial series (**NN**) in the sense that the series (**C5**) contains the same finite natural numbers and in the same order as the series (**NN**), though written in the w-adic number system.

If we now take any other symbolic name instead of the Latin symbolic name 'w' in the series (**C5**), e.g., Greek 'α,' or, 'β,' or, 'γ,' or, 'δ,' etc., then the number of elements and their order in the series (**C5**) will not be changed either. Naturally, if you, carrying out the shown procedure, will reach (if any) the last letter of the Greek alphabet, you will obtain the record of the series (**C5**), i.e., of the initial series (**NN**), in the 1-1–form of Cantor's series (**C2**).

Thus, we have proved *by means of the constructive and effective process*, that the series (**NN**) in the form (**C5**) and Cantor's 'transfinite' series (**C2**) are formally identical. Since the number and the order of all the elements in these two series are the same, they are *similar*, as mathematicians say, and, consequently, *an element-to-element, symbol-to-symbol 1-1–correspondence* exists between the series (**NN**) and (**C2**). Consequently, the series (**NN**) and (**C2**) being isomorphic are indistinguishable from the mathematical point of view, but any distinction of isomorphic structures, in particular, in our subjective interpretation of the *names* '10,' 'w,' 'ω,' and '$\ldots$,' are neither essential nor interesting from a mathematical point of view ([Hilbert 1948], [Bourbaki 1965]).

From a Wittgensteinian perspective "rule-governed" usage associated with the name 'w' in (**C5**), it is a matter of taste whether to consider (or to think of) the series (**C5**) either as a symbolic model of the *potentially* infinite series (**NN**) of the common *finite* natural numbers or as a symbolic model of the *actually* infinite series (**C2**) of Cantor's '*transfinite*' ordinals.

By the way, as is known, Cantor's set of 'transfinite' ordinals up to the ε_0 (**C2**) is countable, so there is little wonder that there is a 1-1–correspondence between (**C2**) and (**NN**). Moreover, there are a number of such different correspondences [Alexandrov 1948], but hitherto all such correspondences *changed* the *natural order* of the natural numbers in the series (**NN**). From this point of view, our 1-1–correspondence between the series (**NN**) in the form (**C4**) and Cantor's series (**C2**) is unique in the sense that this 1-1–correspondence *retains the natural order* of natural numbers in the series (**NN**), i.e., the 1-1–correspondence is the *homothety or similarity* relation. That is made for the first time.

8.3.1. *Remark. Noncommutative version of Peano's arithmetic.* It is easy to show that an isomorphism obtains not only between the series of the common *finite* natural numbers (**NN**) of the form (**C5**) and Cantor's *transfinite* ordinals (**C2**), but also between Cantor's "arithmetic" of transfinite ordinals and Peano's arithmetic qualified somewhat as follows. Consider the presentation of the series (**NN**) in the nonpositional w-adic form (**C5**), and introduce the following two additional (to the classical Peano's axiom system) axioms defining Cantor-like noncommutativity of the addition operation of common finite natural numbers.

Axiom 7. For any $n < \mathrm{w}$, $n+\mathrm{w} = \mathrm{w}$. Consequently: $n+\mathrm{w} = \mathrm{w} \neq \mathrm{w}+n$.

Using 'w' instead of 'ω' in (**Ord**), introduce the more general

Axiom 8. If for any names α and β of the series (**Ord**) $\alpha < \beta$, then $\alpha + \beta = \beta$.

As a result we shall have a new, special, *noncommutative* Peano "arithmetic" of common *finite* natural numbers where, for example, for a fixed radix $\mathrm{w} = 10$: $1+10 = 10$, $9+10 = 10$, i.e., $n+10 = 10 \neq 10+n$, and $n + 10^2 = 10^2$, $n \cdot 10^2 + 10^3 = 10^3$, and so on (see Cantor's theory of transfinite ordinals in [Cantor 1914], [Kleene 1957]).

It is uncertain where such a mutilated, noncommutative "arithmetic" of finite numbers could be applied in mathematics, but it certainly would be useful in the case of certain illegal financial activities: $\$n + \$\mathrm{w} = \$\mathrm{w}$, where the $\$n$ vanishes (from the Revenue Board), though the value itself of that $\$n$ may be quite sizeable.

8.3.2. *Remark. Proof of the consistency of Cantor's 'theory' of transfinite ordinals.* The isomorphism of (**NN**) and (**C2**) in the form (**C5**) proves the obvious *relative* consistency of Cantor's "theory" of transfinite ordinal numbers: the last is certainly consistent if and only if Peano's arithmetic is consistent. In other words, since all mathematics used in Cantor's theory of transfinite ordinals is exhausted by the shown

Aristotle-Peano axiom, Cantor's "theory" of transfinite ordinals simply can't be inconsistent. By the way, the consistency of Cantor's "theory" of transfinite ordinals is proved here for the first time. Though, as it's easy to see, the proven consistency of the "theory" does not guarantee that it has a mathematical sense.

The proven consistency (conditional though) of Cantor's ordinals explains, in particular, why such a deep mathematician as Kronecker could not disprove *mathematically* a quite obvious inanity of Cantor's theory of transfinite ordinals.

8.3.3. *Remark. Natural solution of Burali-Forti's paradox.* According to the Burali-Forti's paradox, the ordinal of all transfinite ordinals leads to a contradiction. In order to avoid the contradiction, Cantor's "naive" and modern "nonnaive" set theories offer a quite questionable way out: they declare, as usual, that such a *contradictory* 'thing' as the 'ordinal of all ordinals' simply does not exist ([Cantor 1914], [Cantor 1985], [Kleene 1967]), i.e., every transfinite ordinal is existent, i.e., actual, but the 'ordinal of all ordinals' is nonexistent, i.e., *unattainable*, i.e., *potential.*

This is of little consequence, however, since following the traditional Cantor's approach (see his argumentation as to the 'ω' introduction and the further construction of the series (**C2**) , above), we can always *denote* the *contradictory* 'ordinal number of all *transfinite* ordinals' a *new name* which so far has not been used in Cantor's set theory, say, 'alpha' ('α'), *call* this name 'α' a *least* ***trans****transfinite integer* that is greater than any 'common' *trans*finite Cantor's ordinals and go on:

$$\alpha, \alpha+1, \alpha+2, \ldots, 2\alpha, 2\alpha+1, 2\alpha+2, \ldots,$$
$$\alpha^2, \ldots, \alpha^\alpha, \ldots, \text{ and so on, (akin to } (\mathbf{C2}))$$

of course, following Cantor's famous slogan "The essence of ... mathematics is in its freedom!" and the Aristotle-Peano axiom "if a thing 'α' is integer then the thing 'α'+1 is integer too." And so on ([Zenkin 1997a], [Zenkin 1997b]).

In one word, "the set theory," as P.Vopenka concludes rightly, "whose energies were directed to the actualization of potential infinity turned out not to be able to eliminate the potentiality, but could only remove it into a more far sphere" [Vopenka 1983].

8.4. **Whether Cantor's ω is an integer? What does it mean to apply '+1' to the name 'ω'?** According to Cantor, ω is an *integer* and that is why the operation '+1', strictly defined (before Cantor) only for the finite numbers, is considered applicable to ω within the framework of Cantor's "naive" as well as modern "nonnaive" set theory. Cantor and modern Cantorians, when they want to defend acceptance of unusual, contradicting normal human's intuition, properties and the frequently quite extravagant peculiarities of transfinite objects, typically accentuate the inadmissibility of transferring properties that are defined for the *finite* numbers to the *transfinite* numbers ([Cantor 1914], [Cantor 1985], [Kleene 1957], [Hodges 1998]).

Let us consider how this important methodological demand of modern metamathematics and axiomatic set theories is maintained in metamathematical practice.

Return to Cantor's diagonal proof, and add to Cantor's 'diagonal rule' (**R**) the following rule (in a usual computer notation):

(**R3**) integer $\mathbf{C}$; $\mathbf{C} := 1$; for any $i > 1$ $[\mathbf{C} := \mathbf{C} + 1]$;

which is defined on and applied to the natural *indexes* of the real numbers in the enumeration (**E**), and is realized *simultaneously* with and (step-by-step) *parallel* to Cantor's rule (**R**) at each step i. In other words, while Cantor creates his *new* real number y_1 according to the rule (**R**), we shall at the same time create a *new* integer $\mathbf{C}$ in accordance with the rule (**R3**). Since the parameter 'i' is the same for both rules, it is obvious that the situation when, say, Cantor has finished the process of creating his new real number y_1, but the process of creating our new integer $\mathbf{C}$ so far is not completed (and vice versa) is impossible.

Therefore only the following two cases are of interest.

(1) If the ith step of Cantor's diagonal process is *finite*, say, $i = n$, then Cantor's (anti)diagonal real number y_1 created towards the ith step by the rule (**R**) is a *finite* binary sequence,

$$y_1 = 0.y_{11}y_{12}y_{13}\ldots y_{1n},$$

which, of course, *differs from each* of the *first* n real numbers in the enumeration (**E**), but itself is not a real number, since the *finite* binary sequence with the length n defines a *finite* subsegment of the initial segment $[0, 1]$ with the finite length $1/(2^n)$, i.e., the sequence defines not an only *individual* real number, but a whole *infinite set* of real numbers which, according to Cantor, has the same cardinality as that of the initial segment $[0, 1]$. Simultaneously, we arrive at $\mathbf{C} = n + 1$ which is a *finite* natural number greater (and, consequently, also *different from*) *each* of the *indexes* of the same *first* n real numbers in (**E**): $n + 1 > 1, 2, 3, \ldots, n$.

(2) If the creation of Cantor's (anti)diagonal real number y_1 is completed, then so too is the creation of the new integer **C**; and its value *can't be finite* (otherwise go to the case (1)), and the question naturally arises: what is the **C** in such a case? Since that **C** was produced by means of the application of the operation '+1' only to *finite* natural numbers (positive integers), it must be an integer, but it can't be finite. So, after Cantor's diagonal algorithm stops its work, i.e., arrives at its 'halting' state, the integer **C** becomes a minimal 'integer' which is *greater than (and, consequently, different from) any finite* natural number. According to Cantor's own definition, the last is a minimal *transfinite* integer, i.e., it is his famous ω ([Cantor 1914], [Cantor 1985]).

Thus, the application of the non-Cantorian 'diagonal rule' (**R3**) to the *indexes* of the real numbers of the enumeration (**E**) produces a *rigorous proof* (anyway, of the same kind and level of the rigor as Cantor's proof does that) of the following quite interesting *conditional* statement.

Theorem 1. If the application of Cantor's 'diagonal rule' (**R**) to the enumeration (**E**) of real numbers generates a new real number as an *individual* mathematical object, then the simultaneous application of the non Cantorian 'diagonal rule' (**R3**) to the *indexes* of *the same* real numbers of *the same* enumeration (**E**) generates Cantor's minimal transfinite integer ω, or, shortly, IF Cantor's (anti)diagonal real number exists THEN *in the same sense* Cantor's ω exists.

8.4.1. *Remark. Rigorous proof of the existence of Cantor's 'omega.'* It is obvious that Theorem 1 gives much more rigorous (conditional though) *proof* of the ω existence than the original Cantor's speculative argumentation cited above in §8.1. Different ways in proving Theorem 1, based on a cognitive semantical visualization of the Continuum Problem and a mirror-like isomorphism of the real numbers and natural numbers graphic representations, are presented and discussed in ([Zenkin 1997a], [Zenkin 1999], [Zenkin 1997d], [Zenkin 1998]).

Now, taking into account 1) that the process of the construction of the binary sequence of digits of Cantor's (anti)diagonal real numbers y_1 is in fact the infinite process of the dichotomy of the segment $[0, 1]$ producing a point as its final result (provided by Cantor's axiom about a system of imbedded segments with the lengths tending to zero [Fichtengolz 1958]); 2) that a point of a segment is an *indivisible* 'thing,' according to Euclid's axiomatic geometry, i.e., it is a 'thing' to which the division operation '/2' is simply not applicable, from the

Process descriptions	Process 1 Generation of the Series of Natural Numbers	Process 2 Dichotomy of the segment $[0,1]$
Initial term	1	$L_1 = \|[0,1]\| = 1$
Operation	'+1'	'/2'
Common term	n	$L_n = 1/(2^{n-1})$
Iteration	$n \to n+1$	$L_n/2 \to L_{n+1}$
"Final" result	Cantor's ω	$L_\infty = 0$ is a point
Further applicability of the operation	$\omega+1, \omega+2, \omega+3, \ldots?$	'$L_\infty/2$' *has no sense*

TABLE 3. Formal isomorphism of the two mathematical processes.

modern "really working" mathematics point of view; and 3) that there is the obvious formal isomorphism between the processes of generating the natural numbers, on the one hand, and the infinite dichotomy of the segment $[0, 1]$, on the other hand (see Table 3), we arrive at the following *conditional* statement.

Corollary 1. If the application of the common operation '/2' to a common point of a segment has a mathematical sense, then the application of the addition operation '+1' to Cantor's minimal transfinite 'integer' ω has that same sense.

8.4.2. *Remark.* A professional nonstandard set theorist might object: within the framework of modern nonstandard analysis, a common Euclidian point is a finite segment, but only a very-very small, i.e., trans-transfinite, one and such a 'point' is divisible as usual finite segments but into smaller transfinite parts [Uspensky 1987]. I would like to emphasize that so far we have here considered foundations of Cantor's "naive" set theory which is far removed from the later inventions of nonstandard analysis. In fact, some of such problems of the nonstandard analysis were considered earlier in [Zenkin 1999].

If we wish to place our trust not in Cantorian verbal directives, but in their real, according to Wittgenstein, "rule-governed" usage within the framework of real metamathematical proofs, we must agree that the application of the operation '+1' to Cantor's 'thing' ω has the same "mathematical" sense as the application of the operation '/2' to the common Euclidian 'individual' geometrical point.

9. A FUNNY DEVELOPING *meta*MATHEMATICAL GAME FOR TWO HONEST TRICKSTERS.

Wittgenstein was very skeptical of Cantor's diagonal argument.

Cantor's proof was suspected of being a senseless metamathematical game (see [Wittgenstein 1956], [Hodges 1998]): a man tries day in, day out to put all irrational numbers into a series; when *all* reals are put into a series and enumerated and this "idiotical work" is finally ended, a trickster (Cantor's diagonal procedure) suddenly appears and says to the man: "Of course, you have just enumerated *all* reals and you have utilized *all* natural numbers, however, please, here is a *new* real, yet you lack even one spare natural number in order to enumerate this single real. Consequently, a number of reals is greater than a number of natural numbers."

Consider now the following natural continuation of this, according to Wittgenstein, "daft [metamathematical] activity" [Zenkin 2002a].

So, we already have one (Cantor's) trickster (hereafter: a Cheat 1) who is able to do the following metamathematical conjuring trick: *After*(!) a given enumeration (**E**) has been presented, Cheat 1 shows (generates, defines, invents, takes out of his sleeve, etc.) a *new* (anti)diagonal Cantor real y_1 which is different from every real of the enumeration (**E**).

Now we introduce a second trickster (hereafter: a Cheat 2) who is able to change the *countable* set of *indexes* $\{1, 2, 3, \dots\}$ in the given enumeration (**E**) to another *countable* set of *indexes*, say $\{2, 3, 4, \dots\}$. Naturally without any modifying of the *number* and *order* of the reals themselves in the initial sequence (**E**).

Since the final result of applying Cantor's diagonal algorithm depends *only* upon the number and order of the reals in (**E**) and does not depends upon any indexing of these reals, Cheat 1 (and all his admirers) simply has not algorithmical tools to observe the artful algorithmical operation of the Cheat 2.

However, I must especially emphasize here that the both of our Cheats are absolutely honest in the sense that they do not violate any law of any logic: simply Cheat 1 plays by the rule based upon the property of *actuality* of the *actually infinite* sequence (**E**) or, what amounts to the same thing, upon Cantor's Axiom; whereas Cheat 2 plays by the rule based upon the property of the *infinity* of the same *actually infinite* sequence (**E**), or, more precisely, upon the transitivity law of the equivalence relation (between all *countable* sets).

Consider the following purely metamathematical game and its implications for Cantor.

The starting state, in accordance with Cantor's diagonal proof, is as follows.

"A given arbitrary enumeration

(**E**) $$x_1, x_2, x_3 \dots$$

contains *all* real numbers of X."

Step 1: Our Cheat 2 with stealth absconds with a single index, say '1,' and reindexes the sequence (**E**) to form the sequence:

$$(\mathbf{E}^*) \qquad x_2, x_3, x_4, \ldots,$$

so that the sequence ($\mathbf{E}^*$) contains *the same* reals in *the same* order: x_2 of ($\mathbf{E}^*$) is equal to x_1 of (**E**), x_3 of ($\mathbf{E}^*$) is equal to x_2 of (**E**), and so on.

Step 2: Cheat 1, using (**E**) or, what is the same, ($\mathbf{E}^*$), creates a *new* real number, say y_1, and makes the following claim: "So, now you lack even one free natural number in order to index my *new* real number, y_1. Consequently, the number of all my real numbers is greater than the number of all your natural numbers, i.e., scientifically speaking, the cardinality of X is greater than the cardinality of N."

Step 3: Cheat 1, again openly, takes index '1' from his sleeve and claims: "Your card is covered; here is a free, spare natural number to index your nonindexed real number y_1."

Step 4: A game referee indexes the new (Cantor's) real number y_1 of Cheat 1 using the number (actually the numeral) '1' of Cheat 2, puts it in its natural first place within the given enumeration ($\mathbf{E}^*$) as follows:

$$(\mathbf{E.1}) \qquad y_1, x_2, x_3, x_4 \ldots$$

and claims: "Since Cheat 1 has no new reals at this moment, the enumeration (**E.1**) now contains *all* the real numbers of X. Draw game: 0:0!"—Consequently, the number of reals is not greater than the number of natural numbers.

It is obvious that now our Cheats are free to return to Step 1 and repeat the steps of the game. Then again. And so on to infinity.

Consequently, and this is shown in [Zenkin 2000a], [Zenkin 2001], [Zenkin 1997a], [Zenkin 1997b], [Zenkin 1997c], [Zenkin 2000c], [Zenkin and Zenkin 2002]), we have the following *potentially* infinite "reasoning" (here $\mathbf{B}$ = "an enumeration contains *all* real numbers of X"):

$$(\mathbf{L3}) \qquad \mathbf{B} \rightarrow \neg\mathbf{B} \rightarrow \mathbf{B} \rightarrow \neg\mathbf{B} \rightarrow \mathbf{B} \rightarrow \neg\mathbf{B} \rightarrow \mathbf{B} \rightarrow \ldots$$

In [Zenkin 2000b], [Zenkin 1997e], the necessary and *sufficient* conditions of paradoxicality as a whole are formulated, and by means of the classical model theory it has been proven rigorously that the true nature of the "Liar" and similar paradoxes is described not by the traditional *finite* form (**L**) but by the following *potentially infinite* "reasoning":

$$\text{(L4)} \qquad \mathbf{A} \to \neg\mathbf{A} \to \mathbf{A} \to \neg\mathbf{A} \to \mathbf{A} \to \neg\mathbf{A} \to \mathbf{A} \to \ldots$$

The formal coincidence of *infinite* paradoxical "reasonings" (**L3**) and (**L4**) is no mere accident. It shows that Cantor's diagonal proof contains the *infinite* fragment (**L3**) which is in fact a new "Liar"-type set theoretic paradox (a paradox for both "naive" and "nonnaive" set theory). Until the *potentially* infinite "reasoning" of (**L3**) is completed, Cantor's conclusion "Consequently, the assumption that X is countable is false" is from the standpoint of classical logic invalid, since the "conclusion" contains a fatal, very insidious logic error called "jump to a (very desired) conclusion." Since the *potentially* infinite "reasoning" (**L3**), according to Aristotle, can never be completed, Cantor's diagonal proof is incapable of completion and, consequently, the theorem concerning the uncountability of the continuum is simply unprovable from the standpoint, that is, of classical logic.

10. Proof of Aristotle's Thesis "*Infinitum Actu Non Datur.*"

As said above, according to Aristotle's definition of 'potentially infinity,' "...the infinite exists through one thing being taken after another, what is taken ***always being finite***, but ever other and other" [Aristotle] (cited by A. W. Moore [Moore 1993]).

What is important about the definition is that any potentially *infinite* set (process) is always presented in reality as *finite*, that is, as a *finite* approximation to an infinite "ideal." All really working mathematics is based on this definition with no real problem.

However, Cantor's proof allows us to generalize Aristotle's definition of 'potentially infinite' in the following way [Zenkin 2000a].

At every step in the process (**L3**) we have an *actually infinite* (**E**)-type sequence of consisting entirely of reals. The application of Cantor's diagonal rule (**R**) to the sequence, explicitly acknowledging the *actuality* of the sequence, generates a *new* real differing *from all pre-existing* reals. Then, using the *infinity* of the same sequence, we can now generate a *new* sequence which includes the preceding one as well as this *new* Cantorian (anti)diagonal real, i.e., we get the *new* sequence which again contains *all* reals. In other words, originating from the initial Cantor's enumeration (**E**), the process (**L3**) proves that no *actually infinite* set of reals can contain all reals, i.e., it refutes the notion that the property itself of the set X of being 'actual' in Cantor's sense, according to which the actual set contains *all* its elements.

It allows us to generalize Aristotle's definition of 'potential infinity' to the case where its 'thing' is not finite in the following manner:

> ... the infinite [of the set of all reals (continuum)] exists through one thing [a sequence of all reals plus new Cantor's real] being taken one after another [a preceding sequence of all reals], what is taken ***being always [actually infinite]***, but ever other and other.

In other words, even if we assume that the set X of all reals is *actually* infinite, i.e., comprises *all* its elements, the process (**L3**) proves by construction that the set X can't contain *all* reals, i.e., the set X is ultimately a *potentially* infinite set.

Thus, the Aristotle's famous thesis "Infinitum Actu Non Datur" refers not only to *finite* 'things,' but also to *actually infinite* 'things' pointing out to a similarity of the properties 'to be finite' and 'to be actual' *within the framework of Cantor's diagonal proof.*

Indeed, there is another important aspect of the process (**L3**), viz. that at every step Cantor, explicitly using the property 'to be *actual*' attributed to an *actually infinite* sequence, constructs a *new real* and proves **not-B**. Now we, explicitly using the property 'to be *infinite*' of the same *actually infinite* sequence, can construct a *new sequence* of reals and prove **B**. From the classical logic point of view this means that the notions 'actual' and 'infinite' are *contradictory.* Yet in fact the only notion which is contradictory to 'infinite' always was and remains now the notion of the 'finite.' Consequently, from a logical point of view, the notions 'actual' and 'finite' are algorithmically identical. The last means that *within the framework of Cantor's diagonal proof* the 'actual infinite' signifies in (metamathematical) practice a 'finite infinite.' In short, the notion 'actual infinity' is, *within the framework of Cantor's diagonal proof*, a self-contradictory notion and therefore its *algorithmical* usage is, according to Gauß, inadmissible in mathematics.

11. Main conclusions: Whether the Lord exists in Cantor's transfinite 'paradise'?

1. The traditional Cantor's theorem regarding the uncountability of the continuum had never two *necessary* conditions of its proof formulated in an explicit form. In classical mathematics such a situation is considered as absurd and simply inadmissible. However, just this quite scandal fact explains why Kronecker, Poincaré and many other outstanding mathematicians could not "physically" disprove Cantor's theorem in a straightforward mathematical way: it is impossible to

prove a mathematical failure of what does not exist at all. The explication of these two conditions and their logical analysis has shown that Cantor's diagonal proof is invalid from the point of view of classical logic and 'really working' mathematics.

2. It has been proved that the notion of 'actual infinity' is self-contradictory, i.e., the set-theoretical properties 'to be actual' and 'to be finite' are *algorithmically* indistinguishable, *within the framework of Cantor's diagonal proof.*

3. It has been proved that Cantor's 'theory' of the *cardinal transfinite* 'integers,' based on Cantor's axiom which states that "all infinite sets are actual" and his theorem on the uncountability of continuum, is invalid, and therefore Cantor's conception itself stating an 'existence' of actually infinite sets differing in their cardinalities and his famous '*aleph*'atic series of increasing transfinite cardinals have no sense from the point of view of classical logic and 'really working' mathematics.

4. It has been proved that Cantor's 'theory' of the *ordinal transfinite* 'integers' is *conditionally* consistent, i.e., if and only if Peano's axiomatics are consistent, but that the mathematical sense of this 'theory' is exhausted by the Aristotle-Peano inductive definition of the common series (**NN**) of the common *finite* natural numbers, and it's quite doubtful that Cantor's ω-adic series (**C2**) of the transfinite ordinals indeed leads us "unto Heaven."

5. The great intuitive *scientific* insight of Aristotle, Gauß, Kronecker, Cauchy, Poincaré, and many other outstanding logicians and mathematicians, "*Infinitum Actu Non Datur*," is *rigorously proved* here for the first time.

6. It is obvious that the disproof of Cantor's theorem on the uncountability of continuum and a posterior unavoidable rejection from the set-theoretical 'distinguishing' of infinite sets by their cardinalities will essentially change the traditional logical and methodological paradigms of modern mathematics and philosophy of infinity, and open a real way to resolve central problems connected with Great Crises I, II, and III in foundations of mathematics.

7. Every metamathematician and set theorist, from a child's age, is familiar with the 'bold' and 'proud' challenge sent by Hilbert (to whom?): "*No one shall expel us from the paradise which Cantor has created for us!*" [Hilbert 1948].

However, I believe that many of them also remember the following, quite enlightening Story (Holy Bible, "Genesis," Chapter 11).

> And the whole earth was of one language, and of one speech. [...]
>
> And it came to pass, as they journeyed from the east, that they found a plain in the land of Shi'-nar; and they dwelt there.
>
> And they said one to another, Go to, let us make brick, and burn them throughly. And they had brick for stone, and slime had they for morter.
>
> And they said, Go to, let us build us a city and a tower, whose top may reach unto heaven; and let us make us a name, lest we be scattered abroad upon the face of the whole earth.
>
> And the Lord came down to see the city and the tower, which the children of men builded.
>
> And the Lord said, Behold, the people is one, and they have all one language; and they begin to do: and now nothing will be restrained from them, which they have imagined to do.

As is well known, this pretentious, ambitious building of Babel 1 "unto heaven" was not liked by the Lord and He confounded "their language, that they may not understand one another's speech.

"So the Lord scattered them abroad from thence upon the face of all the earth: and they left off to build the city" and Babel 1 "whose top may reach unto heaven."

In such a way the Lord punished the people for their crazy and smug pride.

As is known, this Story is placed in the *very beginning* of the Holy Bible. Is the fact accidental? I don't think so. In particular, because ... in the *very end* of the Bible there is another Story as to "The Cretians are always liars, evil beasts, slow bellies" ("The Epistle of Paul to Titus," Chapter 1, Verse 12).

This story is known in science as the "Liar" paradox.

As was shown above (and strictly proved in [Zenkin 2000b], [Zenkin 1997e]), a veritable logical sense of the "Liar" is not the traditional *finite* 'binomial' conjunction (**L**), but the *infinite* "reasoning" (**L4**) which, being written in its explicit *semantic* form, is as follows (here T = true, F = false):

(**L5**) $$\mathbf{T} \to \mathbf{F} \to \mathbf{T} \to \mathbf{F} \to \mathbf{T} \to \mathbf{F} \to \mathbf{T} \to \dots$$

I suspect that the Bible's "Liar" (**L5**) is not a common paradox, it is an Omen and Warning to people to forbear in future from a building of new ambitious Babels, "whose top may reach unto heaven." However this time the Lord, taking into account a possible over-the-limit, 'left-hemispherical,' metamathematical 'progress' in the people's brains, confounded not languages, but truth and falsity that people "may not understand one another's speech" and "[leave] off to build" any new Babels in future.

During two millenniums this Omen was existing in its latent (*potential*) form as a quite harmless logical puzzle. Cantor's ambitious transfinite 'stairway unto heaven' (Babel 2) *actualized* this highly explosive mixture in the form of the infinite "reasoning" (**L3**) and truth and falsity became indistinguishable within the framework of his "Study on Transfinitum" and modern metamathematics and axiomatic set theory.

However today the problem, in all likelihood, overstepped the limits of the abstract, academic sciences and became a global social phenomenon. To understand that it's enough simply to glance around or to look at press-, media- and TV-mirrors of our real life.—Whether anybody is able today surely to distinguish truth from falsehood and vice versa?

"Hath not God made foolish the wisdom of this world?"—With what an aim? ...

"He that hath an ear, let him hear ..."

8. We have a Project as to how to solve all the problems touched upon in the paper and welcome any serious collaboration to realize the Project.

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References

[Alexandrov 1948] P. S. Alexandrov, *Introduction to common theory of sets and functions*, Gostehizdat, Moscow-Leningrad, 1948.

[Aristotle] Aristotle, *Physics*, Book III, p. 6.

[Arnold 1999] V. I. Arnold, "Anti-scientific revolution and mathematics," *Vestnik RAS*, 1999, No. 6, pp. 553–558.

[Bourbaki 1965] N. Bourbaki, *Set theory*, MIR, Moscow, 1965.

[Brouwer 1954] L. E. J. Brouwer, "Points and Spaces," *Canadian J. of Math.*, Vol. 6 (1954), pp. 1–17.

[Cantor 1914] G. Cantor, *Foundations of common study about manifolds*, NEW IDEAS IN MATHEMATICS, No. 6, ed. A. V. Vassil'jev, SPb, 1914.

[Cantor 1985] ——— *Proceedings on set theory*, NAUKA, Moscow, 1985.

[Capiński and Kopp 1999] M. Capiński and E. Kopp, *Measure, integral and probability*, Springer, London, 1999.

[Chaitin 2001] G. J. Chaitin, "The Omega man," *New Scientist* magazine, 10 March 2001.

[Cohen 1966] Paul J. Cohen, *Set theory and the continuum hypothesis*, W. A. Benjamin, Inc., Amsterdam, 1966, Chapter II, §4, "Cardinal Numbers."

[Davis 1980] M. Davis, *Applied non-standard analysis*, MIR, Moscow, 1980.

[Feferman 1998] S. Feferman, *In the light of logic*, Oxford University Press, 1998.

[Fichtengolz 1958] G. M. Fichtengolz, *Course of differential and integral calculus*, Moscow-Leningrad, 1958.

[Fraenkel and Bar-Hillel 1958] A. A. Fraenkel and Y. Bar-Hillel, *Foundation of set theory*, North-Holland Publishing Company, Amsterdam, 1958.

[Hilbert 1948] D. Hilbert, *Foundations of geometry. On infinity*, OGIZ, Moscow-Leningrad, 1948.

[Hodges 1998] W. Hodges, "An editor recalls some hopeless papers," *The Bulletin of Symbolic Logic*, Vol. 4 (1998), No. 1, pp. 1–17.

[Kleene 1957] S. Kleene, *Introduction to metamathematics*, MIR, Moscow, 1957.

[Kleene 1967] ———, *Mathematical logic*, John Wiley & Sons, Inc., 1967.

[Moore 1993] A. W. Moore, "The infinity," *The Problems of Philosophy. Their Past and Present*, ed. Ted Honderich, London, New York, 1993.

[Peregrin] Ja. Peregrin, "Structure and meaning," `http://www.cuni.cz/~peregrin/HTMLTxt/str&mea.htm`.

[Poincaré 1983] H. Poincaré, *On science*, "Science," Moscow, 1983.

[Raatikainen 2001] Panu Raatikainen, "Review of G. Chaitin's books *Exploring Randomness* and *The Unknowable*," *Notices of the AMS*, Vol. 48 (2001), No. 9, pp. 992–996. See at `http://www.ams.org/notices/200109/200109-toc.html`.

[Turchin] V. F. Turchin, "Infinity," `http://pespmc1.vub.ac.be/infinity.html`.

[Uspensky 1987] V. A. Uspensky, *What is a non-standard analysis?* Nauka, Moscow, 1987, p. 117.

[Vopenka 1983] P. Vopenka, "Mathematics in an alternative set theory," *News in a foreign science. Mathematics*, No. 31 (1983), Moscow, p. 124.

[Wikipedia] *Wikipedia, the free encyclopedia.* `http://www.wikipedia.com`, `http://www.wikipedia.com/wiki/Actual+infinity`.

[Wittgenstein 1956] L. Wittgenstein, *Remarks on the foundations of mathematics*, Blackwell, Oxford, 1956.

[Zenkin 1997a] A. A. Zenkin, "Cognitive visualization of some transfinite objects of the classical Cantor set theory," in the collection *Infinity in Mathematics: Philosophical and Historical Aspects*, ed. Prof. A. G. Barabashev, "Janus-K," Moscow, 1997, pp. 77–91, 92–96, 184–189, 221–224.

[Zenkin 1997b] ———, "On logic of some quasi-finite reasonings of set theory and meta-mathematics. New paradox of Cantor's set theory," *News of Artificial Intelligence*, 1997, No. 1, pp. 64–98, 156–160

[Zenkin 1997c] ———, "The time-sharing principle and analysis of one class of quasi-finite plausible reasonings (with G.Cantor's theorem on the uncountability as an example)," *Doklady Mathematics*, Vol. 56 (1997), No. 2, pp. 763–765.

[Zenkin 1997d] ———, "Cognitive visualization of the continuum problem and of the hyper-real numbers theory," International Conference 'Analyse et Logique,' UMH, Mons, Belgia, 1997, *Abstracts*, pp. 93–94.

[Zenkin 1997e] ———, "Automated classification of logical and mathematical paradoxes. On one 'physical' model of the 'Liar' paradox," *News of Artificial Intelligence*, 1997, No. 3, pp. 69–79.

[Zenkin 1997f] ———, "Whether the Lord exists in G.Cantor's transfinite paradise?" *News of Artificial Intelligence*, 1997, No. 1, pp. 156–160.

[Zenkin 1997g] ———, "Transfinite cavitation in the ranks of G.Cantor's ordinals," *News of Artificial Intelligence*, 1997, No. 3, pp. 131–137.

[Zenkin 1998] ———, "Cognitive visualization of the continuum problem and mirror symmetric proofs in transfinite numbers mathematics," ISIS-Symmetry Congress and Exhibition, Book of Abstracts, p. 156, Haifa, Israel, 1998.

[Zenkin 1999] ______, "Cognitive (semantic) visualization of the continuum problem and mirror symmetric proofs in the transfinite numbers theory," *Visual Mathematics: Art and Science Electronic Journal of ISIS-Symmetry*, Vol. 1 (1999), No. 2, `http://members.tripod.com/vismath1/zen/index.html`.

[Zenkin 2000a] ______, "George Cantor's mistake," *Voprosy Filosofii* (Philosophy Problems), 2000, No. 2, pp. 165–168. See at: `http://www.channel1.com/users/srbayne/histanalytic2.htm`, `http://www.com2com.ru/alexzen/papers/The_Cantor_Paradise.htm`.

[Zenkin 2000b] ______, "New approach to analysis of the paradoxes problem," *Voprosy Filosofii* (Philosophy Problems), 2000, No. 10 (October), pp. 79–90. `http://www.com2com.ru/alexzen/papers/vf2/vf2-rus.html`.

[Zenkin 2000c] ______, "On logic and philosophy of G.Cantor's 'Study on transfinity,'" *Modern Logic: Problems of History, Theory and Scientific Applications*, Proceedings of the Sixth All-Russian Conference, Sankt-Peterburg, 2000, pp. 458–461.

[Zenkin 2000d] ______, "Scientific counter-revolution in mathematics," "NG-SCIENCE" supplement to the *Independent Newspaper* (*Nezavisimaya Gazeta*) of 19 July 2000, p. 13. At the Web site: `http://science.ng.ru/magnum/2000-07-19/5_mathem.html` (in Russian) `http://www.com2com.ru/alexzen/papers/ng-02/contr_rev.htm` (in English).

[Zenkin 2001] ______, "Infinitum Actu Non Datur," *Voprosy Filosofii* (Philosophy Problems), 2001, No. 9 (September), pp. 157–169.

[Zenkin 2002a] ______, "Scientific intuition of genii against mytho-'logic' of Cantor's transfinite 'paradise,'" International Symposium 'Philosophical Insights into Logic and Mathematics (PILM 2002): The History and Outcome of Alternative Semantics and Syntax,' 2002, Nancy, France.

[Zenkin 2002b] ______, "Dangerous illness of modern mathematical education," Ninth International Conference 'Mathematics. Computer. Education,' Dubna, Russia, 2002, *Abstracts*, Section 1, Humanitarian and Natural-Scientific Education, p. 316.

[Zenkin 2002c] ———, "Goedel's numbering of multi-modal texts," *The Bulletin of Symbolic Logic*, Vol. 8, No. 1, 2002, p. 180. `http://www.math.ucla.edu/~asl/bsl/0801/0801-006.ps`.

[Zenkin 2003] ———, "Etalon splitting of meta-mathematical consciousness," in the collection *Mathematics and Experience*, ed. A. G. Barabashev, 2003. See `http://www.philosophy.ru/library/math/zen-kog2.pdf` (in Russian).

[Zenkin and Zenkin 2000] Al. Zenkin and An. Zenkin, "Throughout Full Of Gaps Continuum: From the Language of Abstractions to the Language of Images. And Backwards," International Conference 'Non-linear World,' Collection of scientific proceedings, Progress-Traditsija, Moscow, 2000, pp. 172–179.

[Zenkin and Zenkin 2002] ———, "On one reconstruction of Wittgenstein's objection against Cantor's diagonal argument," Seventh Scientific Conference 'Modern Logic: Problems of Theory, History, and Applications in Science,' Sankt-Petersburg, 20–22 June 2002, Conference Proceedings, pp. 320–323.

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