ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 5, 2012

GAUSSIAN MAPS FOR DOUBLE COVERS OF TORIC SURFACES

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1. Introduction. In this paper, we apply the work of Duflot [**3**] on the Gaussian map for double covers of smooth projective varieties to make a cohomological study of Gaussian maps for certain divisors on double covers of smooth toric surfaces, and devote more particular attention to the special case of double covers of Hirzebruch surfaces. These analyses require earlier work on Gaussian maps on smooth toric surfaces and Hirzebruch surfaces done in [**4, 14**]. We focus on cohomological analyses for divisors on such double covers and do not address geometric consequences of the analyses. Such geometric aspects of the Gaussian map are discussed, for example, in [**2, 18–20**]. We plan to return to more geometric considerations in later work.

An outline of the paper is as follows. We give a brief exposition of the subject of smooth toric surfaces, in order to set the notation we use, and also discuss the special case of Hirzebruch surfaces.

We'll next do the following, in successive sections of the paper:

• review the cohomology computations of [**4, 14**] for Hirzebruch surfaces in particular and smooth toric surfaces more generally;

• move on to study multiplication maps for the cohomology of line bundles and 1-forms on toric surfaces;

• recall basic definitions concerning Gaussian maps from Wahl [**18, 19**] and establish surjectivity results for Gaussian maps on Hirzebruch surfaces, and more generally, smooth toric surfaces;

• review the theory of double covers and Gaussian maps for double covers;

• pause to show how to use these computations to study the Gaussian map for the canonical divisor for a double cover of a smooth toric surface; and finally,

Received by the editors on February 9, 2010.
DOI:10.1216/RMJ-2012-42-5-1471

• discuss various Gaussian maps on double covers of smooth toric surfaces, focusing on double covers of Hirzebruch surfaces in particular.

2. Toric surfaces. Here, we set the notation we use for smooth toric surfaces. General references include Fulton [**6**] and Oda [**15**].

We consider a fan Δ in $\mathbf{Z}^2$ formed from $n+2$ vectors with initial side the vector $(a_0, b_0) = (0,1)$, terminal side $(a_{n+1}, b_{n+1}) = (1,0)$, and intermediate vectors arrayed counterclockwise between these two, labeling the vectors consecutively $\{(a_i, b_i)\}_{i=0}^{n+1}$. For any two consecutive vectors, we require that the determinant of their 2×2 matrix be 1. We will call such a fan a nonsingular fan.

From a nonsingular fan, Δ, we can construct a complex manifold $S = S(\Delta)$ as a quotient space of $n+2$ disjoint copies of $\mathbf{C}^2$, $S = \mathbf{C}^2 \coprod \mathbf{C}^2 \coprod \cdots \coprod \mathbf{C}^2 / \sim$. The equivalence relation, $\sim$, is defined as the equivalence relation generated by:

$$(x_0, y_0) \sim (x_i, y_i) \longleftrightarrow (x_i, y_i) = (x_0^p\ y_0^q, x_0^r\ y_0^s)$$

and x_0, y_0 are such that $x_0^p\ y_0^q$ and $x_0^r\ y_0^s$ make sense; p, q, r, s are defined by

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} a_{i-1} & a_i \\ b_{i-1} & b_i \end{bmatrix}^{-1} \doteq A_i^{-1}, \quad 0 \le i \le n+1.$$

In the above equations,(x_i, y_i) is a point in the ith $\mathbf{C}^2$ of the disjoint union, called $\mathbf{C}_i^2$, for $0 \le i \le n+1$. Additionally, the equivalence relation for comparing elements of $\mathbf{C}_i^2$ to $\mathbf{C}_j^2$ is

$$(x_i, y_i) \sim (x_j, y_j) \longleftrightarrow (x_j, y_j) = (x_i^\alpha\ y_i^\beta, x_i^\gamma\ y_i^\delta)$$

where $0 \le i, j \le n+1$ and

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = A_j^{-1} A_i.$$

Let $q : \mathbf{C}_0^2 \coprod \mathbf{C}_1^2 \coprod \cdots \coprod \mathbf{C}_{n+1}^2 \to S$ be the quotient map and $U \subset S$ with $U = \{a \in S \mid q^{-1}(\{a\})$ has exactly $n+2$ elements$\}$. Also, let

$$C_i \doteq \{q(x_i, 0) \mid (x_i, 0) \in \mathbf{C}_i^2\} \cup \{q(0, y_{i+1}) \mid (0, y_{i+1}) \in \mathbf{C}_{i+1}^2\}.$$

Note that $S - U = C_0 \cup C_1 \cup \cdots \cup C_n$ where:

a. $C_i \cap C_j$

$$= \begin{cases} q((0,0)_{i+1}) & \text{if the vectors } (a_i, b_i) \text{ and } (a_j, b_j) \text{ are adjacent,} \\ & j = i+1; \\ \varnothing & \text{if the vectors } (a_i, b_i) \text{ and } (a_j, b_j) \text{ are not adjacent} \\ & \text{and} \end{cases}$$

b. each C_i is isomorphic to $\mathbf{P}^1$.

In the above, $(0,0)_{i+1}$ is the origin in $\mathbf{C}_{i+1}$.

Thus, $S - U$ is equal to the union of $n+2$ $\mathbf{P}^1$s arranged in a cycle where $C_i^1 \cap C_j^1$ is a single point if and only if i and j are consecutive. If $U_i \doteq q(\mathbf{C}_i^2)$, we may define charts on S by noting that

a) U_i is open in S; and

b) if $\varphi_i : \mathbf{C}^2 \to U_i$ is the map defined by $\varphi_i(a,b) = q(a,b) \in U_i$, then the map $\varphi_j^{-1}\varphi_i$,

$$\varphi_i^{-1}(U_i \cap U_j) \xrightarrow{\varphi_j^{-1}\varphi_i} \varphi_j^{-1}(U_i \cap U_j),$$

where $\varphi_i^{-1}(U_i \cap U_j) \subseteq \mathbf{C}^2$ and $\varphi_j^{-1}(U_i \cap U_j) \subseteq \mathbf{C}^2$, is bi-holomorphic.

In the chart $(\mathbf{C}^2, \varphi_i)$, C_i is defined by $y_i = 0$. In the chart $(\mathbf{C}^2, \varphi_{i+1})$, C_i is defined by $x_{i+1} = 0$. Also, we have:

$$C_i \cap U_j = \begin{cases} \{q(x_i, 0) \mid x_i \in \mathbf{C}\} & j = i \\ \{q(0, y_{i+1}) \mid y_{i+1} \in \mathbf{C}\} & j = i+1 \\ \varnothing & j \neq i,\ i+1. \end{cases}$$

We'll later use the coordinates established above on S to present cohomology computations.

Definition 2.1. The Hirzebruch surface, $\mathbf{F}_k$, is the smooth toric surface defined by the four vector nonsingular fans, $(0,1), (-1,0), (k,-1)$ and $(1,0)$ where $k > 0$.

2.1. Divisors on smooth toric surfaces. We summarize divisor computations and facts about divisors on smooth toric surfaces that we will need here. References include Fulton [**6**], Oda [**15**] and Murray [**14**]. However, we use the notation conventions of the previous section.

Lemma 2.2. *Let $S = S(\Delta)$ be a smooth toric surface defined by the nonsingular fan*

$$\Delta = \{(a_i, b_i) \mid 0 \le i \le n+1\}.$$

• *Any divisor on S is linearly equivalent to a unique integer linear combination of curves $C_1, \dots, C_n$ (i.e.,* $\mathrm{Pic}\,(S)$ *is the free abelian group on the set $\{C_1, \dots, C_n\}$).*

• *C_{n+1} is linearly equivalent to $-\sum_{i=1}^n a_i C_i$, and C_0 is linearly equivalent to $-\sum_{i=1}^n b_i C_i$.*

For $\mathbf{F}_k$, this yields $C_0 \sim C_2$ and $C_3 \sim C_1 - kC_2$.

Through an abuse of notation, we often write the relation of linear equivalence between divisors on toric surfaces as an equality, rather than using the symbol "$\sim$."

In addition, in the rest of this paper, given a smooth toric surface S defined by the nonsingular fan $\Delta = \{(a_i, b_i) \mid 0 \le i \le n+1\}$, the curves C_i are exactly those defined in this section, using the coordinates established here.

The canonical divisor of a smooth toric surface S defined by the nonsingular fan, Δ, may be computed as (see, e.g., [**14**])

$$K_S = -C_0 - C_1 - \cdots - C_{n+1} = \sum_{i=1}^{n} (a_i + b_i - 1) C_i;$$

this specializes to the surface $\mathbf{F}_k$ as

$$K_{\mathbf{F}_k} = -C_0 - C_1 - C_2 - C_3 = -2C_1 + (k-2)C_2.$$

The intersection numbers for the cycle of curves C_i on the smooth toric surface defined by the nonsingular fan, Δ, are:

$$C_i \cdot C_j = \begin{cases} C_i \cdot C_i = -(a_{i-1}b_{i+1} - a_{i+1}b_{i-1}) & \text{if } j = i \\ 1 & \text{if } j = i \pm 1 \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

for $\mathbf{F}_k, C_1 \cdot C_1 = k$, $C_1 \cdot C_2 = 1$, $C_0 \cdot C_0 = 0$, $C_3 \cdot C_3 = -k$ and $C_2 \cdot C_2 = 0$. If we regard $\mathbf{F}_k$ as the rational normal scroll S_k (using the notation of [**7**]), rather than a toric surface, we know that it possesses a defining bundle map $S_k \to \mathbf{P}^1$. From this point of view, C_1 corresponds to a zero section of this bundle map, C_2 corresponds to a fiber of the bundle map and C_3 corresponds to the unique irreducible curve on S_k of negative self-intersection. From this picture (see, e.g., [**7**, page 518 ff.]) one may deduce that, if C is an irreducible curve on $\mathbf{F}_k = S_k$, $C \neq C_3$, then, when we write $C \sim m_1 C_1 + m_2 C_2$, we must have $m_1 \geq 0$, $m_2 \geq 0$.

2.2. Polygons and divisors: Results from Oda. In this section, we summarize the results of Demazure and Oda, as presented in Oda [**15**], for the case of the nonsingular fan

$$\Delta = \{(a_i, b_i) \mid 0 \leq i \leq n+1\}.$$

Let S be the smooth toric surface defined by Δ. We define

$$u_i = (a_i, b_i), 0 \leq i \leq n+1,$$

for convenience. We use orthogonality properties in $\mathbf{R}^2$ to simplify notation, and the inner product $\langle *, * \rangle$ is the usual inner product on $\mathbf{R}^2$.

Definition 2.4. The polygon associated to a divisor. Suppose that S is a smooth toric surface, defined by a nonsingular fan

$$\Delta = \{(a_i, b_i) \mid 0 \leq i \leq n+1\}.$$

Given a divisor $E = \sum_{i=0}^{n+1} e_i C_i$ on S (here, "=" means "equals" and not "linearly equivalent to"), define a closed, convex subset P_E (possibly an empty set) of $\mathbf{R}^2$ as follows:

$$P_E \doteq \{(x,y) \in \mathbf{R}^2 \mid \langle (x,y), u_i \rangle \geq -e_i \quad \text{for all } i\}.$$

Now, we have been careful, in the above definition, to distinguish between equality and linear equivalence of divisors. However, we shall often, but not always, blur this distinction as follows. Suppose that

$E = \sum_{i=0}^{n+1} e_i C_i$ is a divisor on S, as in the above definition. We know that $C_0 \sim -\sum_{i=1}^{n} b_i C_i$ and $C_{n+1} \sim -\sum_{i=1}^{n} a_i C_i$. Thus, E is linearly equivalent to the divisor

$$\widetilde{E} = \sum_{i=1}^{n} (e_i - e_0 b_i - e_{n+1} a_i) C_i;$$

noting that the coefficients of C_0 and C_{n+1} in the divisor $\widetilde{E}$ are zero, we see that the polygons P_E and $P_{\tilde{E}}$ are translates of each other (since the fan is nonsingular):

$$P_{\tilde{E}} = P_E - (e_{n+1}, e_0).$$

When we make an assumption that a divisor "equals" a linear combination of the C_is, where the coefficients of C_0 and C_{n+1} are zero, we are replacing the divisor with a linearly equivalent divisor, tacitly, in the above way, and we are leaving to the reader the verification that this does not affect the proofs where this is done.

If E is a divisor, $E = \sum_{i=0}^{n+1} e_i C_i$, the nonsingularity of the fan Δ means that unique $l_i(E) \in \mathbf{Z}^2$ exist for $0 \le i \le n+1$ such that

$$\langle l_i(E), u_i \rangle = -e_i,$$
$$\langle l_i(E), u_{i-1} \rangle = -e_{i-1}$$

for each i. In fact, we see that

$$l_i(E) = \begin{bmatrix} b_i & -b_{i-1} \\ -a_i & a_{i-1} \end{bmatrix} \begin{bmatrix} -e_{i-1} \\ -e_i \end{bmatrix} \in \mathbf{Z}^2,$$

for each i.

Note that, if we apply the conditions $e_0 = e_{n+1} = 0$, since $u_0 = (0,1)$, $u_{n+1} = (1,0)$, then $l_0(E) = (0,0)$, $l_1(E) = (e_1, 0)$, $l_{n+1}(E) = (0, e_n)$ and P_E is contained in the first quadrant.

Definition 2.5. Edges, bounding lines, vertices and geometric vertices. For each i, $0 \le i \le n+1$, if $E = \sum_{i=0}^{n+1} e_i C_i$,

- $\sigma_i(E) \doteq \{t l_{i+1}(E) + (1-t) l_i(E)) \mid t \in [0,1]\}$ is the ith *edge* of P_E.

• $l_i(E)$ is henceforth called the ith *vertex* of P_E.

• $L_i(E) \doteq \{(x,y) \in \mathbf{R}^2 \mid \langle (x,y), u_i \rangle = -e_i\}$ is the ith *bounding line* of P_E.

• A *geometric vertex* of P_E is a point v on the topological boundary of P_E such that no line segment in $\mathbf{R}^2$ containing v in its interior (computed with respect to the subspace topology on the line segment) is entirely contained in P_E. (Note that a line segment is not a point.)

The terminology above does not necessarily mean that $\sigma_i(E)$ is really one-dimensional, or that it is a subset of P_E, for example; or that $l_i(E)$ is an element of P_E.

Definition 2.6. Interior edge points and interior points. If $E = \sum_{i=0}^{n+1} e_i C_i$ is a divisor on the smooth toric surface S defined by the nonsingular fan $\Delta = \{u_i = (a_i, b_i) \mid 0 \le i \le n+1\}$, then for a point $(x,y) \in P_E$:

• if a unique i exists such that $e_i + \langle (x,y), u_i \rangle = 0$ and $e_j + \langle (x,y), u_j \rangle > 0$, for every $j \ne i$, then we say that (x,y) is an *interior edge point* of $\sigma_i(E)$.

• if, for every j, $e_j + \langle (x,y), u_j \rangle > 0$, we say that (x,y) is an *interior point* of the polygon P_E.

We see immediately that, for each i,

• $l_i(E)$ is the unique intersection point of $L_i(E)$ and $L_{i-1}(E)$,

• $\pm(b_i, -a_i)$ are vectors parallel to $L_i(E)$, and

• $\sigma_i(E) \subseteq L_i(E)$.

By direct calculation, using the explicit formula for $l_i(E)$ above, nonsingularity of the fan and the formulas for $C_i \cdot C_j$ given in the previous section, we see that, if

$$\alpha_i(E) \doteq e_{i+1} + e_{i-1} + e_i(C_i \cdot C_i) = E \cdot C_i, \tag{2.7}$$

then, for every i such that $0 \le i \le n+1$,

$$l_{i+1}(E) - l_i(E) = \alpha_i(E)(b_i, -a_i). \tag{2.8}$$

Note also that, if $D = \sum_{i=0}^{n+1} d_i C_i$ and $E = \sum_{i=0}^{n+1} e_i C_i$ are two divisors on S, then

$$l_i(D+E) = l_i(D) + l_i(E),$$
$$\alpha_i(D+E) = \alpha_i(D) + \alpha_i(E),$$

for $0 \le i \le n+1$. In addition, if the two divisors are linearly equivalent, say $E_1 \sim E_2$, then $E_1 \cdot C_i = E_2 \cdot C_i$ for every i; in other words, $\alpha_i(E_1) = \alpha_i(E_2)$, for every i.

Theorem 2.11 (Demazure's theorem [**15**, Corollary 2.15, subsection 2.3, page 83]). *Let S be a smooth toric surface defined by the nonsingular fan $\Delta = \{u_i = (a_i, b_i) \mid 0 \le i \le n+1\}$. Let $E = \sum_{i=0}^{n+1} e_i C_i$ be a divisor on S. Let $h_E : \mathbf{R}^2 \to \mathbf{R}$ be the unique function such that $h_E(u_i) = -e_i$ and $h_E(t_1 u_i + t_2 u_{i+1}) = -(t_1 e_i + t_2 e_{i+1})$, for every i and every $t_1, t_2 \ge 0$ in $\mathbf{R}$. The following are equivalent*:

a. *E is ample.*

b. *E is very ample.*

c. *For every $(x, y) \in \mathbf{R}^2$, $\langle l_i(E), (x, y)\rangle \ge h_E(x, y)$, for every i; furthermore, $\langle l_i(E), (x, y)\rangle = h_E(x, y)$ if and only if (x, y) is in the positive cone spanned by u_i and u_{i+1}.*

d. *P_E is a two-dimensional compact convex set, $\{l_i(E) \mid 0 \le i \le n+1\}$ is the complete set of geometric vertices of P_E, and $l_i(E) \ne l_j(E)$ if $i \ne j$. Furthermore, P_E is the convex hull of the set of $n+2$ vertices $l_0(E), l_1(E), \dots, l_{n+1}(E)$.*

Oda has shown that the Nakai criterion for ample divisors gives rise to the "toric Nakai criterion"; in the notation of this paper this becomes:

Theorem 2.12 (the toric Nakai criterion [**15**]). *Let S be a smooth toric surface defined by the nonsingular fan $\Delta = \{u_i = (a_i, b_i) \mid 0 \le i \le n+1\}$, and suppose that $E = \sum_{i=0}^{n+1} e_i C_i$ is a divisor on S (and the C_is are defined using the coordinates of this section). Then E is ample if and only if $\alpha_i(E) = e_{i+1} + e_{i-1} + e_i(C_i \cdot C_i) > 0$, for $0 \le i \le n+1$.*

For the Hirzebruch surfaces, this criteria resolves into the well-known:

Corollary 2.13. *Suppose $\mathbf{F}_k$ is the Hirzebruch surface defined using the coordinates of this section, with corresponding bases C_1, C_2 of the Picard group. Then, a divisor linearly equivalent to $d_1C_1 + d_2C_2$ on $\mathbf{F}_k$ is ample if and only if it is very ample if and only if $d_1 > 0$ and $d_2 > 0$.*

We remark that, if D, E are two ample divisors on S, then $mD + nE$ is ample, for every $m \geq 0$, $n \geq 0$, $(m, n) \neq (0, 0)$.

Demazure's theorem and the toric Nakai criterion have some consequences, proofs omitted, for the geometry of the polygon P_E for an ample divisor $E = \sum_{i=0}^{n+1} e_iC_i$ on the smooth toric surface defined by the nonsingular fan $\Delta = \{u_i = (a_i, b_i) \mid 0 \leq i \leq n+1\}$:

• the nonzero vector $v_i(E) \doteq l_{i+1}(E) - l_i(E)$ is a vector parallel to the line $L_i(E)$ and the line segment $\sigma_i(E)$,

• $v_i(E)$ is a positive scalar multiple $\alpha_i(E)$ of $(b_i, -a_i)$,

• the vector $u_i = (a_i, b_i)$ is orthogonal to both the line $L_i(E)$ and the line segment $\sigma_i(E)$, and the orientation of the ordered pair $v_i(E), u_i$ is positive, i.e., the third coordinate of $v_i(E) \times u_i$ is positive: since $v_i(E) = \alpha_i(E)(b_i, -a_i)$ and $\alpha_i(E) > 0$ (by the toric Nakai criterion), the third coordinate of $v_i \times u_i$ is equal to $\alpha_i(E)(a_i^2 + b_i^2) > 0$.

• the vector u_i is an "inward" pointing normal vector with respect to the polygon P_E and the edge $\sigma_i(E)$,

• If $e_0 = e_{n+1} = 0$ and θ_i is equal to the angle that the vector $l_i(E)$ makes with the x-axis, then $0 = \theta_0 < \theta_1 < \cdots < \theta_n < \theta_{n+1} = \pi/2$, so that the points $l_i(E)$ are arranged in "counterclockwise" order after $l_0(E) = (0, 0)$.

Additional consequences are the following geometric facts, which we do prove here in a simple way:

Lemma 2.14. *Let $D = \sum_{i=0}^{n+1} d_iC_i$ and $E = \sum_{i=0}^{n+1} e_iC_i$ be ample divisors on the smooth toric surface S defined by the nonsingular fan $\Delta = \{u_i = (a_i, b_i) \mid 0 \leq i \leq n+1\}$.*

a) *For each i such that $0 \leq i \leq n+1$, $\sigma_i(D) = L_i(D) \cap P_D$.*

b) *For each i such that $0 \leq i \leq n+1$, $\sigma_i(D+E) = \sigma_i(D) + \sigma_i(E)$.*

c) $P_{D+E} = P_D + P_E$. (*This is true if D and E satisfy the weaker condition of being basepoint free divisors, see* [**6**].)

Proof. For a), Demazure's theorem tells us that $l_{i+1}(D)$ and $l_i(D)$ are in P_D, for each i, so that the line segment containing these two points, $\sigma_i(D)$, is entirely contained in P_D. On the other hand, if $\xi \in L_i(D) \cap P_D$, then since $v_i(D) \doteq l_{i+1}(D) - l_i(D)$ is a vector parallel to $L_i(D)$, and $l_i(D)$ is clearly a point on $L_i(D)$, we must have $\xi = l_i(D) + tv_i(D)$, for some $t \in \mathbf{R}$. Then,

$$(1-t)\langle l_i(D), u_{i+1}\rangle - td_{i+1} = \langle \xi, u_{i+1}\rangle \geq -d_{i+1},$$

so that

$$(t-1)d_{i+1} \leq (1-t)\langle l_i(D), u_{i+1}\rangle.$$

If $t=1$, then $\xi = l_{i+1}(D) \in P_D$; if $t=0, \xi = l_i(D) \in P_D$. If $t-1>0$, then $-d_{i+1} \geq \langle l_i(D), u_{i+1}\rangle$; since $l_i(D) \in P_D$, $\langle l_i(D), u_{i+1}\rangle \geq -d_{i+1}$. Thus, $l_i(D) \in L_i(D) \cap L_{i+1}(D) = \{l_{i+1}(D)\}$, a contradiction. If $t<0$, a similar argument shows that $\langle l_{i+1}(D), u_{i-1}\rangle = -d_{i-1}$, so that $l_{i+1}(D) \in L_i(D) \cap L_{i-1}(D) = \{l_i(D)\}$, again a contradiction. Therefore, $t \in [0,1]$, and we are done.

For b), it is clear that $\sigma_i(D+E) \subseteq \sigma_i(D)+\sigma_i(E)$. To see the opposite inclusion, first recall that $\alpha_i(D)$ and $\alpha_i(E)$ are positive integers. Then, note that the line segment $\sigma_i(D)$ is parallel to the line segment $\sigma_i(E)$; in fact, since $v_i(D) \doteq l_{i+1}(D)-l_i(D)$, $v_i(E) \doteq l_{i+1}(E)-l_i(E)$ for every i,

$$v_i(D) = \alpha_i(D)(b_i, -a_i)$$

and

$$v_i(E) = \alpha_i(E)(b_i, -a_i),$$
$$v_i(D) = \frac{\alpha_i(D)}{\alpha_i(E)} v_i(E).$$

Let $\alpha_i = [\alpha_i(D)/\alpha_i(E)]$ for each i; this is a positive rational number.

Thus, if $u \in \sigma_i(D)$ and $\widetilde{u} \in \sigma_i(E)$, $s,t \in [0,1]$ exist such that $u = sl_{i+1}(D) + (1-s)l_i(D)$, $\widetilde{u} = tl_{i+1}(E) + (1-t)l_i(E)$. Let

$$t_1 = \frac{s}{\alpha_i+1} + \frac{\alpha_i t}{\alpha_i+1}.$$

Then, $t_1 \in [0,1]$, and

$$t_1 l_{i+1}(D+E) + (1-t_1) l_i(D+E) = u + \widetilde{u}.$$

For c), note that $P_D + P_E \subseteq P_{D+E}$, for any divisors D and E (no ampleness required). If D and E are ample, then so is $D+E$, and clearly, for each i, $l_i(D+E) = l_i(D) + l_i(E)$. Since $P_D + P_E$ is convex, contains $l_i(D+E)$ for every i and P_{D+E} is the convex hull of the $n+2$ points $l_0(D+E), \ldots, l_{n+1}(D+E)$, we have the other containment. □

Note that, as a consequence of the above geometry: If $E = \sum_{i=0}^{n+1} e_i C_i$ is an ample divisor on the smooth toric surface S defined by the nonsingular fan $\Delta = \{u_i = (a_i, b_i) \mid 0 \le i \le n+1\}$, then for every point $(x,y) \in P_E$, one and only one of the following three options hold:

- A unique i exists such that $(x,y) = l_i(E)$. In this case, as defined previously, (x,y) is a vertex (and a geometric vertex) of P_E.

- A unique i exists such that $e_i + \langle (x,y), u_i \rangle = 0$ and $e_j + \langle (x,y), u_j \rangle > 0$, for every $j \neq i$. In other words, $(x,y) \in \sigma_i(E)$ and (x,y) is an interior edge point of $\sigma_i(E)$.

- For every j, $e_j + \langle (x,y), u_j \rangle > 0$. In other words, (x,y) is an interior point of the polygon P_E.

2.3. Lattice points in polygons P_E. Given the divisor $E = \sum_{i=0}^{n+1} e_i C_i$ on the smooth toric surface S defined by nonsingular fan

$$\Delta = \{u_i = (a_i, b_i) \mid 0 \le i \le n+1\},$$

we may form the polygon P_E, and its integer lattice points $P_E \cap \mathbf{Z}^2$.

We will need the following theorems about these lattice points later. The first theorem originates with Fakhruddin ([**5**]); later proofs of this theorem are given in [**8, 11, 16**].

Theorem 2.15 (Fakhruddin's theorem [**5, 8, 11, 16**]). *If D and E are divisors on the smooth toric surface S, with D ample and E generated by sections, then $P_{D+E} \cap \mathbf{Z}^2 = (P_D \cap \mathbf{Z}^2) + (P_E \cap \mathbf{Z}^2)$.*

Note that it is always true that $P_{D+E} \cap \mathbf{Z}^2 \supseteq (P_D \cap \mathbf{Z}^2) + (P_E \cap \mathbf{Z}^2)$, by definition of the polygons.

A very simple proof of this theorem in the case of the Hirzebruch surface $\mathbf{F}_k$, a bit more general, is given below. Recall that $\mathbf{F}_k$ is defined by the fan $\Delta_k = \{(0,1),(-1,0),(k,-1),(0,1)\}$. We may write divisors D and E on $\mathbf{F}_k$ in the form

$$D = d_1C_1 + d_2C_2, E = e_1C_1 + e_2C_2.$$

Here we agree that $d_0 = d_3 = e_0 = e_3 = 0$, so that the polygons P_D, P_E and P_{D+E} are contained in the first quadrant; P_D consists of the points $(x,y) \in \mathbf{R}^2$ such that

- $0 \leq x$ and $x \leq d_1$,
- $0 \leq y$ and $y \leq d_2 + kx$.

Note that, if $d_1 < 0$, or if $d_2 + d_1k < 0$, then P_D is empty. Also, for this case of the Hirzebruch surface, using Corollary 2.13, the divisor D is ample if and only if $d_1 > 0$, $d_2 > 0$.

Similarly, if $e_1 \geq 0$ and $e_2 + e_2k \geq 0$, P_E consists of the points $(x,y) \in \mathbf{R}^2$ such that

- $0 \leq x \leq e_1$,
- $0 \leq y \leq e_2 + kx$

and is empty otherwise.

Lemma 2.16. *Given the divisors $D = d_1C_1+d_2C_2, E = e_1C_1+e_2C_2$, on the Hirzebruch surface $\mathbf{F}_k$, such that $d_1 \geq 0$, $d_2 \geq 0$, $e_1 \geq 0$, $e_2 \geq 0$, we have*

$$P_{D+E} \cap \mathbf{Z}^2 = (P_D \cap \mathbf{Z}^2) + (P_E \cap \mathbf{Z}^2).$$

Proof. Given the hypotheses, $d_1 + e_1 \geq 0$, $d_2 + e_2 \geq 0$, consider $(M,N) \in P_{D+E} \cap \mathbf{Z}^2$, so that we must have

$$0 \leq M \leq d_1 + e_1,$$
$$0 \leq N \leq d_2 + e_2 + kM.$$

Since $d_1, e_1 \geq 0$, integers $c, \widehat{c}$ exist such that $0 \leq c \leq d_1$ and $0 \leq \widehat{c} \leq e_1$ with $c + \widehat{c} = M$. Then

$$0 \leq N \leq (d_2 + kc) + (e_2 + k\widehat{c}).$$

Now since $d_2 \geq 0$, $e_2 \geq 0$, $k \geq 0$, and $c, \widehat{c} \geq 0$, both $d_2 + kc \geq 0$ and $e_2 + k\widehat{c} \geq 0$. Then integers $d, \widehat{d}$ exist such that

$$0 \leq d \leq d_2 + kc,$$
$$0 \leq \widehat{d} \leq e_2 + k\widehat{c},$$

and

$$0 \leq d + \widehat{d} \leq (d_2 + kc) + (e_2 + k\widehat{c})$$

with $d + \widehat{d} = N$. Thus, $(c, d) + (\widehat{c}, \widehat{d}) = (M, N)$; $(c, d) \in P_D \cap \mathbf{Z}^2$, $(\widehat{c}, \widehat{d}) \in P_E \cap \mathbf{Z}^2$. $\square$

We will later make use of the following lemma, which may be deduced directly from Fakhruddin's theorem, but we offer here an independent proof.

Lemma 2.17. *Let $D = \sum_{i=0}^{n+1} d_i C_i$ and $E = \sum_{i=0}^{n+1} e_i C_i$ be ample divisors on the smooth toric surface S defined by the nonsingular fan $\Delta = \{u_i = (a_i, b_i) \mid 0 \leq i \leq n+1\}$. Then, for every i, $\sigma_i(D+E) \cap \mathbf{Z}^2 = (\sigma_i(D) \cap \mathbf{Z}^2) + (\sigma_i(E) \cap \mathbf{Z}^2)$.*

Proof. This proof is elementary except for the fact that it uses the toric Nakai criterion. Note that $D + E$ is ample. The line segment $\sigma_i(D+E)$ has two distinct end points, $l_i(D+E) = l_i(D) + l_i(E)$, and $l_{i+1}(D+E) = l_{i+1}(D) + l_{i+1}(E)$ by definition. So we need only consider interior edge points p in $\sigma_i(D+E) \cap \mathbf{Z}^2$. Then there is a unique $t \in (0,1)$ such that

$$\begin{aligned} p &= l_i(D+E) + t(l_{i+1}(D+E) - l_i(D+E)) \\ &= l_i(D) + l_i(E) + t\alpha_i(D+E)(b_i, -a_i). \end{aligned}$$

(Recall $\alpha_i(E) = E \cdot C_i > 0$ for every i.)

Using subscripts to denote first and second coordinates of points, we see that

$$p_1 - l_i(D+E)_1 = t(-b_i)\alpha_i(D+E),$$
$$p_2 - l_i(D+E)_2 = t a_i \alpha_i(D+E).$$

Thus, if $a_i, b_i \neq 0$,

$$\frac{p_1 - l_i(D+E)_1}{-b_i} = \frac{p_2 - l_i(D+E)_2}{a_i},$$

and in any case,

$$a_i(p_1 - l_i(D+E)_1) = -b_i(p_2 - l_i(D+E)_2).$$

Case 1. $a_i, b_i \neq 0$. In this case, a_i and b_i are relatively prime, due to the nonsingularity of the fan. Therefore, we have the following divisor relations among integers:

$$\begin{aligned} a_i &\mid (p_2 - l_i(D+E)_2), \\ b_i &\mid (p_1 - l_i(D+E)_2). \end{aligned}$$

Thus,

$$X_i \doteq \frac{p_1 - l_i(D+E)_1}{-b_i} = \frac{p_2 - l_i(D+E)_2}{a_i} \in \mathbf{Z},$$

and

$$t = \frac{X_i}{\alpha_i(D+E)} = \frac{X_i}{\alpha_i(D) + \alpha_i(E)}.$$

Now, let

$$r = \frac{\alpha_i(D)}{\alpha_i(D) + \alpha_i(E)},$$

so that

$$1 - r = \frac{\alpha_i(E)}{\alpha_i(D) + \alpha_i(E)}$$

and

$$X_i r + X_i(1-r) = X_i \in \mathbf{Z}.$$

Let $T_1 = [X_i r]$, $T_2 = [X_i(1-r)]$, so that $T_1 \leq X_i r < T_1 + 1$, $T_2 \leq X_i(1-r) < T_2 + 1$, and $T_1 + T_2 \leq X_i < T_1 + T_2 + 1$. Since $X_i \in \mathbf{Z}$, we must have

a. $T_1 + T_2 = X_i$ or

b. $T_1 + T_2 + 1 = X_i$.

Note that $0 \leq T_1 \leq X_i[\alpha_i(D)/\alpha_i(D) + \alpha_i(E)]$, so that $0 \leq (T_1/\alpha_i(D)) \leq t < 1$; similarly, $0 \leq (T_2/\alpha_i(E)) < 1$.

Case 1a. Let

$$\begin{aligned} p_D &= l_i(D) + \frac{T_1}{\alpha_i(D)}(l_{i+1}(D) - l_i(D)) \\ &= l_i(D) + \frac{T_1}{\alpha_i(D)}(\alpha_i(D)(-b_i, a_i)) \\ &= l_i(D) + T_1(-b_i, a_i) \in \sigma_i(D) \cap \mathbf{Z}^2 \end{aligned}$$

and

$$\begin{aligned} p_E &= l_i(E) + \frac{T_2}{\alpha_i(E)}(l_{i+1}(E) - l_i(E)) \\ &= l_i(E) + \frac{T_2}{\alpha_i(E)}(\alpha_i(E)(-b_i, a_i)) \\ &= l_i(E) + T_2(-b_i, a_i) \in \sigma_i(E) \cap \mathbf{Z}^2; \end{aligned}$$

then

$$p_D + p_E = l_i(D+E) + (T_1+T_2)(-b_i, a_i) = l_i(D+E) + X_i(-b_i, a_i) = p.$$

Case 1b. $T_1 + T_2 + 1 = X_i$. Now,

$$\frac{T_1 + 1}{\alpha_i(D)} \leq 1,$$

or

$$\frac{T_2 + 1}{\alpha_i(E)} \leq 1.$$

This is true because if $[T_1 + 1/\alpha_i(D)] > 1$ and $[T_2 + 1/\alpha_i(E)] > 1$, then $T_1 + T_2 + 1 > \alpha_i(D) + \alpha_i(E)$, so that $X_i = t(\alpha_i(D) + \alpha_i(E)) > \alpha_i(D) + \alpha_i(E)$, implying $t > 1$, a contradiction. We suppose that $[T_1 + 1/\alpha_i(D)] \leq 1$, the other case being handled in a symmetric fashion.

Let

$$\begin{aligned} p_D &= l_i(D) + \frac{T_1+1}{\alpha_i(D)}(l_{i+1}(D) - l_i(D)) \\ &= l_i(D) + (T_1+1)(-b_i, a_i) \in \sigma_i(D) \cap \mathbf{Z}^2 \end{aligned}$$

and

$$\begin{aligned} p_E &= l_i(E) + \frac{T_2}{\alpha_i(E)}(l_{i+1}(E) - l_i(E)) \\ &= l_i(E) + T_2(-b_i, a_i) \in \sigma_i(E) \cap \mathbf{Z}^2; \end{aligned}$$

then, as before,

$$p_D + p_E = p.$$

If $a_i = 0$, then $b_i \neq 0$, and $p_2 = l_i(D+E)_2$; if $b_i = 0$, then $a_i \neq 0$ and $p_1 = l_i(D+E)_1$. Since the two possibilities are symmetric, we consider only the first. If $a_i = 0$, then since $a_{i-1}b_i - a_ib_{i-1} = 1$ and $a_i = 0$, we see that $b_i = \pm 1$. Thus,

$$X_i \doteq \frac{p_1 - l_i(D+E)_1}{-b_i} \in \mathbf{Z},$$

and we may proceed exactly as in Case 1 to arrive at points $p_D \in \sigma_i(D)$, $p_E \in \sigma_i(E)$ such that $p_D + p_E = p$. $\square$

3. Cohomology computations: Smooth toric surfaces. We collect here needed cohomology computations. We separate out the more general discussions for smooth toric surfaces from the special cases for Hirzebruch surfaces mostly for the reader's convenience. However, computations for the Hirzebruch surfaces are often slightly improved versions of those for general toric surfaces. These "improvements" become useful in later computations.

3.1. Computations of H^i: $\mathcal{O}(D), \Omega^1(D)$. We use the notation established in Section 2; coordinates are as in that section as well.

Thus, let S be a smooth toric surface defined by the nonsingular fan $\Delta = \{(a_i, b_i)\}_{i=0}^{n+1}$ as in Section 2. Recall that, when we write a divisor D as $\sum_{i=0}^{n+1} d_iC_i$, we sometimes assume that $d_0 = d_{n+1} = 0$. As we have seen, this condition forces the polygon P_D to be a subset of the first quadrant in $\mathbf{R}^2$. Note that we have already seen how any divisor $\widetilde{D}$, written as linear combinations of $C_0, \dots, C_{n+1}$, is linearly equivalent to a divisor D in the above form; and the polygons for the two divisors are isometric affine (integer) translates.

For any divisor D the well-known correspondence between $H^0(S, \mathcal{O}_S(D))$ and P_D is given by:

Lemma 3.1 (see, e.g., [**6, 14, 15**]). *Let S be a smooth toric surface given by a nonsingular fan $\Delta = \{(a_i, b_i)\}_{i=0}^{n+1}$. Suppose that $D = \sum_{i=0}^{n+1} d_i C_i$ is a divisor on S. Then*

$$\dim H^0(S, \mathcal{O}(D)) = \#(P_D \cap \mathbf{Z}^2).$$

More precisely, using the coordinates established in Section 2, a basis for the vector space $H^0(S, \mathcal{O}(D))$ is

$$\{x_0^c y_0^d \mid (c,d) \in P_D \cap \mathbf{Z}^2\}.$$

Moving on to consider $H^0(S, \Omega^1(D))$, we will use the following theorem from Murray, [**14**]. This theorem is stated in terms of the coordinates on S set up in Section 2.

Theorem 3.2 ([**14**, Theorem 1, page 196]). *Let S be a smooth toric surface given by the nonsingular fan $\Delta = \{(a_i, b_i) \mid 0 \le i \le n+1\}$. Suppose that $D = \sum_{i=0}^{n+1} d_i C_i$ is a divisor on S determining the polygon P_D. We may then decompose*

$$H^0(S, \Omega^1(D)) = \bigoplus_{(c,d) \in P_D \cap \mathbf{Z}^2} H^0(\Omega^1(D))_{(c,d)}$$

and

- *If (c,d) is an interior point of $P_D \cap \mathbf{Z}^2$, then $H^0(\Omega^1(D))_{(c,d)}$ has dimension two, with basis*

$$\left\{ x_0^c y_0^d \frac{dx_0}{x_0}, x_0^c y_0^d \frac{dy_0}{y_0} \right\}.$$

• *If* (c,d) *is an interior edge point of* $P_D \cap \mathbf{Z}^2$*, say* $(c,d) \in \sigma_i(D)$*, then* $H^0(\Omega^1(D))_{(c,d)}$ *has dimension one, with basis*

$$\left\{ x_0^c y_0^d \left(b_i \frac{dx_0}{x_0} - a_i \frac{dy_0}{y_0} \right) \right\}.$$

• *For all other points* $(c,d) \in P_D \cap \mathbf{Z}^2$*,* $H^0(\Omega^1(D))_{(c,d)} = 0$*.*

Thus, if there are no interior points or interior edge points in P_D, $H^0(S, \Omega^1(D)) = 0$. Additionally, we have the following [**6, 14, 15**]:

Theorem 3.3. *Suppose that* S *is a smooth toric surface, defined by a nonsingular fan* Δ *of* $n+2$ *vectors. Let* D *be an ample divisor on* S*. Then,*

a. $h^1(S, \mathcal{O}(D)) = h^2(S, \mathcal{O}(D)) = 0$.

b. $h^1(S, \Omega^1(D)) = h^2(S, \Omega^1(D)) = 0$.

c. $h^0(S, \mathcal{O}(D)) = (1/2) D \cdot (D - K_S) + 1 = \chi(\mathcal{O}(D))$.

d. $h^0(S, \Omega^1(D)) = D^2 - n = \chi(\Omega^1(D))$.

One uses the Riemann-Roch theorem to obtain c and d in the theorem above.

3.2. $H^i : \mathcal{O}(D), \Omega^1(D)$-Hirzebruch surfaces.

Theorem 3.3 from the previous section may be slightly "improved" for Hirzebruch surfaces.

We use the notation established in Section 2; coordinates are as in that section as well.

We have the following theorem from [**4**], giving more detailed versions of Lemma 3.1 and Theorem 3.2:

Lemma 3.4. *Consider the Hirzebruch surface* $\mathbf{F}_k$*, defined by the nonsingular fan*

$$\Delta_k = \{(0,1), (-1,0), (k,-1), (1,0)\}.$$

Let $D = d_1C_1 + d_2C_2$ be a divisor on $\mathbf{F}_k$ ($d_0 = d_3 = 0$).

1. *$H^0(\mathbf{F}_k, \mathcal{O}(D))$ is zero if either d_1 or $d_2 + d_1k$ is negative.*

2. *If d_1 and $d_2 + d_1k$ are nonnegative, then $H^0(\mathbf{F}_k, \mathcal{O}(D))$ has basis $\{x_0^c y_0^d | 0 \leq c \leq d_1, 0 \leq d \leq d_2 + ck\}$.*

3. *$H^0(\mathbf{F}_k, \Omega^1(D))$ is zero if either d_1 or $d_2 + d_1k$ is negative.*

4. *If $d_1 \geq 0$ and $d_2 + d_1k \geq 0$, then $H^0(\mathbf{F}_k, \Omega^1(d_1C_1 + d_2C_2)) = \mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{M}$, where*

$$\mathcal{X} \doteq \langle\{x_0^i y_0^j dx_0 \mid 0 \leq i \leq d_1 - 2, 0 \leq j \leq d_2 + ik + k - 1\}\rangle$$
$$\mathcal{Y} \doteq \langle\{x_0^\alpha y_0^\beta dy_0 \mid 0 \leq \alpha \leq d_1, 0 \leq \beta \leq d_2 + k\alpha - 2\}\rangle$$

and

$$\mathcal{M} \doteq \langle\{x_0^i y_0^{d_2+ki+k-1}(y_0dx_0 + kx_0dy_0) \mid \\ 0 \leq i \leq d_1 - 2, d_2 + ki + k - 1 \geq 0\}\rangle.$$

Note that sometimes one of more of the sets $\mathcal{X}, \mathcal{Y}$ or $\mathcal{M}$ will be empty. For example, $\mathcal{M}$ is zero if $d_1 < 2$.

Corollary 3.5. *With the same hypotheses as the above theorem, in addition assuming that $d_1 \geq 0$ and $d_2 \geq 0$, then*

a. $\dim H^0(\mathbf{F}_k, \mathcal{O}(D)) = (d_1 + 1)((k/2)d_1 + d_2 + 1) = \chi(\mathcal{O}(D))$.

b. *If $d_1 \geq 2$ and $d_2 + k - 1 \geq 0$, then*

$$\dim \mathcal{X} = d_2(d_1 - 1) + \frac{d_1(d_1 - 1)}{2}k.$$

c. *If $d_1 \geq 0$ and $d_2 \geq 2$, then*

$$\dim \mathcal{Y} = (d_2 - 1)(d_1 + 1) + k\frac{d_1(d_1 + 1)}{2}.$$

d. *If $d_1 \geq 2$ and $d_2 + k - 1 \geq 0$, then*

$$\dim \mathcal{M} = d_1 - 1.$$

e. *If* $d_1 \geq 1$ *and* $d_2 \geq 1$, *then*

$$\dim H^0(\Omega^1(D)) = 2d_1d_2 + kd_1^2 - 2.$$

Given the computations for $H^0(\mathbf{F}_k, \mathcal{O}(D))$ and $H^0(\mathbf{F}_k, \Omega^1(D))$, and using the Kodaira-Serre duality, we can now compute H^2. Using the usual four vector fan Δ_k, and recalling the canonical divisor $K_{\mathbf{F}_k} = -2C_1 + (k-2)C_2$, we get

$$\begin{aligned} h^2(\mathbf{F}_k, \mathcal{O}(D)) &= h^0(\mathbf{F}_k, \mathcal{O}(K_{\mathbf{F}_k} - D)) \\ &= h^0(\mathbf{F}_k, \mathcal{O}(-(d_1+2)C_1 + (k - d_2 - 2)C_2)). \end{aligned}$$

From Lemma 3.4 and Corollary 3.5, we have the following corollaries:

Lemma 3.6. *Consider the Hirzebruch surface* $\mathbf{F}_k$, *defined by the nonsingular fan*

$$\Delta_k = \{(0,1), (-1,0), (k,-1), (1,0)\}.$$

Let $D = d_1C_1 + d_2C_2$ *be a divisor on* $\mathbf{F}_k$ $(d_0 = d_3 = 0)$.

a. *If* $d_1 \geq -1$ *or* $d_2 + kd_1 \geq -k - 1$,

$$H^2(\mathbf{F}_k, \mathcal{O}(D)) = 0.$$

b. *If* $d_1 \geq 1$ *or* $d_2 + kd_1 \geq 1$, *then*

$$H^2(\mathbf{F}_k, \Omega^1(D)) = 0.$$

c. *If* $d_1 \geq 0$ *and* $d_2 \geq 0$, *then* $H^1(\mathbf{F}_k, \mathcal{O}(D)) = 0$.

4. Multiplication maps.

4.1. General definition.

Definition 4.1. In general, given a smooth projective variety X, a coherent sheaf $\mathcal{F}$ and a line bundle $\widehat{\mathcal{F}}$ on X, the multiplication map

$$\mu: \ H^0(X, \mathcal{F}) \otimes H^0(X, \widehat{\mathcal{F}}) \longrightarrow H^0(X, \mathcal{F} \otimes \widehat{\mathcal{F}})$$

is defined by

$$\mu(s \otimes t) = st$$

where s and t are sections of $\mathcal{F}$ and $\widehat{\mathcal{F}}$, respectively.

4.2. Multiplication maps, $\mathcal{O}, \Omega^1$. Let S be a smooth toric surface given by the nonsingular fan $\Delta = \{(a_i, b_i)\}_{i=0}^{n+1}$ with $D = \sum_{i=0}^{n+1} d_i C_i$, $E = \sum_{i=0}^{n+1} e_i C_i$ divisors on S. We do not necessarily always assume that $d_0 = e_0 = d_{n+1} = e_{n+1} = 0$.

The multiplication map

$$\mu : H^0(S, \mathcal{O}(D)) \otimes H^0(S, \mathcal{O}(E)) \longrightarrow H^0(S, \mathcal{O}(D + E))$$

is given by

$$(x_0^c y_0^d) \otimes (x_0^{\hat{c}} y_0^{\hat{d}}) \longmapsto x_0^{c+\hat{c}} y_0^{d+\hat{d}}.$$

Using the correspondence between divisors and polygons described in Section 2, Fakhruddin's theorem 2.15 immediately gives

Theorem 4.2 (Fakhruddin [**5**]). *Let S be a smooth toric surface, given by the nonsingular fan $\Delta = \{(a_i, b_i)\}_{i=0}^{n+1}$ with $D = \sum_{i=0}^{n+1} d_i C_i$ an ample divisor and $E = \sum_{i=0}^{n+1} e_i C_i$ generated by global sections. Then the multiplication map*

$$\mu : H^0(S, \mathcal{O}(D)) \otimes H^0(S, \mathcal{O}(E)) \longrightarrow H^0(S, \mathcal{O}(D + E))$$

given by

$$(x_0^c y_0^d) \otimes (x_0^{\hat{c}} y_0^{\hat{d}}) \longmapsto x_0^{c+\hat{c}} y_0^{d+\hat{d}}$$

is surjective.

Now we consider multiplication maps

$$\widehat{\mu} : H^0(S, \Omega^1(D)) \otimes H^0(S, \mathcal{O}(E)) \longrightarrow H^0(S, \Omega^1(D+E)).$$

As a consequence of Murray's calculation of $H^0(\Omega^1(D))$ (see Theorem 3.2 of this paper) and Fakhruddin's theorem, we prove:

Theorem 4.3. *Let S be a smooth toric surface given by the nonsingular fan $\Delta = \{(a_i, b_i)\}_{i=0}^{n+1}$. Suppose that D and E are ample divisors on S such that $D + K_S$ and $D + K_S + C_j$ are ample, for every $j, 0 \leq j \leq n+1$. Then, the multiplication map*

$$\widehat{\mu} : H^0(S, \Omega^1(D)) \otimes H^0(S, \mathcal{O}(E)) \longrightarrow H^0(S, \Omega^1(D+E))$$

is surjective.

Proof. We use coordinates as established in Section 2. We may assume $D = \sum_{i=1}^n d_i C_i$, $E = \sum_{i=1}^n e_i C_i$. with $d_0 = e_0 = d_{n+1} = e_{n+1} = 0$.

Consider the description of $H^0(S, \Omega^1(D+E))$ given by Murray's theorem 3.2:

$$H^0(S, \Omega^1(D+E)) = \oplus_{(c,d) \in P_{D+E} \cap \mathbf{Z}^2} H^0(\Omega^1(D+E))_{(c,d)}.$$

For a given $(c, d) \in P_{D+E} \cap \mathbf{Z}^2$, Murray's theorem 3.2 gives us two alternatives that we need to consider here: (c, d) is an interior point of P_{D+E}, or (c, d) is an interior edge point of P_{D+E}.

Assuming that (c, d) is an interior point of P_{D+E}, we have two basis vectors $x_0^c y_0^d (dx_0/x_0)$ and $x_0^c y_0^d (dy_0/y_0)$ contained in $H^0(\Omega^1(D+E))_{(c,d)}$. By definition of an interior point, we must have

- $c > 0, d > 0$ and
- $d_j + e_j + a_j c + b_j d > 0, 1 \leq j \leq n$.

However, this means that

- $-1 + c \geq 0, -1 + d \geq 0$ and
- $(d_j - 1) + e_j + a_j c + b_j d \geq 0, 1 \leq j \leq n$.

In other words, using the definition of $P_{D+K_S+E} \cap \mathbf{Z}^2$ and the fact that $K_S = -(\sum_{i=0}^{n+1} C_i)$,

$$(c, d) \in P_{D+K_S+E} \cap \mathbf{Z}^2.$$

Using our ampleness assumption and Fakhruddin's theorem 2.15, $(f_1,g_1) \in P_{D+K_S} \cap \mathbf{Z}^2$ and $(f_2,g_2) \in P_E \cap \mathbf{Z}^2$ exist such that $(f_1,g_1)+(f_2,g_2)=(c,d)$. But, using the description of P_{D+K_S}, we have

- $-1+f_1 \geq 0, -1+g_1 \geq 0$ and
- $(d_j-1)+a_jf_1+b_jg_1 \geq 0, 1 \leq j \leq n$.

In other words,

- $f_1>0, g_1>0$ and
- $d_j+a_jf_1+b_jg_1>0, 1 \leq j \leq n$

and (f_1,g_1) is an interior point of P_D. Thus, $x_0^{f_1}y_0^{g_1}(dx_0/x_0) \in H^0(S,\Omega^1(D))$, $x_0^{f_2}y_0^{g_2} \in H^0(S,\mathcal{O}(E))$ and $\widehat{\mu}(x_0^{f_1}y_0^{g_1}(dx_0/x_0) \otimes x_0^{f_2}y_0^{g_2}) = x_0^cy_0^d(dx_0/x_0)$.

Also, $\widehat{\mu}(x_0^{f_1}y_0^{g_1}(dy_0/y_0) \otimes x_0^{f_2}y_0^{g_2}) = x_0^cy_0^d(dy_0/y_0)$.

Now, assume that (c,d) is an interior edge point of P_{D+E}. Since $D+E$ is ample, there is exactly one edge $\sigma_i(D+E)$ such that (c,d) is interior to $\sigma_i(D+E)$ and (c,d) corresponds to the basis element $x_0^cy_0^d(b_i(dx_0/x_0)-a_i(dy_0/y_0))$ of $H^0(\Omega^1(D+E))_{(c,d)}$. This means that

- $d_i+e_i+a_ic+b_id=0$ and
- for every $j \neq i$, $0 \leq j \leq n+1$, $d_j+e_j+a_jc+b_jd>0$.

Therefore,

- $d_i+e_i+a_ic+b_id=0$ and
- for all $j \neq i$, $0 \leq j \leq n+1$, $(d_j-1)+e_j+a_jc+b_jd \geq 0$,

so that (c,d) is an edge point corresponding to the edge $\sigma_i(D+K_S+C_i)$ of the polygon $P_{D+K_S+C_i}$. Since $D+K_S+C_i$ and E are ample, Lemma 2.17 tells us that

$$(\sigma_i(D+K_S+C_i) \cap \mathbf{Z}^2) + (\sigma_i(E) \cap \mathbf{Z}^2) = \sigma_i(D+K_S+C_i+E) \cap \mathbf{Z}^2.$$

So, $(f_1,g_1) \in \sigma_i(D+K_S+C_i) \cap \mathbf{Z}^2$ and $(f_2,g_2) \in \sigma_i(E) \cap \mathbf{Z}^2$ exist such that $(f_1,g_1)+(f_2,g_2)=(c,d)$. In other words,

- $d_j-1+a_jf_1+b_jg_1 \geq 0$, $j \neq i$ and
- $d_i+a_if_1+b_ig_1=0$.

Therefore, (f_1,g_1) is an interior edge point in P_D corresponding to the edge $\sigma_i(D)$, and

$$x_0^{f_1} y_0^{g_1} \left(b_i \frac{dx_0}{x_0} - a_i \frac{dy_0}{y_0} \right) \in H^0(\Omega^1(D))_{(f_1, g_1)}.$$

Also, $x_0^{f_2} y_0^{g_2} \in H^0(S, \mathcal{O}(E))$, and

$$x_0^{f_2} y_0^{g_2} \left(x_0^{f_1} y_0^{g_1} \left(b_i \frac{dx_0}{x_0} - a_i \frac{dy_0}{y_0} \right) \right) = x_0^c y_0^d \left(b_i \frac{dx_0}{x_0} - a_i \frac{dy_0}{y_0} \right). \qquad \square$$

4.3. Multiplication maps and Hirzebruch surfaces. Consider the Hirzebruch surface $\mathbf{F}_k$, defined by the nonsingular fan

$$\Delta_i = \{(0,1), (-1,0), (k,-1), (1,0)\}.$$

Let $D = d_1 C_1 + d_2 C_2$ and $E = e_1 C_1 + e_2 C_2$ be divisors on $\mathbf{F}_k$ (and, $e_0 = e_3 = d_0 = d_3 = 0$).

As we have seen,

$$H^0(\mathbf{F}_k, \mathcal{O}(D))$$

has basis

$$\{x_0^c y_0^d \mid 0 \le c \le d_1,\ 0 \le d \le d_2 + ck\}$$

if $d_1 \ge 0$ and $d_2 + d_1 k \ge 0$, and is zero otherwise.

The multiplication map, in the above coordinates, is

$$\begin{aligned} \mu : H^0(\mathbf{F}_k, \mathcal{O}(D)) \otimes H^0(\mathbf{F}_k, \mathcal{O}(E)) &\longrightarrow H^0(\mathbf{F}_i, \mathcal{O}(D+E)) \\ (x_0^c y_0^d) \otimes (x_0^{\hat{c}} y_0^{\hat{d}}) &\longmapsto x_0^{c+\hat{c}} y_0^{d+\hat{d}} \end{aligned}$$

where $d_1, e_1 \ge 0$, $d_2 + d_1 k \ge 0, e_2 + e_1 k \ge 0$.

Note. If $d_1 < 0$ or $e_1 < 0$ or $d_2 + d_1 k < 0$ or $e_2 + e_1 k < 0$, then the domain of the multiplication map is zero.

Using the correspondence between divisors and polygons described in Section 2, the slightly improved version of Fakhruddin's theorem for Hirzebruch surfaces (Lemma 2.16) yields:

Lemma 4.4. *Given the divisors* $D = d_1C_1 + d_2C_2$, $E = e_1C_1 + e_2C_2$, *on the Hirzebruch surface* $\mathbf{F}_k$, *such that* $d_1 \geq 0$, $d_2 \geq 0$, $e_1 \geq 0$, $e_2 \geq 0$, *the multiplication map*

$$\mu : H^0(\mathbf{F}_k, \mathcal{O}(D)) \otimes H^0(\mathbf{F}_k, \mathcal{O}(E)) \longrightarrow H^0(\mathbf{F}_i, \mathcal{O}(D + E))$$

is surjective.

Next consider the multiplication maps

$$\widehat{\mu} : H^0(\mathbf{F}_k, \Omega^1(D)) \otimes H^0(\mathbf{F}_k, \mathcal{O}(E)) \longrightarrow H^0(\mathbf{F}_k, \Omega^1(D + E)).$$

We have the following "improvement" of Theorem 4.3; the proof given here is a direct one and does not rely on Fakhruddin's theorem.

Theorem 4.5. *Given the divisors* $D = d_1C_1 + d_2C_2, E = e_1C_1 + e_2C_2$, *on the Hirzebruch surface* $\mathbf{F}_k$, *such that* $d_1 \geq 2$, $d_2 \geq 2$, $e_1 \geq 0$ *and* $e_2 \geq 0$, *the multiplication map*

$$\widehat{\mu} : H^0(\mathbf{F}_k, \Omega^1(D)) \otimes H^0(\mathbf{F}_k, \mathcal{O}(E)) \longrightarrow H^0(\mathbf{F}_k, \Omega^1(D + E)).$$

is surjective.

Proof. Note that, in any case, $D + E$ is ample.

Case 1. Consider an interior point $(M, N) \in P_{D+E} \cap \mathbf{Z}^2$, giving rise to the basis element $x_0^M y_0^N (dw/w) \in H^0(\Omega^1(D + E))_{(M,N)}$, where w may be either x_0 or y_0. Then, $0 < M < d_1 + e_1$, $0 < N < d_2 + e_2 + kM$. Since $d_1 \geq 2$ and $e_1 \geq 0$, integers a, b exist such that $0 < a < d_1$, $0 \leq b \leq e_1$ and $a + b = M$. Since $0 < N < (d_2 + ka) + (e_2 + kb)$ and $d_2 + ka \geq 2$, $e_2 + kb \geq 0$, integers c, d exist such that $0 < c < d_2 + ka$, $0 \leq d \leq e_2 + kb$ and $c + d = N$.

Therefore, (a, c) is an interior point in $P_D \cap \mathbf{Z}^2$, giving rise to the basis element $x_0^a y_0^c (dw/w) \in H^0(\Omega^1(D))_{(a,c)}$. Also, (b, d) is a point in $P_E \cap \mathbf{Z}^2$, giving rise to the basis element $x_0^b y_0^d$ of $H^0(\mathbf{F}_k, \mathcal{O}(E))$, and

$$x_0^a y_0^c \frac{dw}{w} \cdot x_0^b y_0^d = x_0^M y_0^N \frac{dw}{w}.$$

Case 2. Consider an interior edge point in $\sigma_3(D+E)\cap \mathbf{Z}^2$, which must be of the form $(0,N)$, with $0 < N < d_2+e_2$. This gives rise to the basis element $y_0^N(dy_0/y_0)$ of $H^0(\Omega^1(D+E))_{(0,N)}$. Integers a,b exist with $0 < a < d_2$,$0 \leq b \leq e_2$ with $a+b=N$. Thus, since $(0,a)$ is an interior point of the edge $\sigma_3(D)$, $y_0^a(dy_0/y_0)$ is a basis element of $H^0(\Omega^1(D))_{(0,a)}$. Also, $y_0^b \in H^0(\mathbf{F}_k, \mathcal{O}(E))$, and

$$y_0^a \frac{dy_0}{y_0} \cdot y_0^b = y_0^N \frac{dy_0}{y_0}.$$

Case 3. Consider an interior edge point in $\sigma_0(D+E)\cap \mathbf{Z}^2$, which must be of the form $(M,0)$, with $0 < M < d_1+e_1$. This gives rise to the basis element $x_0^M(dx_0/x_0)$ of $H^0(\Omega^1(D+E))_{(M,0)}$. Integers a,b exist with $0 < a < d_1$, $0 \leq b \leq e_1$ with $a+b=M$. Thus, since $(a,0)$ is an interior point of the edge $\sigma_0(D)$, $x_0^a(dx_0/x_0)$ is a basis element of $H^0(\Omega^1(D))_{(a,0)}$. Also, $x_0^b \in H^0(\mathbf{F}_k, \mathcal{O}(E))$, and

$$x_0^a \frac{dx_0}{x_0} \cdot x_0^b = x_0^M \frac{dx_0}{x_0}.$$

Case 4. Consider an interior edge point in $\sigma_1(D+E)\cap \mathbf{Z}^2$, which must be of the form (d_1+e_1,N), with $0<N<d_2+e_2+k(d_1+e_1)$. This gives rise to the basis element $x_0^{d_1+e_1}y_0^N(dy_0/y_0)$ of $H^0(\Omega^1(D+E))_{(d_1+e_1,N)}$. Integers a,b exist with $0<a<d_2+kd_1$, $0\leq b\leq e_2+ke_1$ with $a+b=N$. Thus, since (d_1,a) is an interior point of the edge $\sigma_1(D)$, $x_0^{d_1}y_0^a(dy_0/y_0)$ is a basis element of $H^0(\Omega^1(D))_{(d_1+e_1,N)}$. Also, $x_0^{e_1}y_0^b \in H^0(\mathbf{F}_k, \mathcal{O}(E))$, and

$$x_0^{d_1}y_0^a \frac{dy_0}{y_0} \cdot x_0^{e_1}y_0^b = x_0^{d_1+e_1}y_0^N \frac{dy_0}{y_0}.$$

Case 5. Consider an interior edge point in $\sigma_2(D+E)\cap \mathbf{Z}^2$, which must be of the form (M,N), with $0<M<d_1+e_1$ and $N=d_2+e_2+kM$. This gives rise to the basis element $x_0^M y_0^{d_2+e+2+kM}(-(dx_0/x_0)-k(dy_0/y_0))$ of $H^0(\Omega^1(D+E))_{(M,d_2+e_2+kM)}$. Integers a,b exist with $0<a<d_1$, $0\leq b\leq e_1$ with $a+b=M$. Thus, since (a,d_2+ka) is an interior point of the edge $\sigma_2(D)$, $x_0^a y_0^{d_2+ka}(-(dx_0/x_0)-k(dy_0/y_0))$ is a basis element of $H^0(\Omega^1(D))_{(a,d_2+ka)}$. Also, $x_0^b y_0^{e_2+kb} \in H^0(\mathbf{F}_k, \mathcal{O}(E))$, and

$$x_0^a y_0^{d_2+ka}\left(-\frac{dx_0}{x_0}-k\frac{dy_0}{y_0}\right)\cdot x_0^b y_0^{e_2+kb} = x_0^M y_0^{d_2+e_2+kM}\left(-\frac{dx_0}{x_0}-k\frac{dy_0}{y_0}\right). \square$$

Surjectivity results for multiplication maps on toric varieties are also more generally considered in [**10, 12**], in the context of Castelnuovo-Mumford regularity.

5. The Gaussian map. We collect here definitions, basic facts and notation, following Wahl [**18–20**] closely.

Consider a multiplication map μ, where X is a smooth projective variety and $\mathcal{F}$ and $\mathcal{G}$ are line bundles on X:

$$\mu : H^0(X,\mathcal{F}) \otimes H^0(X,\mathcal{G}) \longrightarrow H^0(X,\mathcal{F}\otimes\mathcal{G}).$$

Definition 5.1. Given the multiplication map μ, $\ker\mu \doteq \mathcal{R}(\mathcal{F},\mathcal{G})$ [**18**].

Given an open set $U \subset X$, over which $\mathcal{F}|_U$ is trivial, let T be a generator of $\mathcal{F}|_U$. Let $\alpha = \sum \sigma_i \otimes \tau_i \in \mathcal{R}(\mathcal{F},\mathcal{G})$. We can write $\sigma_i = f_i T$ locally for some $f_i \in \mathcal{O}(U)$. Given a generator S of $\mathcal{G}|_U$, we can write $\tau_i = g_i S$ locally for some $g_i \in (U)$. (Note that $\sum f_i g_i = 0$.) Then we can define the Gaussian map

$$\Phi_{X,\mathcal{F},\mathcal{G}}(\alpha) \doteq \sum (f_i dg_i - g_i df_i) \otimes T \otimes S \in H^0(\Omega^1_X \otimes \mathcal{F} \otimes \mathcal{G}).$$

This is well defined, proof in Wahl [**20**, page 123].

If $\mathcal{F} = \mathcal{G}$, then $\Lambda^2 H^0(\mathcal{F}) \subset \mathcal{R}(\mathcal{F},\mathcal{F})$ by identifying $\sigma\wedge\tau$ with $1/2(\sigma\otimes\tau - \tau\otimes\sigma)$. We restrict the domain of $\Phi_{X,\mathcal{F},\mathcal{F}}$ to $\Lambda^2 H^0(X,\mathcal{F})$, writing $\Phi_{X,\mathcal{F},\mathcal{F}}|_{\Lambda^2 H^0(X,\mathcal{F})} \doteq \Phi_{X,\mathcal{F}}$ for simplicity of notation. Then, if $\sigma = fT$ and $\tau = gT$ locally, the Gaussian map

$$\Phi_{X,\mathcal{F}} : \Lambda^2 H^0(X,\mathcal{F}) \longrightarrow H^0(X,\Omega^1_X \otimes \mathcal{F}^2)$$

is given by

$$\Phi_{X,\mathcal{F}}(\sigma\wedge\tau) = (f\,dg - g\,df) \otimes T \otimes T$$

[**18**]. Hence, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{R}(\mathcal{F},\mathcal{F}) \arrow[rr, "\Phi_{X,\mathcal{F},\mathcal{F}}"] & & H^0(X,\Omega^1_X\otimes\mathcal{F}^2) \\
& \Lambda^2 H^0(X,\mathcal{F}) \arrow[ul] \arrow[ur, "\Phi_{X,\mathcal{F}}"'] &
\end{tikzcd} \tag{5.2}$$

and

$$\operatorname{im}\Phi_{X,\mathcal{F}} = \operatorname{im}\Phi_{X,\mathcal{F},\mathcal{F}}.$$

If $\mathcal{L}, \mathcal{M}$ and $\mathcal{N}$ are line bundles, then there is a commutative diagram:

$$\begin{tikzcd} \mathcal{R}(\mathcal{L},\mathcal{M}) \otimes H^0(\mathcal{N}) \arrow[r, "\Phi_{X,\mathcal{L},\mathcal{M}}\otimes id"] \arrow[d, "a"] & H^0(\Omega^1 \otimes \mathcal{L} \otimes \mathcal{M}) \otimes H^0(\mathcal{N}) \arrow[d, "\hat{\mu}"] \\ \mathcal{R}(\mathcal{L},\mathcal{M}\otimes\mathcal{N}) \arrow[r, "\Phi_{X,\mathcal{L},\mathcal{M}\otimes\mathcal{N}}"'] & H^0(\Omega^1 \otimes \mathcal{L} \otimes \mathcal{M} \otimes \mathcal{N}), \end{tikzcd} \tag{5.3}$$

where the horizontal maps are defined using Gaussian maps as indicated and the vertical map, $\widehat{\mu}$ is a multiplication map for forms, defined as

$$\omega \otimes h \longmapsto h\omega,$$

for $\omega \in H^0(\Omega^1 \otimes \mathcal{L} \otimes \mathcal{M})$ and $h \in H^0(\mathcal{N})$. Map a is defined by

$$\left(\sum \sigma_i \otimes \tau_i\right) \otimes hT' \overset{a}{\longmapsto} \sum(\sigma_i \otimes h\tau_i),$$

if T' is a local generator of $\mathcal{N}$.

Finally, we note the naturality of the Gaussian maps: for an appropriate map $f : X \to Y$, and sheaves $\mathcal{G}$, $\widehat{\mathcal{G}}$ on Y, we construct the sheaves $f^*\mathcal{G}$, and $f^*\widehat{\mathcal{G}}$ on X. Then, there exists a commutative diagram:

$$\begin{tikzcd} \mathcal{R}(\mathcal{G},\widehat{\mathcal{G}}) \arrow[r, "\Phi_{Y,\mathcal{G},\hat{\mathcal{G}}}"] \arrow[d] & H^0(\Omega^1_Y \otimes \mathcal{G} \otimes \widehat{\mathcal{G}}) \arrow[d] \\ \mathcal{R}(f^*\mathcal{G}, f^*\hat{\mathcal{G}}) \arrow[r, "\Phi_{X,f^*\mathcal{G},f^*\hat{\mathcal{G}}}"] & H^0(\Omega^1_X \otimes f^*\mathcal{G} \otimes f^*\widehat{\mathcal{G}}) \end{tikzcd} \tag{5.4}$$

6. Surjectivity of the Gaussian map.

6.1. Gaussian maps for Hirzebruch surfaces. Consider the Hirzebruch surface $\mathbf{F}_k$, defined by the nonsingular fan

$$\Delta_k = \{(0,1), (-1,0), (k,-1), (1,0)\}.$$

Given the divisors D, E on $\mathbf{F}_k$, we have the multiplication map,

$$\mu = \mu(D,E) : H^0(\mathbf{F}_k, \mathcal{O}(D)) \otimes H^0(\mathbf{F}_k, \mathcal{O}(E)) \longrightarrow H^0(\mathbf{F}_k, \mathcal{O}(D+E)).$$

We have defined the kernel of this map as

$$\ker \mu(D,E) \doteq \mathcal{R}(D,E).$$

We have the following proposition:

Proposition 6.1 [**4, 14**]. *Given the Hirzebruch surface $\mathbf{F}_k$, defined by the nonsingular fan*

$$\Delta_k = \{(0,1), (-1,0), (k,-1), (1,0)\},$$

let D be a divisor on $\mathbf{F}_k$. If D is ample, then the Gaussian map

$$\Phi_{\mathbf{F}_k, D} : \mathcal{R}(D,D) \longrightarrow H^0(\mathbf{F}_k, \Omega^1(2D))$$

is surjective.

In order to analyze Gaussian maps on double covers of Hirzebruch surfaces, we will see that we have to think about more general Gaussian maps

$$\Phi_{\mathbf{F}_k, D, E} : \mathcal{R}(D,E) \longrightarrow H^0(\mathbf{F}_k, \Omega^1(D+E)).$$

The main result for this section is:

Proposition 6.2. *Given the Hirzebruch surface $\mathbf{F}_k$, defined by the nonsingular fan*

$$\Delta_k = \{(0,1), (-1,0), (k,-1), (1,0)\},$$

let D and E be divisors on $\mathbf{F}_k$. Then, if D and E are ample, the Gaussian map

$$\Phi_{\mathbf{F}_k, D, E} : \mathcal{R}(D,E) \longrightarrow H^0(\mathbf{F}_k, \Omega^1(D+E))$$

is surjective.

Proof. We assume that $D = d_1C_1 + d_2C_2, E = e_1C_1 + e_2C_2$, with $d_0 = e_0 = d_3 = e_3 = 0$. The ampleness hypotheses, as we have previously noted, are equivalent to requiring that $d_1 > 0$, $d_2 > 0$, $e_1 > 0$, $e_2 > 0$. In the argument below, if a divisor $w_1C_1 + w_2C_2$ is given in terms of the basis C_1, C_2 of the Picard group of $\mathbf{F}_k$, we write the divisor as an ordered pair (w_1, w_2).

We construct two commutative diagrams using Diagram 5.3. We omit denoting the surface $\mathbf{F}_k$ in the cohomology groups.

Diagram 1.

$$\begin{tikzcd}
\mathcal{R}((1,1),(d_1,d_2)) \arrow[r, "\Phi_1"] & H^0(\Omega^1(d_1+1,d_2+1)) \\
\mathcal{R}((1,1),(1,1)) \otimes H^0(\mathcal{O}(d_1-1,d_2-1)) \arrow[u, "a_1"] \arrow[r, "\Phi_2\otimes\mathrm{id}"] & H^0(\Omega^1(2,2)) \otimes H^0(\mathcal{O}(d_1-1,d_2-1)) \arrow[u, "\widehat{\mu_1}"']
\end{tikzcd}$$

Diagram 2.

$$\begin{tikzcd}
\mathcal{R}((d_1,d_2),(e_1,e_2)) \arrow[r, "\Phi_3"] & H^0(\Omega^1(d_1+e_2,d_2+e_2)) \\
\mathcal{R}((d_1,d_2),(1,1)) \otimes H^0(\mathcal{O}(e_1-1,e_2-1)) \arrow[u, "a_2"] \arrow[r, "\Phi_1\otimes id"] & H^0(\Omega^1(d_1+1,d_2+1)) \otimes H^0(\mathcal{O}(e_1-1,e_2-1)) \arrow[u, "\hat{\mu}_2"']
\end{tikzcd}$$

Here $\Phi_1 = \Phi_{\mathbf{F}_k, C_1+C_2, D}$, $\Phi_2 = \Phi_{\mathbf{F}_k, C_1+C_2}$ and $\Phi_3 = \Phi_{\mathbf{F}_k, D, E}$.

Since $C_1 + C_2$ is ample, looking at Diagram 1, we see that $\Phi_2 \otimes id$ is surjective by Proposition 6.1. Also, $\widehat{\mu_1}$ is surjective using Proposition 4.5. Therefore, the commutativity of the diagram forces

$$\mathcal{R}((1,1),(d_1,d_2)) \xrightarrow{\Phi_1} H^0(\mathbf{F}_k, \Omega^1(d_1+1, d_2+1))$$

to be surjective.

Next, considering Diagram 2, we have just seen that $\Phi_1 \otimes id$ is surjective. Since $D + C_1 + C_2 = (d_1+1)C_1 + (d_2+1)C_2$ and

$E - (C_1 + C_2) = (e_1 - 1)C_1 + (e_2 - 1)C_2$ satisfy the hypotheses of Proposition 4.5, $\widehat{\mu_2}$ is surjective by that proposition.

Thus, the commutativity of Diagram 2 forces the Gaussian map

$$\Phi_{\mathbf{F}_k,D,E} : \mathcal{R}(D,E) \longrightarrow H^0(\mathbf{F}_k, \Omega^1(D+E))$$

to be surjective. $\square$

Note that the only case of Proposition 6.1 used in the above proof is the very simple case of $D = C_1 + C_2$. Thus, we may view Proposition 6.2 as giving an alternative proof of Proposition 6.1, once the simple case of $D = C_1 + C_2$ only is proved in 6.1.

6.2. Gaussian maps for general toric surfaces. Let S be a smooth toric surface given by the nonsingular fan $\Delta = \{(a_i, b_i)\}_{i=0}^{n+1}$ with $D = \sum_{i=0}^{n+1} d_i C_i$, $E = \sum_{i=0}^{n+1} e_i C_i$ divisors on S. (Recall the definition of the curves C_i from Section 2.)

Here again, we have the multiplication map

$$\mu : H^0(S, \mathcal{O}(D)) \otimes H^0(S, \mathcal{O}(E)) \longrightarrow H^0(S, \mathcal{O}(D+E)),$$

and its kernel

$$\ker \mu \doteq \mathcal{R}(D,E).$$

Corresponding to Proposition 6.1, we have Murray's theorem ([**14**], Theorem 3]) for smooth toric surfaces:

Theorem 6.3 [**14**]. *Let S be a smooth toric surface given by the nonsingular fan $\Delta = \{(a_i, b_i)\}_{i=0}^{n+1}$, and let $D = \sum_{i=0}^{n+1} d_i C_i$ be an ample divisor on S. Then the Gaussian map*

$$\mathcal{R}(D,D) \xrightarrow{\Phi} H^0(S, \Omega^1(2D))$$

is surjective.

We don't formulate here such a neat analog of Proposition 6.2 for general toric surfaces, but a useful version for the purposes of this paper is given below.

Proposition 6.4. *Let S be a smooth toric surface given by the nonsingular fan $\Delta = \{(a_i, b_i)\}_{i=0}^{n+1}$. Suppose that L is a divisor such that L, $L + K_S$ and $L + K_S + C_i$ are ample, for every i. Suppose that G is a divisor such that $G - 2L$ is ample. Then the Gaussian maps*

$$\mathcal{R}(L, G) \xrightarrow{\Phi} H^0(S, \Omega^1(G + L))$$

and

$$\mathcal{R}(G, G - L) \xrightarrow{\Phi} H^0(S, \Omega^1(2G - L))$$

are surjective.

Proof. Note that the ampleness hypotheses imply that the following divisors are ample: $G - L = G - 2L + L$, $G = G - L + L$, $G + L$, $G + L + K_S$, $G + L + K_S + C_i$ (for every i), $2L + K_S = L + K_S + L$ and $2L + K_S + C_i = L + K_S + C_i + L$ (for every i).

We construct two commutative diagrams using Diagram 5.3, as in Proposition 6.2.

Diagram 1.

$$\begin{tikzcd}
\mathcal{R}(L, G) \arrow[r, "\Phi_1"] & H^0(S, \Omega^1(G + L)) \\
\mathcal{R}(L, L) \otimes H^0(S, \mathcal{O}(G - L)) \arrow[u, "a_1"] \arrow[r, "\Phi_2 \otimes \mathrm{id}"] & H^0(S, \Omega^1(2L)) \otimes H^0(S, \mathcal{O}(G - L)); \arrow[u, "\hat{\mu}_1"']
\end{tikzcd}$$

Diagram 2.

$$\begin{tikzcd}
\mathcal{R}(G, G - L) \arrow[r, "\Phi_3"] & H^0(S, \Omega^1(2G - L)) \\
\mathcal{R}(L, G) \otimes H^0(S, \mathcal{O}(G - 2L)) \arrow[u, "a_2"] \arrow[r, "\Phi_1 \otimes \mathrm{id}"] & H^0(S, \Omega^1(G + L)) \otimes H^0(S, \mathcal{O}(G - 2L)). \arrow[u, "\hat{\mu}_2"']
\end{tikzcd}$$

Now, looking at Diagram 1, the necessary ampleness hypotheses are satisfied so that we may apply Theorem 4.3, yielding the surjectivity of $\widehat{\mu}_1$. We also have the necessary ampleness hypotheses to apply

Theorem 6.3 above, yielding the surjectivity of $\Phi_2 \otimes \mathrm{id}$. Thus, Φ_1 is surjective.

Passing to Diagram 2, the necessary ampleness hypotheses are satisfied so that we may apply Theorem 4.3 again, to get surjectivity for $\widehat{\mu}_2$; since we have just seen that Φ_1 is surjective, this yields the surjectivity of Φ_3. □

We note the following consequences of Diagram 5.3 and Fakhruddin's theorem, but we will not use these in the rest of the paper.

Lemma 6.5. *Let S be a smooth toric surface. Suppose that L, M and N are all ample divisors on S. Then, the map*

$$a : \mathcal{R}(L,M) \otimes H^0(S,\mathcal{O}(N)) \longrightarrow \mathcal{R}(L,M+N)$$

is surjective.

Proof. Using the ampleness hypotheses and Fakhruddin's theorem 4.2, we see that the diagram below has short exact sequences as rows (as indicated) and that the center and right maps are surjective. Since the diagram is commutative, this forces the left map to be surjective. (We delete mention of S in the diagram.)

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{R}(L,M) \otimes H^0(\mathcal{O}(N)) \arrow[r] \arrow[d, "a"] & H^0(\mathcal{O}(L)) \otimes H^0(\mathcal{O}(M)) \otimes H^0(\mathcal{O}(N)) \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\mu"] & H^0(\mathcal{O}(L+M)) \otimes H^0(\mathcal{O}(N)) \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & \mathcal{R}(L,M+N) \arrow[r] & H^0(\mathcal{O}(L)) \otimes H^0(\mathcal{O}(M+N)) \arrow[r, "\mu\otimes 1"] & H^0(\mathcal{O}(L+M+N)) \arrow[r] & 0.
\end{tikzcd}$$ □

Corollary 6.6. *With the hypotheses of Lemma 6.5, the Gaussian map $\Phi : \mathcal{R}(L,L+N) \to H^0(S,\Omega^1(2L+N))$ is surjective if and only if the multiplication map $\widehat{\mu} : H^0(S,\Omega^1(2L)) \otimes H^0(S,\mathcal{O}(N)) \to H^0(S,\Omega^1(2L+N))$ is surjective.*

Proof. Consider the commutative diagram below:

$$\begin{tikzcd}
\mathcal{R}(L,L+N) \arrow[r, "\Phi"] & H^0(S,\Omega^1(2L+N)) \\
\mathcal{R}(L,L) \otimes H^0(S,\mathcal{O}(N)) \arrow[r, "\Phi_1\otimes\mathrm{id}"] \arrow[u, "a"] & H^0(S,\Omega^1(2L)) \otimes H^0(S,\mathcal{O}(N)). \arrow[u, "\hat{\mu}"]
\end{tikzcd}$$

Murray's theorem 6.3 on the surjectivity of the Gaussian map gives us that Φ_1 is surjective, and since a is surjective by the above lemma, we see that Φ is surjective if and only if $\widehat{\mu}$ is surjective. ◻

7. Double covers.

7.1. General definitions and theorems concerning double covers. A general reference for the notation and results of most of this section is [**1**].

Lifting the discussion directly from [**1**], we let Y be a smooth projective surface and $\mathcal{L}$ a line bundle on Y such that $\mathcal{L}^{\otimes 2} = \mathcal{L} \otimes \mathcal{L}$ has a section s. Let $D \subseteq Y$ be the divisor corresponding to the zeroes of the section s. Let $\widetilde{Y}$ be the total space of $\mathcal{L}$ with $p : \widetilde{Y} \to Y$ the line bundle projection. Then the pullback bundle $p^*\mathcal{L}$ is a line bundle on $\widetilde{Y}$.

$$
(7.1) \qquad p^*\mathcal{L}\left\{ \begin{tikzcd} p^*\widetilde{Y} \arrow[rr] \arrow[d, shift right, "\hat{p}"'] & & \widetilde{Y} \arrow[d, "p"] \\ \widetilde{Y} \arrow[u, shift right, "t"'] \arrow[rr, "p"] & & \widetilde{Y} \end{tikzcd} \right\}\mathcal{L}
$$

By definition, $p^*\widetilde{Y} = \{(a,b) \in \widetilde{Y} \times \widetilde{Y} \mid p(a) = p(b)\}$. Now, there is a section t of $p^*\mathcal{L}$ defined by $t(e) = (e,e) \in p^*\widetilde{Y}$, where $e \in \widetilde{Y}$ and t is a section of $p^*\mathcal{L}$, since $\widehat{p}$ is defined by $\widehat{p}(a,b) = p(a)$. Define $X = \{z \in \widetilde{Y} \mid (p^*s - t^2)(z) = 0\}$. Then $X \subseteq \widetilde{Y}$. Define $\pi : X \to Y$ as $p|_X$.

Locally, X is defined by an equation $s = t^2$. Over a point of Y where $s \neq 0$, we have 2 points of X. Over a point of Y where $s = 0$, we have only one point of X. Then $\{s = 0\} \subset Y$ is the branch divisor, D. D is a divisor, non-negative, in the linear system of $\mathcal{L}^{\otimes 2}$.

So double covers of Y are determined by a line bundle $\mathcal{L}$ and an effective divisor D in the linear system determined by $\mathcal{L}^{\otimes 2}$. If the divisor D is locally defined by $s = 0$, then the double cover is locally defined by $t^2 = s$. In fact, the following holds:

Proposition 7.2. *Let Y be a smooth compact surface and $\mathcal{L}$ a line bundle on Y such that $\mathcal{L}^{\otimes 2}$ has a global section s, not identically zero. Then a surface X and a map $\pi : X \to Y$ exist such that*:

1. $\mathcal{O}_Y(D) = \mathcal{L}^{\otimes 2}$.

2. *D is the divisor on Y corresponding to section s.*

3. *D is an effective divisor in the linear system of $\mathcal{L}^{\otimes 2}$.*

4. *X is smooth at x_0 if and only if s is smooth at $\pi(x_0)$. Thus, X is smooth if and only if D is smooth.*

Definition 7.3. Let X and Y be surfaces, D the divisor on Y, and $\pi : X \to Y$ as defined in Proposition 7.3. Then $\pi : X \to Y$ is defined as the double covering of Y branched along the divisor D and determined by the line bundle $\mathcal{L}$.

Further results from [**1**] include:

Lemma 7.4. *Let $\pi : X \to Y$ be the double covering of Y branched along a smooth divisor D and determined by the line bundle $\mathcal{L}$, i.e., $\mathcal{L}^{\otimes 2} = \mathcal{O}_Y(D)$. Then*

1. $K_X = \pi^*(K_Y \otimes \mathcal{L})$.

2. $\pi_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{L}^{-1}$.

Theorem 7.5 (The projection formula, applied to double covers). *Let $\pi : X \to Y$ be a double cover branched along a smooth divisor D and determined by $\mathcal{L}$, with $\mathcal{D} \cong \mathcal{L}^{\otimes 2}$. If $\mathcal{F}$ is a sheaf on X and is a locally free $\mathcal{O}_Y$-module of finite rank on Y (i.e., a vector bundle on Y), then*

$$\pi_*(\mathcal{F} \otimes \pi^*\mathcal{G}) \cong \pi_*\mathcal{F} \otimes \mathcal{G}$$

as sheaves.

In particular, if $\mathcal{G}$ is a locally free sheaf on Y and $\mathcal{F} = \mathcal{O}_X$, then

$$\pi_*\pi^*\mathcal{G} = \pi_*(\mathcal{O}_X \otimes \pi^*\mathcal{G}) \cong \pi_*\mathcal{O}_X \otimes \mathcal{G} = (\mathcal{O}_Y \oplus \mathcal{L}^{-1}) \otimes \mathcal{G} = \mathcal{G} \oplus (\mathcal{L}^{-1} \otimes \mathcal{G}).$$

We will also need to compute the cohomology of sheaves of the form $\pi_*\Omega^1_X$, for a double cover $\pi : X \to Y$. This requires the introduction (see the Appendix) of the sheaves $\Omega^1_Y(\log D)$, in view of the following theorem:

Lemma 7.6 (see, e.g., Duflot [**3**]). *Let $\pi : X \to Y$ be the double covering of Y branched along a smooth divisor D and determined by the line bundle $\mathcal{L}$, i.e., $\mathcal{L}^{\otimes 2} = \mathcal{O}_Y(D)$. Then*

$$\pi_* \Omega^1_X = \Omega^1_Y \oplus (\Omega^1_Y(\log D) \otimes \mathcal{L}^{-1}).$$

Now passing from sheaves to their cohomology, we first restate some general theorems about sheaf cohomology, applied to the double cover case, without proof. References include [**1, 7, 9**].

Note that if $\pi : X \to Y$ is a double cover branched along a smooth divisor D, determined by $\mathcal{L}$, with $\mathcal{D} \cong \mathcal{L}^{\otimes 2}$, and if $\mathcal{F}$ is a sheaf on X, then, since π has finite fibers, the Leray spectral sequence for π degenerates, and

$$H^i(Y, \pi_* \mathcal{F}) \cong H^i(X, \mathcal{F})$$

for every $i \geq 0$.

Letting $\mathcal{G}$ be a locally free sheaf on Y, we see that

$$H^i(X, \pi^*(\mathcal{G})) \cong H^i(Y, \pi_* \pi^*(\mathcal{G})) \cong H^i(Y, \mathcal{G}) \oplus H^i(Y, \mathcal{L}^{-1} \otimes \mathcal{G}).$$

We'll generally use additive notation for line bundles, viewing them in terms of their associated divisors. Thus, we get:

Theorem 7.7. *Let $\pi : X \to Y$ be a double cover branched along a smooth divisor D and determined by the line bundle $\mathcal{O}(L)$, where $D = 2L$. Suppose that $\widetilde{G}$ is a divisor on Y, and $\widetilde{E} = \pi^* \widetilde{G}$. Then*

a. $H^i(X, \mathcal{O}_X(\widetilde{E})) \cong H^i(Y, \mathcal{O}_Y(\widetilde{G})) \oplus H^i(Y, \mathcal{O}_Y(\widetilde{G} - L))$.

b. $H^i(X, \Omega^1_X(\widetilde{E})) \cong H^i(Y, \Omega^1_Y(\widetilde{G})) \oplus H^i(Y, \mathcal{O}_Y(\widetilde{G} - L) \otimes \Omega^1_Y(\log D))$.

Proof. We leave the proof of a to the reader. For b, we use the various theorems cited in the exposition of this section:

$$\begin{aligned} H^i(X, \Omega^1_X(\widetilde{E})) &\cong H^i(Y, \pi_*(\Omega^1_X(\widetilde{E}))) \\ &\cong H^i(Y, \pi_*(\Omega^1_X \otimes \mathcal{O}_X(\widetilde{E}))) \\ &\cong H^i(Y, \pi_*(\Omega^1_X \otimes \pi^* \mathcal{O}_Y(\widetilde{G}))) \end{aligned}$$

$$\cong H^i(Y, \pi_*\Omega^1_X \otimes \mathcal{O}_Y(\widetilde{G})).$$

Now, by Lemma 7.6,

$$\pi_*\Omega^1_X = \Omega^1_Y \oplus (\Omega^1_Y(\log D) \otimes \mathcal{L}^{-1}),$$

so we obtain b by switching to additive notation. □

8. Gaussian maps for double covers.

8.1. Review of the results of [3]. Consider smooth projective varieties X and Y of the same dimension with X a double cover of Y with smooth branch locus D, and the covering map $\pi : X \to Y$; the divisor L is such that $D = 2L$.

Given the map $\pi : X \to Y$, let G be a divisor on Y. Consider the following Gaussian map:

$$\Lambda^2 H^0(X, \pi^*\mathcal{O}(G)) \xrightarrow{\Phi_{X,\pi^*\mathcal{O}(G)}} H^0(X, (\pi^*\mathcal{O}(G))^2 \otimes \Omega^1_X). \tag{8.1}$$

Using the discussion and isomorphisms of subsection 7.1, we may identify $\Phi_{X,\pi^*\mathcal{O}(G)}$ with

$$\begin{aligned} &\Lambda^2 H^0(Y, \mathcal{O}(G)) \oplus \Lambda^2 H^0(Y, \mathcal{O}(G-L)) \\ &\qquad \oplus (H^0(Y, \mathcal{O}(G)) \otimes H^0(Y, \mathcal{O}(G-L))) \\ &\qquad \xrightarrow{\Phi_{X,\pi^*\mathcal{O}(G)}} H^0(Y, \Omega^1_Y(2G)) \\ &\qquad\qquad \oplus H^0(Y, \mathcal{O}(2G-L) \otimes \Omega^1_Y(\log D)). \end{aligned} \tag{8.2}$$

For ease of reference, we will refer to the various components of this map as follows. Let:

$$\begin{aligned} V_0 &= \Lambda^2 H^0(Y, \mathcal{O}(G)), \\ W_0 &= \Lambda^2 H^0(Y, \mathcal{O}(G-L)), \\ V_1 &= H^0(Y, \mathcal{O}(G)) \otimes H^0(Y, \mathcal{O}(G-L)), \end{aligned}$$

and

$$\begin{aligned} A_0 &= H^0(Y, \Omega^1_Y(2G)), \\ A_1 &= H^0(Y, \mathcal{O}(2G-L) \otimes \Omega^1_Y(\log D)). \end{aligned}$$

We have the following theorem from Duflot [**3**]:

Theorem 8.3 [**3**]. *Suppose that X a double cover of Y with smooth branch locus D, the construction giving the covering map $\pi : X \to Y$; and using the divisor L such that $D = 2L$. Given the Gaussian map*

$$\begin{aligned}\Lambda^2 H^0(Y, \mathcal{O}(G)) \oplus \Lambda^2 H^0(Y, \mathcal{O}(G-L)) \oplus (H^0(Y, \mathcal{O}(G)) \\ \otimes H^0(Y, \mathcal{O}(G-L))) \\ \xrightarrow{\Phi_{X,\pi^*\mathcal{O}(G)}} H^0(Y, \Omega^1_Y(2G)) \oplus H^0(Y, \mathcal{O}(2G-L) \\ \otimes \Omega^1_Y(\log D)),\end{aligned}$$

then we have:

$$\begin{aligned}\Phi_{X,\pi^*\mathcal{O}(G)}|_{V_0} &: V_0 \longrightarrow A_0 \\ \Phi_{X,\pi^*\mathcal{O}(G)}|_{V_1} &: V_1 \longrightarrow A_1 \\ \Phi_{X,\pi^*\mathcal{O}(G)}|_{W_0} &: W_0 \longrightarrow A_0,\end{aligned} \tag{8.3}$$

and

$$\Phi_{X,\pi^*\mathcal{O}(G)}|_{V_0} = \Phi_{Y,\mathcal{O}(G)}.$$

Let

$$\mu_{G,G-L} : H^0(Y, \mathcal{O}(G)) \otimes H^0(Y, \mathcal{O}(G-L)) \to H^0(Y, \mathcal{O}(2G-L))$$

be the multiplication map as indicated. We also have the following proposition:

Proposition 8.3 [**3**]. *With the same hypotheses as Theorem 8.2, there is a commutative diagram of exact sequences*

(8.4)

$$\begin{tikzcd} R(G,G-L) \arrow[r, hook] \arrow[d, "\Phi_{G,G-L}"] & H^0(Y,\mathcal{O}(G)) \otimes H^0(Y,\mathcal{O}(G-L)) \arrow[r, "\mu_{G,G-L}/2"] \arrow[d, "\Phi_{X,\pi^*\mathcal{O}(G)}|_{V_1}"] & H^0(Y,\mathcal{O}(2G-L)) \arrow[d, "r"] \\ H^0(Y,\Omega^1(2G-L)) \arrow[r, hook] & H^0(Y,\Omega^1(\log D)\otimes\mathcal{O}(2G-L)) \arrow[r] & H^0(Y,\mathcal{O}_D(2G-L)). \end{tikzcd}$$

Moreover, if $\mu_{G,G-L}$ *is surjective, and* $H^1(Y, \Omega^1_Y(2G-L)) = 0$*, then this is a commutative diagram of short exact sequences.*

This leads to the following corollary:

Corollary 8.5 [**3**]. *With the same hypotheses as Theorem* 8.2 *plus, assuming that* $\mu_{G,G-L}$ *is surjective, and* $H^1(Y, \Omega^1_Y(2G-L)) = 0$*, then*:

a. *The snake lemma gives an exact sequence*

$$\begin{aligned} 0 \longrightarrow \ker \Phi_{G,G-L} \longrightarrow \ker \Phi_{X,\pi*\mathcal{O}(G)}|_{V_1} \longrightarrow \ker r \\ \longrightarrow \operatorname{cok} \Phi_{G,G-L} \longrightarrow \operatorname{cok} \Phi_{X,\pi*\mathcal{O}(G)}|_{V_1} \longrightarrow \operatorname{cok} r \longrightarrow 0. \end{aligned}$$

b. *If* $H^1(Y, \Omega^1_Y(2G-L)) = 0$*, then* $\operatorname{coker} r = H^1(Y, \mathcal{O}(2G-3L))$.

c. *If* $H^1(Y, \mathcal{O}(2G-L)) = 0$ *and* $\Phi_{Y,G,G-L}$ *is surjective, then* $\operatorname{cok} \Phi_{X,\pi*\mathcal{O}(G)}|_{V_1} \cong H^1(Y, \mathcal{O}(2G-3L))$.

d. *If* $H^1(Y, \mathcal{O}(2G-L)) = 0$*, and* $\Phi_{Y,G,G-L}$ *and* $\Phi_{Y,G}$ *are surjective, then*

$$\operatorname{corank} \Phi_{X,\pi^*\mathcal{O}(G)} = h^1(Y, 2G-3L).$$

9. Gaussian maps: Canonical divisors of double covers. Again consider smooth projective surfaces X and Y with X a double cover of Y with smooth branch locus D, and the covering map $\pi : X \to Y$; the divisor L is such that $D = 2L$.

As we have noted previously, $K_X = \pi^*(K_Y + L)$.

Theorem 9.1. *Given smooth surfaces* X *and* Y *with* X *a double cover of* Y *with smooth branch locus* D*, covering map* $\pi : X \to Y$ *and defining divisor* L *such that* $D = 2L$: *If* $H^0(Y, K_Y) = p_g(Y) = 0$*, then*

$$\operatorname{corank} \Phi_{X,K_X} = \operatorname{corank} \Phi_{Y,K_Y+L} + h^0(Y, \Omega^1(\log D) \otimes \mathcal{O}(2K_Y + L)).$$

Proof. We use the notation of the previous section. Since $H^0(Y, K_Y) = p_g(Y) = 0$, $W_0 = 0$ and $V_1 = 0$. The result is obtained from a direct application of Theorem 8.2. $\square$

9.1. Double covers of general toric surfaces. Let S be a smooth toric surface, defined by a fan of $n+2$ vectors as usual. Since $p_g(S) = 0$, we have the following corollary to Theorem 9.1:

Corollary 9.2. *If S is a smooth toric surface, $\pi : X \to S$ is a double cover of S branched along a smooth curve D, constructed from the line bundle L such that $2L = D$, then*

$$\operatorname{corank} \Phi_{X,K_X} = \operatorname{corank} \Phi_{S,K_S+L} + h^0(S, \Omega^1(\log D) \otimes \mathcal{O}_S(2K_S + L)).$$

From Murray's result on the surjectivity of the Gaussian map for smooth toric surfaces, Theorem 6.3, we have:

Lemma 9.3 [**14**]. *If S is a smooth toric surface, L is a divisor on S and $K_S + L$ is ample, then*

$$\operatorname{corank} \Phi_{S,K_S+L} = 0.$$

Combining this lemma with the previous corollary yields:

Corollary 9.4. *Suppose that S is a smooth toric surface. Let $\pi : X \to S$ be a double cover of S branched along a smooth curve D, constructed with the divisor L such that $2L = D$. If $K_S + L$ is ample, then*

$$\operatorname{corank} \Phi_{X,K_X} = h^0(S, \Omega^1(\log D) \otimes \mathcal{O}_S(2K_S + L)).$$

We may compute $h^0(S, \Omega^1(\log\ D) \otimes \mathcal{O}_S(2K_S + L))$ using results from the Appendix. Applying Theorem 11.8 to Corollary 9.2, with $E = 2K_S + L, D = 2L$ yields:

Theorem 9.5. *Suppose that S is a smooth toric surface. Let $\pi : X \to S$ be a double cover of S branched along a smooth curve D, constructed with the divisor L such that $2L = D$. If $h^1(S, \mathcal{O}_S(2K_S + L)) = 0$ and $h^1(S, \Omega^1_S(2K_S + L)) = 0$, then* a.

$$\begin{aligned} h^0(S,\Omega^1_S(\log\ 2L)(2K_S+L)) = h^0(S,\Omega^1_S(2K_S+L)) \\ + h^0(S,\mathcal{O}_S(2K_S+L)) \\ - \chi(2K_S-L) \\ + h^0(S,\mathcal{O}_S(L-K_S)). \end{aligned}$$

b.

$$\begin{aligned} \operatorname{corank}\Phi_{X,K_X} &= \operatorname{corank}\Phi_{S,K_S+L} + h^0(S,\Omega^1(2K_S+L)) \\ &\quad + h^0(S,\mathcal{O}_S(2K_S+L)) - \chi(2K_S-L) \\ &\quad + h^0(S,\mathcal{O}_S(L-K_S)). \end{aligned}$$

Theorem 9.5 combined with Theorem 9.3 yields:

Theorem 9.6. *Suppose that S is a smooth toric surface. Let $\pi : X \to S$ be a double cover of S branched along a smooth curve D, constructed with the divisor L such that $2L = D$. If $h^1(S,\mathcal{O}_S(2K_S+L)) = 0$, $h^1(S,\Omega^1_S(2K_S+L)) = 0$ and K_S+L is ample, then*

$$\begin{aligned} \operatorname{corank}\Phi_{X,K_X} &= h^0(S,\Omega^1_S(2K_S+L)) + h^0(S,\mathcal{O}_S(2K_S+L)) \\ &\quad - \chi(2K_S-L) + h^0(S,\mathcal{O}_S(L-K_S)). \end{aligned}$$

Combining Theorem 9.6 with Lemma 3.3 gives:

Corollary 9.7. *Suppose that S is a smooth toric surface. Let $\pi : X \to S$ be a double cover of S branched along a smooth curve D, constructed with the divisor L such that $2L = D$. If K_S+L and $2K_S+L$ are ample, then*

$$\begin{aligned} \operatorname{corank}\Phi_{X,K_X} &= \chi(\mathcal{O}_S(2K_S+L)) + \chi(\Omega^1(S,2K_S+L)) \\ &\quad - \chi(2K_S-L) + h^0(S,\mathcal{O}_S(L-K_S)). \end{aligned}$$

One can use this corollary to compute specific values for the corank; for example, since we have, using the Riemann-Roch theorem, that

$$\chi(2K_S+L) = \frac{1}{2}(2K_S+L)(K_S+L)+1$$

and

$$\chi(2K_S - L) = \frac{1}{2}(2K_S - L)(K_S - L) + 1,$$

and Theorem 3.3 tells us that $\chi(\Omega^1(S, 2K_S + L)) = (2K_S + L)^2 - n$, we get that

$$\begin{aligned}\chi(\mathcal{O}_S(2K_S + L)) + \chi(\Omega^1(S, 2K_S + L)) - \chi(\mathcal{O}_S(2K_S - L))\\ = \frac{1}{2}(2K_S + L)(K_S + L) + 1\\ - \frac{1}{2}(2K_S - L)(K_S - L) - 1 + (2K_S + L)^2 - n\\ = 40 - 5n + 7K_S \cdot L + L^2.\end{aligned}$$

Hence,

Corollary 9.8. *With the hypotheses as in Corollary* 9.7*, then*

$$\operatorname{corank} \Phi_{X,K_X} = 40 - 5n + 7K_S \cdot L + L \cdot L + h^0(S, \mathcal{O}_S(L - K_S)).$$

9.2. Double covers of Hirzebruch surfaces. Let $\pi : X \to \mathbf{F}_k$ be a double cover of $\mathbf{F}_k$ branched along a smooth curve D, such that $D \sim 2\alpha C_1 + 2\beta C_2$, $\alpha \geq 0$, $\beta \geq 0$. The divisor L satisfies $2L = D$; $L \sim \alpha C_1 + \beta C_2$.

Adapting Corollary 9.8 for Hirzebruch surfaces, we have:

Corollary 9.9. *If $X, \mathbf{F}_k$, and L are as described above, and $\alpha \geq 5$, $\beta + 2k \geq 5$ and $\beta + 2 - k \geq 0$, then*

$$\operatorname{corank} \Phi_{X,K_X} = 39 + \frac{11}{2} K_{\mathbf{F}_k} \cdot L + \frac{3}{2} L^2.$$

Proof. We will use Corollary 9.8. To have $K_{\mathbf{F}_k} + L = (-2 + \alpha)C_1 + (k - 2 + \beta)C_2$ and $2K_{\mathbf{F}_k} + L = (-4 + \alpha)C_1 + (2k - 4 + \beta)C_2$ ample using Theorem 2.13, we need $\alpha \geq 5$ and $\beta + 2k \geq 5$, which we assume. Then Corollary 9.8 says

$$\operatorname{corank} \Phi_{X,K_X} = 30 + 7K_{\mathbf{F}_k} \cdot L + L^2 + h^0(\mathbf{F}_k, \mathcal{O}_{\mathbf{F}_k}(L - K_{\mathbf{F}_k})).$$

We have that $L - K_{\mathbf{F}_k} = (\alpha+2)C_1 + (\beta+2-k)C_2$. Using Lemma 3.5, since $\alpha \geq 5$, $\alpha + 2 \geq 0$. By supposition, $\beta + 2 - k \geq 0$, therefore we can compute

$$\begin{aligned} h^0(\mathbf{F}_k, \mathcal{O}_{\mathbf{F}_k}(L - K_{\mathbf{F}_k})) &= \chi(L - K) \\ &= \frac{1}{2}(L - K_{\mathbf{F}_k})(L - 2K_{\mathbf{F}_k}) + 1 \\ &= \frac{1}{2}(L^2 - 3L \cdot K_{\mathbf{F}_k} + 2K_{\mathbf{F}_k}^2) + 1. \end{aligned}$$

Substituting this into the above yields:

$$\begin{aligned} \operatorname{corank} \Phi_{X,K_X} &= 30 + 7K_{\mathbf{F}_k} \cdot L + L^2 \\ &\quad + \frac{1}{2}(L^2 - 3L \cdot K_{\mathbf{F}_k} + 2K_{\mathbf{F}_k}^2) + 1 \\ &= 39 + \frac{11}{2}K_{\mathbf{F}_k} \cdot L + \frac{3}{2}L^2. \qquad \square \end{aligned}$$

Remark. Even if $\beta + 2 - k < 0$, we can still compute $h^0(\mathbf{F}_k, \mathcal{O}_{\mathbf{F}_k}(L - K_{\mathbf{F}_k}))$ using Lemma 3.4, but we do not do this here.

10. Gaussian maps on double covers of toric surfaces.

10.1. Double covers of Hirzebruch surfaces. In this section and the next, we put together results from essentially all of the preceding sections. We consider Gaussian maps for "large" divisors on double covers of Hirzebruch surfaces. We use without further comment notation from Section 8.

Lemma 10.1. *Let $\pi : X \to \mathbf{F}_k$ be a double cover of $\mathbf{F}_k$ branched along a smooth irreducible curve D, constructed with the line bundle L such that $D = 2L$. Consider a divisor G on $\mathbf{F}_k$. Then, if $G - L$ is ample*:

a. $\Phi_{X,\pi^*\mathcal{O}(G)}\ |_{V_0} = \Phi_{\mathbf{F}_k,G} : V_0 \to A_0$ *is surjective. Thus,* $\operatorname{corank}\Phi_{X,\pi^*\mathcal{O}(G)} = \operatorname{corank}\Phi_{X,\pi^*\mathcal{O}(G)}\ |_{V_1}$.

b. $\mu : H^0(\mathbf{F}_k, \mathcal{O}(G)) \otimes H^0(\mathbf{F}_k, \mathcal{O}(G-L)) \to H^0(\mathbf{F}_k, \mathcal{O}(2G-L))$ *is surjective.*

c. $H^1(\mathbf{F}_k, \Omega^1(2G-L)) = 0$.

d. $H^1(\mathbf{F}_k, \mathcal{O}(2G-L)) = 0$.

e. $\Phi_{\mathbf{F}_k,G,G-L}$ *is surjective.*

Proof. For part a: this follows from Theorem 8.2 and Proposition 6.1, since $\Phi_{X,\pi^*\mathcal{O}(G)}\ |_{V_0} = \Phi_{\mathbf{F}_k,G}$; note that the hypothesis that $G-L$ is ample implies that $G = (G-L)+L$ is also ample. We have seen that, considering the line bundle L used to construct the double cover, with $2L = D$, and writing $L \sim \alpha C_1 + \beta C_2$, we have $\alpha \geq 0$, $\beta \geq 0$. Thus, $G-L$ ample implies $G-L+L$ is ample as well.

Part b follows from Proposition 4.4 since G and $G-L$ are ample.

Parts c and d follow from Lemma 3.3 since $2G-L = G+(G-L)$ is ample.

Finally, part e follows from Proposition 6.2 since G and $G-L$ are ample. $\square$

Combining Lemma 10.1 with Corollary 8.5 yields one of our main theorems:

Theorem 10.2. *Let $\pi : X \to \mathbf{F}_k$ be a double cover of $\mathbf{F}_k$ branched along a smooth irreducible curve D, constructed with the line bundle L such that $D = 2L$. Consider a divisor G on $\mathbf{F}_k$. Then, if $G-L$ is ample,*

$$\operatorname{corank}\Phi_{X,\pi^*\mathcal{O}(G)} = h^1(\mathbf{F}_k, \mathcal{O}(2G-3L)).$$

We would like to compute the corank of the above Gaussian maps more precisely. As remarked previously, considering the line bundle L used to construct the double cover $\pi : X \to \mathbf{F}_k$ branched over the irreducible smooth curve D, with $2L = D$, and writing $L \sim \alpha C_1 + \beta C_2$, we have $\alpha \geq 0$, $\beta \geq 0$. Also, if G is a divisor on $\mathbf{F}_k$, with $G-L$ ample, and we write $G \sim m_1C_1 + m_2C_2$, we must have, by Corollary 2.13, $m_1 > \alpha \geq 0, m_2 > \beta \geq 0$.

Corollary 10.3. *With the hypotheses of Theorem* 10.2*, write* $L \sim \alpha C_1 + \beta C_2$ *and* $G \sim m_1C_1 + m_2C_2$*. If* $(2m_1)/3 \geq \alpha$ *and* $(2m_2)/3 \geq \beta$*, then*

$$\operatorname{corank} \Phi_{X,\pi^*\mathcal{O}(G)} = 0.$$

Proof. Since $m_1 > (2m_1)/3 \geq \alpha$ and $m_2 > (2m_2)/3 \geq \beta$, then by Theorem 10.2,

$$\operatorname{corank} \Phi_{X,\pi^*\mathcal{O}(G)} = h^1(\mathbf{F}_k, \mathcal{O}(2m_1 - 3\alpha, 2m_2 - 3\beta)).$$

Also, $h^1(\mathbf{F}_k, \mathcal{O}(2m_1 - 3\alpha, 2m_2 - 3\beta)) = 0$ by Lemma 3.6. □

On the other hand, the corank above is not always zero. For example, suppose that X is a double cover of $\mathbf{F}_3$, branched along a smooth curve D linearly equivalent to $14C_1+2C_2$; thus, $L = 7C_1+C_2$. Now, suppose that $G \sim 8C_1+3C_2$. Then, the conditions of Theorem 10.2 are satisfied.

For simplicity, we will write $h^i(\mathbf{F}_3, \mathcal{O}(aC_1 + bC_2))$ as $h^i(a, b)$ from now on. We compute $h^1(-5, 3)$ as follows: We know that

$$\begin{aligned} h^1(-5,3) &= -\chi(-5,3) + h^0(-5,3) + h^2(-5,3) \\ &= -\chi(-5,3) + 0 + h^0(3,-2), \end{aligned}$$

using the Kodaira-Serre duality and Lemma 3.4, part 1. Now,

$$\chi(-5,3) = \frac{(-5,3)(-3,2)}{2} + 1 = \frac{45 - 9 - 10}{2} + 1 = 14,$$

and, using Lemma 3.4, part 2, we compute

$$h^0(3,-2) = 15.$$

Thus,

$$\operatorname{corank} \Phi_{X,\pi^*(8,3)} = 1.$$

10.2. Smooth toric surfaces. Let S be a smooth toric surface, defined by a fan of $n+2$ vectors as usual. Recall the definition of the curves C_i from Section 2. Corresponding to Lemma 10.1, we have:

Lemma 10.4. *Let $\pi : X \to S$ be a double cover of S branched along a smooth irreducible curve D, constructed with the line bundle L such that $D = 2L$. Consider a divisor G on S. Then, if $L, L+K_S, L+K_S+C_i$ (for every i) and $G-2L = G-D$ are all ample*:

a. $\Phi_{X,\pi^*\mathcal{O}(G)} \mid_{V_0} = \Phi_{S,G} : V_0 \to A_0$ *is surjective. Thus,* corank $\Phi_{X,\pi^*\mathcal{O}(G)} = \operatorname{corank} \Phi_{X,\pi^*\mathcal{O}(G)} \mid_{V_1}$.

b. $\mu : H^0(S, \mathcal{O}(G)) \otimes H^0(S, \mathcal{O}(G-L)) \to H^0(S, \mathcal{O}(2G-L))$ *is surjective.*

c. $H^1(S, \Omega^1(2G-L)) = 0$.

d. $H^1(S, \mathcal{O}(2G-L)) = 0$.

e. $\Phi_{S,G,G-L}$ *is surjective.*

Proof. Note that the ampleness hypotheses imply that the following divisors are ample: $G-L = G-2L+L$, $G = G-L+L$, $G+L$, $G+L+K_S$, $G+L+K_S+C_i$ (for every i), $2L+K_S = L+K_S+L$ and $2L+K_S+C_i = L+K_S+C_i+L$ (for every i).

For part a: this follows from Theorem 6.3 and Lemma 8.2, since $\Phi_{X,\pi^*\mathcal{O}(G)} \mid_{V_0} = \Phi_{S,G}$. Part b follows from Fakhruddin's theorem 4.2 since G and $G-L$ are ample. Parts c and d follow from Theorem 3.3 since $2G-L = G+(G-L)$ is ample. Finally, part e follows from Proposition 6.4. $\square$

Combining Lemma 10.4 with Corollary 8.5 yields:

Theorem 10.5. *Let $\pi : X \to S$ be a double cover of S branched along a smooth irreducible curve D, constructed with the line bundle L such that $D = 2L$. Consider a divisor G on S. Then, if $L, L+K_S, L+K_S+C_i$ (for every i) and $G-2L = G-D$ are all ample*:

$$\operatorname{corank} \Phi_{X,\pi^*\mathcal{O}(G)} = 0.$$

Proof. Using Corollary 8.5, $\operatorname{corank}\Phi_{X,\pi^*\mathcal{O}(G)} = h^1(S,\mathcal{O}(2G-3L))$. However, since $G-2L$ and $G-L$ are both ample, so is $2G-3L$. Thus, by Theorem 3.3, $h^1(S,\mathcal{O}(2G-3L))=0$. $\square$

APPENDIX

11. $\Omega^1_M(\log\ D)$. The following discussion is excerpted from Saito [**17**] for the reader's convenience.

Theorem 11.1 [**17**]. *Let M be an n-dimensional complex manifold, and $V\subset M$ a hypersurface of M defined by an equation $h(z)=0$, where h is holomorphic on M. Let ω be a meromorphic q-form on M, which may have poles only along V. Then the following four conditions for ω are equivalent*:

1. *$h\omega$ and $h\,d\omega$ are holomorphic on M.*

2. *$h\omega$ and $dh\wedge\omega$ are holomorphic on M.*

3. *A holomorphic function $g(z)$, a holomorphic $(q-1)$-form ξ and a holomorphic q-form η on M exist such that*:

(a) $\dim_C V\cap\{z\in S : g(z)=0\}\le n-2$.

(b) $g\omega=(dh/h)\wedge\xi+\eta$.

4. *An $(n-2)$-dimensional analytic set $A\subset V$ exists such that the germ of ω at any point $p\in V-A$ belongs to $(dh/h)\wedge\Omega^{q-1}_{M,p}+\Omega^q_{M,p}$, where $\Omega^q_{M,p}$ denotes the module of germs of holomorphic q-forms on M at p.*

This leads to the following definition:

Definition 11.2 [**17**]. A meromorphic q-form on M is called a q-form with logarithmic pole along V or logarithmic q-form if it satisfies the equivalent conditions of Theorem 11.1. Let $h_p=0$ be a reduced equation for V, locally at $p\in V$. A meromorphic q-form is logarithmic along V at p if $h_p\omega$ and $h_p d\omega$ are holomorphic.

We denote

$$\Omega^q_{M,p}(\log V)\doteq\{\text{germs of logarithmic } q\text{-forms at } p\},$$
$$\Omega^q_M(\log V)\doteq\cup_{p\in M}\Omega^q_{M,p}(\log V).$$

Theorem 11.3 [**17**]. *If M is a complex manifold, V a smooth hypersurface of M defined by an equation $h(z) = 0$ where h is holomorphic on M, then $\Omega^q_M(\log V)$, $q = 0, 1, \dots, n$, are coherent $\mathcal{O}_M$-modules.*

Saito defines the residue morphism as follows:

Definition 11.4. If ω is a meromorphic q-form on a complex manifold M and a holomorphic function $g(z)$, a holomorphic $(q-1)$-form ξ and a holomorphic q-form η on M exist such that $g\omega = (dh/h) \wedge \xi + \eta$, then the residue morphism, res, is a sheaf homomorphism:

$$\begin{aligned} \mathrm{res} : \Omega^q_M(\log V) &\longrightarrow \mathcal{O}_V \\ \omega &\overset{\mathrm{res}}{\longmapsto} \frac{1}{g}\xi. \end{aligned}$$

Using Saito's definition of the residue morphism, we have the residue exact sequence for logarithmic q-forms:

Theorem 11.5 (see, e.g., [**17**, page 276]). *If M is a complex manifold and V is a smooth hypersurface on M, then the sequence*

$$0 \longrightarrow \Omega^q_M \longrightarrow \Omega^q_M(\log V) \xrightarrow{\mathrm{res}} \mathcal{O}_V \longrightarrow 0$$

is exact.

Applying this short exact sequence to a general surface Y, D a smooth curve on Y and any divisor E on Y, we get the short exact sequence

$$0 \longrightarrow \Omega^1_Y(E) \longrightarrow \Omega^1_Y(\log D)(E) \longrightarrow \mathcal{O}(E)|_D \longrightarrow 0. \tag{11.6}$$

Recall the standard short exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-D)(E) \longrightarrow \mathcal{O}_Y(E) \longrightarrow \mathcal{O}(E)|_D \longrightarrow 0. \tag{11.7}$$

Using the associated long exact sequences gives us the following theorem:

Theorem 11.8. *Suppose that Y is a smooth surface, E a divisor, D a smooth curve on Y.*

a. *If $h^1(Y, \Omega^1_Y(E)) = 0$, then*

$$h^0(Y, \Omega^1_Y(\log D)(E)) = h^0(Y, \mathcal{O}_Y(E)|_D) + h^0(Y, \Omega^1_Y(E)).$$

b. *If $h^1(Y, \mathcal{O}_Y(E)) = 0$, then*

$$h^0(Y, \mathcal{O}_Y(E)|_D) = h^0(Y, \mathcal{O}_Y(E)) - \chi(E-D) + h^0(Y, \mathcal{O}_Y(K_Y+D-E)).$$

c. *If $h^1(Y, \mathcal{O}_Y(E)) = 0$ and $h^1(Y, \Omega^1_Y(E)) = 0$, then*

$$\begin{aligned} h^0(Y, \Omega^1_Y(\log D)(E)) = h^0(Y, \Omega^1_Y(E)) &+ h^0(Y, \mathcal{O}_Y(E)) \\ &- \chi(E-D) + h^0(Y, \mathcal{O}_Y(K_Y + D - E)). \end{aligned}$$

Proof. Note that $\chi(E-D) = h^0(Y, \mathcal{O}_Y(E-D)) - h^1(Y, \mathcal{O}_Y(E-D)) + h^2(Y, \mathcal{O}_Y(E-D))$ and $h^2(Y, \mathcal{O}_Y(E-D)) = h^0(Y, \mathcal{O}_Y(K_Y+D-E)$ by the Kodaira-Serre duality. If $h^1(Y, \mathcal{O}_Y(E)) = 0$, then the sequence

$$\begin{aligned} 0 \longrightarrow H^0(\mathcal{O}_Y(E-D)) &\longrightarrow H^0(Y, \mathcal{O}_Y(E)) \\ \longrightarrow H^0(Y, \mathcal{O}_Y(E)|_D) &\longrightarrow H^1(Y, \mathcal{O}_Y(E-D)) \longrightarrow 0 \end{aligned}$$

is exact. Thus,

$$h^0(\mathcal{O}_Y(E-D)) - h^0(Y, \mathcal{O}_Y(E)) + h^0(Y, \mathcal{O}_Y(E)|_D - h^1(Y, \mathcal{O}_Y(E-D)) = 0.$$

Therefore,

$$h^0(Y, \mathcal{O}_Y(E)|_D = h^1(Y, \mathcal{O}_Y(E-D) - h^0(\mathcal{O}_Y(E-D)) + h^0(Y, \mathcal{O}_Y(E)). \quad \square$$

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