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ON THE ELLIPTIC CURVES ARISING FROM 4-BAR MECHANISMS

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ABSTRACT. We show that the residual curve $\mathcal{R}$ and the Darboux curve $\mathcal{R}'$ associated to a virtual 4-bar mechanism are two smooth elliptic curves defined over $\mathbf{R}$, isomorphic over $\mathbf{C}$ but not isomorphic over $\mathbf{R}$, see Theorem 2. The same question seems to be open for elliptic curves coming from an actual 4-bar mechanism. In this case we show that the corresponding real curves $\mathcal{R}(\mathbf{R})$ and $\mathcal{R}'(\mathbf{R})$ are diffeomorphic, see Theorem 3.

1. Introduction. A planar 4-bar mechanism $\mathcal{M}$ consists of a moving quadrilateral whose lengths of sides are given by positive real numbers d_1, d_2, d_3, d_4 while the directions of the sides are given by unit complex numbers z_1, z_2, z_3, z_4, respectively. The closure condition for the mechanism $\mathcal{M}$ is given by a single complex relation

$$d_1z_1 + d_2z_2 + d_3z_3 + d_4z_4 = 0. \tag{1}$$

If we write $z_k = x_k + iy_k \in S^1 \subset \mathbf{C}$ for all $1 \leq k \leq 4$, then the corresponding real relations are expressed by the equations

$$\begin{cases} d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4 = 0 \\ d_1y_1 + d_2y_2 + d_3y_3 + d_4y_4 = 0 \\ x_k^2 + y_k^2 = 1 & 1 \leq k \leq 4. \end{cases} \tag{2}$$

Naturally, we would like to ensure that the equations (2) do possess at least one real solution, i.e., that there exists a quadrilateral with sides of lengths d_1, d_2, d_3, d_4. The required condition is then

$$2d_k \leq d_1 + d_2 + d_3 + d_4 \tag{3}$$

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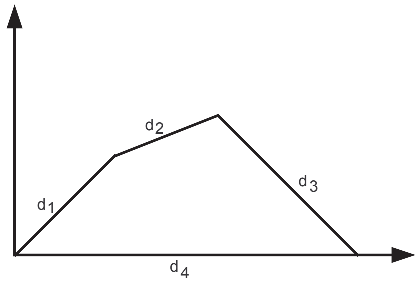

for all $1 \leq k \leq 4$. If these conditions are satisfied, we say that we are in the *constructible case* and that $\mathcal{M}$ is an *actual* mechanism; otherwise, we are in the *non-constructible case* and $\mathcal{M}$ is a *virtual* mechanism.

For the quadrilateral to move with one degree of freedom, in other words, if we classify such quadrilaterals up to similitudes of the plane, we can fix the direction of one side, say the fourth, and we normalize it by taking $d_4 = 1$, $x_4 = -1$, $y_4 = 0$.

Then the equations (2) become

$$\begin{cases} d_1x_1 + d_2x_2 + d_3x_3 = 1 \\ d_1y_1 + d_2y_2 + d_3y_3 = 0 \\ x_k^2 + y_k^2 = 1 & 1 \leq k \leq 3. \end{cases} \tag{4}$$

We thus get 5 equations in $\mathbf{R}^6$. If we complexify and projectivize, then the above equations become

$$\mathcal{C} : \begin{cases} d_1x_1 + d_2x_2 + d_3x_3 = w \\ d_1y_1 + d_2y_2 + d_3y_3 = 0 \\ x_k^2 + y_k^2 = w^2 & 1 \leq k \leq 3. \end{cases} \tag{5}$$

The associated curve $\mathcal{C} \subset \mathbf{P}^6$ is called the *linkage curve*, see [**2**] for more details on this curve. Let $\mathcal{H}_\infty : w = 0$ be the hyperplane at infinity. Then $\mathcal{H}_\infty \cap \mathcal{C} = \mathcal{L} \sqcup \overline{\mathcal{L}}$, where the two complex conjugate lines are given by

$$\mathcal{L} : \begin{cases} d_1x_1 + d_2x_2 + d_3x_3 = 0 \\ x_k = iy_k & 1 \leq k \leq 3 \end{cases} \tag{6}$$

and

$$\overline{\mathcal{L}}:\begin{cases} d_1x_1+d_2x_2+d_3x_3=0 \\ x_k=-iy_k & 1\le k\le 3. \end{cases} \tag{7}$$

Since $\mathcal{C}$ is a complete intersection, we have $\deg\mathcal{C}=1\cdot 1\cdot 2\cdot 2\cdot 2=8$. It follows that $\mathcal{C}=\mathcal{R}\cup\mathcal{L}\cup\overline{\mathcal{L}}$, where curve $\mathcal{R}$ is called the *residual curve* and one has $\deg\mathcal{R}=6$. The singularities of this curve $\mathcal{R}$ are discussed in [**2**], where the following results are established.

Corollary 1. *In the generic case, i.e., when the Grashof equalities*

$$\pm d_1\pm d_2\pm d_3\pm d_4=0$$

all fail, the residual curve $\mathcal{R}$ *is a smooth elliptic curve, i.e., the genus of* $\mathcal{R}$ *is* $g(\mathcal{R})=1$.

We now recall the *Darboux construction* of elliptic curves associated to a 4-bar mechanism $\mathcal{M}$. Note that $z\in S^1$ if and only if the complex conjugate $\overline{z}$ coincides to the inverse z^{-1}. It follows that equation (2) and its conjugate yield the following equations in the complex projective space $\mathbf{P}^3$, defining a plane P and a cubic surface S.

$$\begin{aligned} &P: d_1z_1+d_2z_2+d_3z_3+d_4z_4=0 \\ &S: \frac{d_1}{z_1}+\frac{d_2}{z_2}+\frac{d_3}{z_3}+\frac{d_4}{z_4}=0. \end{aligned} \tag{8}$$

Classically $\mathcal{R}'=P\cap S$ is called the plane cubic curve associated to $\mathcal{M}$ or the *Darboux cubic curve* of $\mathcal{M}$. The residual curve of $\mathcal{M}$ and the Darboux curve of $\mathcal{M}$ are related by the following result, see [**2**].

Theorem 1. *The residual curve* $\mathcal{R}$ *and the Darboux cubic curve* $\mathcal{R}'$ *are isomorphic. More precisely, using the normalization given above by* $d_4=1$, $z_4=-1$ *for both* $\mathcal{R}$ *and* $\mathcal{R}'$, *the morphism*

$$\begin{aligned} \phi:\mathcal{R}\setminus(\mathcal{R}\cap\mathcal{H}_\infty) &\longrightarrow \mathcal{R}'\setminus(\mathcal{R}'\cap\{z_4=0\}) \\ (x_1:y_1:x_2:y_2:x_3:y_3:1) &\longmapsto (x_1+iy_1:x_2+iy_2:x_3+iy_3:-1) \end{aligned}$$

induces an explicit isomorphism between these two curves.

Note that ϕ is in fact a bi-rational morphism, but any bi-rational morphism between projective curves whose indeterminacy points are smooth points extends to a regular isomorphism, and this is the case for ϕ as explained in [**2**].

Even if $\mathcal{R}$ and $\mathcal{R}'$ are both defined over the real numbers $\mathbf{R}$, the above isomorphism ϕ is not defined over $\mathbf{R}$. So one may ask whether these elliptic curves are isomorphic over $\mathbf{R}$. One case when it is easy to answer this question is when the real points $\mathcal{R}(\mathbf{R})$ and $\mathcal{R}'(\mathbf{R})$ of the two curves have distinct topology. In fact, we have the following result.

Theorem 2. *In the generic non-constructible case, the elliptic curves $\mathcal{R}$ and $\mathcal{R}'$ are not isomorphic over $\mathbf{R}$. More precisely, in this case $\mathcal{R}(\mathbf{R})$ is the empty set, while $\mathcal{R}'(\mathbf{R})$ is the disjoint union of two circles S^1.*

In the constructible case, we don't know whether the elliptic curves $\mathcal{R}$ and $\mathcal{R}'$ are isomorphic over $\mathbf{R}$. The above simple idea does not work, since one has the following.

Theorem 3. *Let $0 < a \leq b \leq c \leq d$ be the numbers d_1, d_2, d_3, d_4 written in increasing order of magnitude. Then, in the generic constructible case, one has one of the following two possibilities.*

(i) *Either $a + d > b + c$ and then both real curves $\mathcal{R}(\mathbf{R})$ and $\mathcal{R}'(\mathbf{R})$ are diffeomorphic to S^1, or*

(ii) *$a + d < b + c$ and then both real curves $\mathcal{R}(\mathbf{R})$ and $\mathcal{R}'(\mathbf{R})$ are diffeomorphic to the disjoint union of two circles S^1.*

The above claim for the real curve $\mathcal{R}(\mathbf{R})$ is well-known, see [**2, 3**].

2. The proofs in the smooth case.

2.1. Weierstrass normal forms. We recall here briefly the basic facts on the Weierstrass normal form of a plane cubic curve, see for instance [**1**]. By a Weierstrass normal form, we mean a cubic in $\mathbf{P}^2$ defined by an equation of the form $y^2 z = P(x, z)$ where $P(x, z)$ is a non-zero binary cubic form in x, z. In other words, in the affine open set $z = 1$, such a curve has the defining equation $y^2 = g(x)$ with $g(x)$ a cubic polynomial in x.

Lemma 1. *Any smooth cubic curve $\mathcal{C}$ in $\mathbf{P}^2$ is projectively equivalent to a cubic $y^2z = 4x^3 - \alpha xz^2 - \beta z^3$ in Weierstrass normal form, for some $\alpha, \beta \in \mathcal{C}$ with $\alpha^3 \neq 27\beta^2$.*

Proof. Let $F = 0$ be an equation for $\mathcal{C}$. Choose the linear coordinates on $\mathbf{P}^2$ such that $p = (0 : 1 : 0)$ is a flex of the cubic $\mathcal{C}$ and the tangent to $\mathcal{C}$ at the point p is $z = 0$. The condition for which $z = 0$ is an inflectional tangent to the cubic $\mathcal{C}$ at p is $F(x, y, 0)$, which should have $(0 : 1)$ as a root of multiplicity 3, i.e., the coefficients of x^2y, y^2x, y^3 must vanish. Thus, F has the form

$$F = ax^3 + bx^2z + cxyz + dy^2z + ez^2x + fz^2y + gz^3.$$

Note that $a \neq 0$ (since otherwise F has factor z) and $d \neq 0$ (since p is a non-singular point).

Now replace y by $y - [c/2d]x - [f/2d]z$; that forces the coefficients of xyz, yz^2 to vanish, and does not reintroduce any monomials so far excluded. After that, replace x by $x-[b/3a]z$; that forces the coefficient of zx^2 to vanish, and likewise does not reintroduce any monomials so far excluded. We can assume $d = 1$, $a = -4$. Then our cubic is given by the equation

$$zy^2 = 4x^3 - \alpha z^2x - \beta z^3,$$

which is the required Weierstrass normal form. It remains to show that the curve $\mathcal{C}$ is non-singular if and only if $\alpha^3 \neq 27\beta^2$. The cubic meets the line $z = 0$ only at point p so any singular point has to be in the affine chart $z = 1$. This affine curve $y^2 = 4x^3 - \alpha x - \beta$ is singular if and only if the cubic equation $4x^3 - \alpha x - \beta$ has a multiple root, and this is the case if and only if $\alpha^3 = 27\beta^2$. □

Remark 1. Any smooth plane cubic curve $\mathcal{C}$ has nine points of inflexion, at least one of them real if $\mathcal{C}$ is defined over $\mathbf{R}$. It follows that such a cubic has at least one real Weierstrass normal form $y^2 = g(x)$. If $\mathcal{C}$ is defined over $\mathbf{R}$, then the number of connected components of the real curve $\mathcal{C}(\mathbf{R})$ is one (respectively two) exactly when the cubic equation $g(x) = 0$ has one real root (respectively three real roots).

2.2. Weierstrass normal form of the Darboux cubic. Let us write d_1, d_2, d_3, d_4 in increasing order of magnitude a, b, c, d and normalize by setting $d = 1$. Note that this normalization is different in general from the normalization used above, e.g., in Theorem 1. Using the equations (8), and identifying the plane P to $\mathbf{P}^2$, we see that the Darboux cubic curve $\mathcal{R}'$ is defined in $\mathbf{P}^2$ by the equation

$$(9)\quad \mathcal{R}' : ay(1+ax+by)+bx(1+ax+by)-c^2xy+xy(1+ax+by)=0.$$

Its Weierstrass normal form can be computed using the software MAPLE, and we get $y^2 = g(x)$ where

$$\begin{aligned} g(x) = {} & \tfrac{-1}{864}(12x+a^4+b^4+c^4-2a^2b^2-2b^2c^2-2a^2c^2-2a^2-2b^2-2c^2+1) \\ & \times(-72x^2+6x(a^4+b^4+c^4-2a^2+2b^2+2c^2+2a^2b^2+2b^2c^2+2a^2c^2+1) \\ & + a^8+b^8+c^8-4a^6b^2-4a^6c^2-4a^2b^6-4b^6c^2-4a^2c^6-4b^2c^6 \\ & + 6a^4b^4+6a^6c^6+6b^4c^4+4a^4b^2c^2+4a^2b^4c^2 \\ & + 4a^2b^2c^4-4a^6-4b^6-4c^6+4a^4b^2+4a^4c^2 \\ & + 4a^2b^4+4b^4c^2+4a^2c^4+4b^2c^4-48a^2b^2c^2 \\ & +6a^4+6b^4+6c^4+4a^2b^2+4a^2c^2+4b^2c^2-4a^2-4b^2-4c^2+1) \\ = {} & \ell(x)\cdot q(x). \end{aligned}$$

Here $\ell(x)$ is the linear form corresponding to the first factor, and $q(x)$ is the quadratic form corresponding to the second factor, when we forget about the unit $-1/864$.

Using MAPLE again, we calculate the discriminant Δ of the quadratic form $q(x)$ in the above factorization and get

$$\begin{aligned} \Delta = {} & 324(b+a+1-c)(b-a+1-c)(b+1+a+c)(b+a+c-1) \\ & \times(b+1-a+c)(b-1+a-c)(b-1-a-c)(b+c-a-1). \end{aligned}$$

Since $0 < a \le b \le c \le 1$, it follows that $(b+a+1-c) > 0$, $(b-a+1-c) > 0$, $(b+a+1+c) > 0$, $(b-a+1+c) > 0$, $(b-1+a-c) < 0$ and $(b-1-a-c) < 0$.

In the non-constructible case, one has $(b+a-1+c) < 0$ and hence $(b+c-a-1) < 0$ as well. It follows that $\Delta > 0$ and hence the real curve $\mathcal{R}'(\mathbf{R})$ has two connected components, each one of them diffeomorphic to a circle S^1. In this case $\mathcal{R}(\mathbf{R}) = \varnothing$, and this completes the proof of Theorem 2.

In the constructible case, one has $(b + a - 1 + c) > 0$, and hence all depends upon the sign of $(b + c - a - 1)$. It follows that, if $b + c > a + 1$, then $\Delta > 0$, and hence the real curve $\mathcal{R}'(\mathbf{R})$ has two connected components, each one of them diffeomorphic to a circle S^1. In this case $\mathcal{R}(\mathbf{R})$ also has two connected components, each one of them diffeomorphic to a circle S^1, see [**2, 3**]. In the opposite case, i.e., when $b + c < a + 1$, then $\Delta < 0$, and hence the real curve $\mathcal{R}'(\mathbf{R})$ has one connected component, diffeomorphic to a circle S^1. The same holds for the real curve $\mathcal{R}(\mathbf{R})$, see [**2, 3**]. This completes the proof of Theorem 3.

Remark 2. Since $\ell(x) = 12x + 1 - 2b^2 + b^4 - 2a^2 - 2a^2b^2 + a^4 - 2c^2 - 2c^2b^2 - 2c^2a^2 + c^4$, we can use MAPLE to find the root x_1 of the equation $\ell(x) = 0$ and then we can compute $q(x_1)$ and get

$$q(x_1) = -72a^2b^2c^2 \neq 0.$$

In other words, the equation $g(x) = 0$ has multiple roots if and only if $\Delta = 0$. This can be derived from Theorem 1, since the condition $\Delta = 0$ is exactly equivalent to the Grashof equation, i.e., to the fact that the residual curve $\mathcal{R}$ is singular.

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