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ON A GENERAL THEORY OF FACTORIZATION IN INTEGRAL DOMAINS

D.D. ANDERSON AND ANDREA M. FRAZIER

ABSTRACT. This paper introduces a general theory of factorization of elements in integral domains. This theory subsumes most if not all previously studied cases such as the usual factorization into irreducible elements or into prime elements; the factorization into distinguished classes of elements such as prime powers, primary, or t-pure elements, and the comaximal factorizations of McAdam and Swan.

Let D be an integral domain, let $D^{\#} = D - (U(D) \cup \{0\})$ where $U(D)$ is the group of units of D, and let τ be a relation on $D^{\#}$. The key ideas are the notions of a *τ-factorization of* $a \in D^{\#}$ ($a = \lambda a_1 \cdots a_n$ where $\lambda \in U(D)$, $a_i \in D^{\#}$ and $a_i \tau a_j$ for $i \neq j$), τ-divides $|_\tau$ ($a|_\tau b$ if a occurs in a τ-factorization of b), a *τ-irreducible element* (the only τ-factorizations of a are the trivial ones $a = \lambda(\lambda^{-1}a)$), a *$\tau$-prime element* ($a|\lambda a_1 \cdots a_n$, a τ-factorization, then $a|a_i$ for some i) and a *$|_\tau$-prime element* ($a|_\tau \lambda a_1 \cdots a_n$, a τ-factorization, then $a|_\tau a_i$ for some i). Numerous examples are given to illustrate the theory.

1. Introduction. The notion of factorization of an element of an integral domain plays a central role in algebra. The last 15 years has seen an explosion of research concerning factorization. For example, one can note the work of the first author, D.F. Anderson, Zafrullah and others on generalizations of unique factorization, the work of Chapman, Coykendall, Smith and others on half-factorial domains, the work of D.F. Anderson and others on elasticity, the work of Geroldinger, and Halter-Koch and others on lengths of factorizations, the work of Hassler and Kainrath, and the recent work of McAdam and Swan on comaximal factorization. The purpose of this paper is to lay the foundation for a general theory of factorization of the nonzero nonunit elements of an integral domain. We believe that our theory subsumes most if not all of the various types of factorizations that have been studied. For

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example, it includes the usual factorization into irreducible elements or into prime elements, the factorization into elements that behave like powers of primes such as primary elements or t-pure elements, and the recent comaximal factorizations of McAdam and Swan [**36**]. Our theory is based on the notions of τ-factorizations, τ-divides, τ-atoms, τ-primes and τ-divides primes where τ is a relation on the nonzero nonunit elements of the integral domain in question. Before defining these terms, we give a brief overview of factorization in integral domains.

Throughout, D will denote an integral domain with quotient field K, group of units $U(D)$, and $D^{\#}$ its set of nonzero nonunits. By a *factorization* of $a \in D^{\#}$ we mean $a = a_1 \cdots a_n$ where each $a_i \in D^{\#}$. Sometimes it is useful to write a factorization as $a = \lambda a_1 \cdots a_n$ where $\lambda \in U(D)$ and $a_i \in D^{\#}$. An element $a \in D^{\#}$ is *irreducible* or an *atom* if it can not be factored as $a = bc$ where $b, c \in D^{\#}$ and is *prime* if $a|bc$ implies $a|b$ or $a|c$. The domain D is *atomic* if each element of $D^{\#}$ has a factorization into atoms. Of course a UFD is an atomic domain in which the factorization of each nonzero nonunit into atoms is unique up to associates and order of factors, or equivalently, an integral domain in which each nonzero nonunit element is a product of prime elements. For atomic domains one is interested in how well-behaved factorizations into atoms must be. For example, atomic domains in which any two factorizations of an element into atoms have the same length, called *half-factorial domains*, have received wide attention [**23**]. Various other factorization properties on atomic domains weaker than unique factorization were defined in [**6**]. The definitions of these domains will be reviewed in Section 2 and then extended to τ-factorizations.

We can also study (unique) factorization in nonatomic integral domains. Now UFD's can be characterized by the property that each nonzero nonunit has the form $\lambda p_1^{a_1} \cdots p_n^{a_n}$ where λ is a unit, $p_1, \dots, p_n$ are nonassociate prime elements, and each $a_i \geq 1$. Each of the $p_i^{a_i}$, in addition to being a power of a prime, has other properties, each of which is subject to generalization. For example, each $p_i^{a_i}$ is primary, each is contained in a unique maximal t-ideal (defined below), and the $p_i^{a_i}$ are pairwise coprime. *Weakly factorial domains*, integral domains in which every nonzero nonunit is a product of primary elements, were introduced in [**10**]. Let D be an integral domain. Call a nonzero nonunit $x \in D$ *t-pure* if it is contained in a unique maximal

t-ideal (as is the case for a nonzero primary element). It turns out that every element of $D^{\#}$ is a product of t-pure elements (respectively, primary elements) if and only if the intersection $D = \bigcap_{P \in t\text{-}\max(D)} D_P$ (respectively, $D = \bigcap_{P \in X^{(1)}(D)} D_P$) is locally finite, is independent (i.e., distinct maximal t-ideals contain no common nonzero prime ideal (this, of course, always holds for the second intersection)), and has t-class group $C\ell_t(D) = 0$. (This result is generalized to arbitrary finite character star-operations in Theorem 4.6.) Here t-max(D) is the set of maximal t-ideals of D and $X^{(1)}(D)$ is the set of height-one prime ideals of D. The t-class group is defined below. This factorization into t-pure elements (respectively, primary elements) is unique once elements contained in the same maximal t-ideal are combined, or equivalently, when the elements in the product are pairwise v-coprime. For more on factorizations induced by independent locally finite intersections of localizations, see [**17**]. For more on factorization into t-pure elements and for factorizations into other types of elements such as homogenous elements and rigid elements see [**2, 12, 37**–**39**] and [**8**]. See [**4**] for a survey on non-atomic (unique) factorization.

Recently McAdam and Swan [**36**] studied comaximal factorizations. For $a \in D^{\#}$, D an integral domain, a *comaximal factorization of* a is a factorization $a = a_1 \cdots a_n$ where $(a_i, a_j) = D$ for $i \neq j$. They defined $a \in D^{\#}$ to be *pseudo-irreducible* (respectively, *pseudo-prime*) if a does not have a comaximal factorization $a = bc$ (respectively, if $a|bc$ where b and c are comaximal, then $a|b$ or $a|c$). They showed that an integral domain D with the property that each element of $D^{\#}$ has a comaximal factorization into pseudo-irreducible elements has unique comaximal factorization into pseudo-irreducible elements if and only if every two-generated invertible ideal of D is principal. In [**4**], the first author following suggestions of Zafrullah extended these definitions to general star-operations. Recall that a *star-operation* $\star$ is a closure operation on the set $F(D)$ of nonzero fractional ideals of D with $D^\star = D$ that satisfies $(aA)^\star = aA^\star$ for all $a \in K^* = K - \{0\}$ and $A \in F(D)$. Here a $\star$-*comaximal factorization* of a is a factorization $a = a_1 \cdots a_n$ where $(a_i, a_j)^\star = D$ for $i \neq j$. And $a \in D^{\#}$ is $\star$-*pseudo-irreducible* (resp., $\star$-*pseudo-prime*) if a does not have a $\star$-comaximal factorization $a = bc$ (respectively, if $a|bc$ where $(b, c)^\star = D$, then $a|b$ or $a|c$). If we take $\star$ to be the *d-operation* $A \to A_d = A$ we get the comaximal factorizations of

McAdam and Swan. Also of interest is the case of the *t-operation* $A_t = \bigcup\{B_v = (B^{-1})^{-1} | B \in F(D),\ B \subseteq A \text{ is finitely generated}\}$.

For an introduction to star-operations, see [**29**]; while for a more detailed account the reader may consult [**33**] and the references given there. We recall a few more definitions and facts concerning star-operations. Let D be an integral domain and $\star$ a star-operation on D. A fractional ideal $A \in F(D)$ is called a $\star$-*ideal* if $A = A^\star$. A fractional ideal $A \in F(D)$ is $\star$-*invertible* if there is a $B \in F(D)$ with $(AB)^\star = D$. We can then take $B = A^{-1}$. The set $T_\star(D)$ of $\star$-invertible $\star$-ideals of D forms a group under the $\star$-product $A \star B = (AB)^\star$. Let $Princ(D)$ be its subgroup of nonzero principal fractional ideals. Then the quotient group $C\ell_\star(D) = T_\star(D)/Princ(D)$ is called the $\star$-*class group* of D. For $\star = d$, $C\ell_\star(D) = Pic(D)$ and for $\star = t$, $C\ell_t(D)$ is the *t-class group* or just the *class group* of D. For more on $\star$-invertibility and $\star$-class groups, especially in the case of $\star = t$, see [**20, 40**]. Given two star-operations $\star_1$ and $\star_2$ on D, we write $\star_1 \leq \star_2$ if $A^{\star_1} \subseteq A^{\star_2}$ for all $A \in F(D)$. Now $\star$ has *finite character* if for each $A \in F(D)$, $A^\star = \bigcup\{B^\star | B \in F(D), B \subseteq A \text{ is finitely generated}\}$. So the d-operation and t-operation have finite character. Suppose that $\star$ has finite character. Then each proper integral $\star$-ideal is contained in a maximal $\star$-ideal and a maximal $\star$-ideal is prime. Let $\star$-max(D) denote the set of maximal $\star$-ideals. We always have $D = \bigcap_{P \in \star\text{-}\max(D)} D_P$.

We now define the key notions of our theory. Let D be an integral domain and τ a symmetric relation on $D^\#$. For $a \in D^\#$, a τ-*factorization* is a factorization $a = \lambda a_1 \cdots a_n$ where $\lambda \in U(D)$ and $a_i \tau a_j$ for $i \neq j$. In this case we call a_i a τ-*factor* of a and say that a_i τ-*divides* a, written $a_i |_\tau a$. Call $a \in D^\#$ τ-*irreducible* or a τ-*atom* if $a = \lambda(\lambda^{-1} a)$ $(\lambda \in U(D))$ are the only τ-factorizations of a, and call D τ-*atomic* if each element of $D^\#$ has a τ-factorization into τ-irreducibles. Finally, $a \in D^\#$ is τ-*prime* (respectively, $|_\tau$-*prime*) if whenever $a | \lambda a_1 \cdots a_n$ (respectively, $a |_\tau \lambda a_1 \cdots a_n$) where $\lambda a_1 \cdots a_n$ is a τ-factorization, then $a | a_i$ (respectively, $a |_\tau a_i$) for some i.

Let us see how the previously mentioned factorization schemes relate to τ-factorizations. If we let $\tau = D^\# \times D^\#$, a τ-factorization is just a usual factorization, $|_\tau$ is the usual divides $|$, τ-irreducible is just irreducible, and τ-prime and $|_\tau$-prime are just prime. Next suppose we want to study factorization into a set S of distinguished elements such as atoms, primes, or primaries. Define $a \tau b \Leftrightarrow a,\ b \in S$. Here a non-trivial τ-factorization is just a factorization $a = \lambda a_1 \cdots a_n$ where each $a_i \in S$. Finally, let $\star$ be a star-operation on D. Define $a\tau_\star b \Leftrightarrow (a,b)^\star = D$. Then a $\tau_\star$-factorization is a $\star$-comaximal factorization and a $\star$-pseudo-irreducible (respectively, $\star$-pseudo-prime) element is a $\tau_\star$-atom (respectively, $\tau_\star$-prime).

Section 2 develops the general theory of τ-factorizations while the remaining sections cover particular examples. Section 2 begins with the relevant definitions and then gives a number of examples. Of particular importance is the notion of a divisive relation τ on $D^\#$ for the integral domain D. Here τ is *divisive* if, for $a, a', b, b' \in D^\#$ with $a'|a$, $b'|b$ and $a\tau b$, then $a'\tau b'$. The notions of τ-UFD, τ-FFD, τ-HFD, τ-BFD, τ-ACCP, and τ-atomic are defined and studied. For example, it is shown that τ-UFD $\Rightarrow$ τ-FFD $\Rightarrow$ τ-BFD $\Rightarrow$ τ-ACCP $\Rightarrow$ τ-atomic and that the various factorization properties such as UFD or BFD imply τ-UFD or τ-BFD. Several of these implications require τ to be divisive (see Figure 2). Perhaps the most interesting result of Section 2 is that for τ divisive, a UFD is a τ-UFD (Theorem 2.11).

Section 3 covers the relation ∂ on $D[X]$, D an integral domain, defined by $f\partial g \Leftrightarrow \deg f = \deg g$. Note that ∂ is not divisive. It is shown (Theorem 3.1) that $D[X]$ is a ∂-UFD $\Leftrightarrow$ $D[X]$ is ∂-atomic $\Leftrightarrow$ $D = K$ is algebraically closed. More generally (Theorem 3.3), every $f \in D[X]$ with $\deg f \geq 1$ has a ∂-atomic factorization $\Leftrightarrow$ every indecomposable polynomial of $D[X]$ has degree one $\Leftrightarrow$ D is a Schreier domain with algebraically closed quotient field. The relation ∂ on $\mathbf{R}[X]$, $\mathbf{R}$ the reals, is used to exhibit bad behavior that can occur when the relation is not divisive (Example 3.2). While the reader may question the naturalness of ∂-factorizations, they are useful for providing counterexamples and their study lead us to Theorems 3.1 and 3.3.

Section 4 covers $\star$-comaximal factorizations (which we will call $\star$-atomic factorizations) where $\star$ is a (finite character) star-operation. This generalizes the work of McAdam and Swan [**36**] on comaximal factorizations. Perhaps the main result of this section is Theorem 4.6 which characterizes the $\tau_\star$-UFDs with the property that each $\star$-irreducible element is contained in a unique maximal $\star$-ideal: a domain D is a $\tau_\star$-UFD in which each $\star$-irreducible is contained in a unique maximal $\star$-ideal if and only if $\star$-max(D) is independent of finite character and $Cl_\star(D) = 0$.

Section 5 studies the relation τ_n on $\mathbf{Z}^\#$, $\mathbf{Z}$ the integers, where $a\tau_n b \Leftrightarrow a \equiv b \bmod n$. This topic is developed in more detail in the second author's dissertation [**26, 27**]. The relation τ_n is never divisive and hence exhibits some bad behavior. For example, $\mathbf{Z}$ is never a τ_n-UFD and need not be a τ_n-HFD or even τ_n-atomic. Also, the usual primes of $\mathbf{Z}$ (while being τ_n-primes) are rarely $|_{\tau_n}$-primes. We feel that τ_n-factorization gives a particularly good illustration of the topics discussed in this article and can be used to provide some interesting research projects for undergraduates.

The theory put forth in this article has several natural extensions. First, the definitions have natural generalizations (often in several ways) to commutative rings with zero divisors and to modules. For the theory of (ordinary) factorization in commutative rings, see [**14, 15**] and for factorization in modules over commutative rings, see [**15**]. Second, since much of factorization in an integral domain only involves the multiplicative monoid of the integral domain, we could have developed our theory in the context of commutative cancellative monoids. Third, we could consider τ-factorizations of proper ideals of a domain (or ring) D where τ is now a relation on the proper ideals of D.

For terms and notation not defined here the reader is referred to [**29**]. The conference proceedings [**1, 22**] and the book of survey articles [**24**] are particularly good sources for articles on factorization. Highly recommended is the recent book by Geroldinger and Halter-Koch [**28**]. It contains an extensive bibliography.

2. τ-factorizations.

Let D be an integral domain with quotient field K. Let $D^* = D - \{0\}$, $U(D)$ the group of units of D, and $D^\# = D^* - U(D)$, the nonzero nonunits of D. As usual, $a \sim b$ means that a and b are associates. Let τ be a relation on $D^\#$, that is, $\tau \subseteq D^\# \times D^\#$. We call τ *multiplicative* (respectively, *divisive*) if for $a, b, c \in D^\#$ (respectively, $a, a', b, b' \in D^\#$), $a\tau b$ and $a\tau c$ imply $a\tau bc$ and $b\tau a$ and $c\tau a$ imply $bc\tau a$ (respectively, $a\tau b$, $a'|a$ and $b'|b$ imply $a'\tau b'$). We say that τ is *associate-preserving* if for $a, b, b' \in D^\#$ with $b \sim b'$, $a\tau b$ implies $a\tau b'$ and $b\tau a$ implies $b'\tau a$.

For $a \in D^\#$, we define $a = \lambda a_1 \cdots a_n$, $\lambda \in U(D)$, $a_i \in D^\#$, to be a (an *ordered*) *τ-factorization* of a if $a_i \tau a_j$ for each $i \neq j$ ($i < j$). We say that a is (an *ordered*) *τ-product* of the a_i and that a_i is a (an *ordered*)

τ*-factor* of a. For $a, b \in D^{\#}$, we say that a τ*-(order-) divides* b, written $a|_\tau b$ ($a|_\tau^{\text{ord}} b$), if there exist $\lambda \in U(D)$, $c_1, \dots, c_n \in D^{\#}$, $n \geq 0$, so that $\lambda c_1 \cdots c_i a c_{i+1} \cdots c_n$ is a (an ordered) τ-factorization of b. We call $a = \lambda(\lambda^{-1}a)$ a *trivial* τ*-factorization* of a. Note that if $a = \lambda a_1 \cdots a_n$ is a τ-factorization, then so is each rearrangement $a = \lambda a_{\sigma(1)} \cdots a_{\sigma(n)}$, $\sigma \in S_n$. This need not be true for ordered τ-factorizations.

We pause to give several examples, some of which will be considered in more detail later.

Example 2.1. Throughout D will be an integral domain. (1) $\tau = D^{\#} \times D^{\#}$. This gives the usual notions of factorization and divides. Of course, τ is both multiplicative and divisive.

(2) $\tau = \varnothing$. Here $a \in D^{\#}$ has only the trivial τ-factorization and $a|_\tau b \Leftrightarrow a \sim b$. Vacuously, τ is both multiplicative and divisive.

(3) Let S be a non-empty subset of $D^{\#}$ and take $\tau = S \times S$, so $a \tau b \Leftrightarrow a, b \in S$. Here τ is multiplicative (divisive) if and only if S is multiplicatively closed (closed under nonunit factors). A non-trivial τ-factorization is up to unit factors just a factorization into elements from S. Thus if we take S to be the set of atoms of D we get the usual factorization of an element into irreducible factors. In this case, every element of $D^{\#}$ is an atom or has a non-trivial τ-factorization if and only if D is atomic. We could also take S to be the set of prime elements, prime power elements, primary elements, or other distinguished elements such as rigid elements or t-pure elements (see [**12**] for definitions). Or we could replace S by a subset S' where for each $s \in S$ there exists exactly one $s' \in S'$ with $s' \sim s$. For example, for $D = \mathbf{Z}$, take S to be the set of prime elements and take $S' = \{n \in \mathbf{N} \mid n$ is prime$\}$. Here τ is not associate-preserving. Examples of this type are one of the reasons we chose to include a unit factor in the definition of a τ-factorization. Sometimes it is of interest to replace $S \times S$ by $S \times S - \Delta = \{(s,t) \in S \times S \mid s \neq t\}$. For example, if $P = \{p_\alpha\}$ is a set of nonassociate primes, take $S = \{p_\alpha^k \mid p_\alpha \in P, k \geq 1\}$. Then for $\tau = S \times S - \Delta$, a non-trivial τ-factorization is just a product $\lambda p_{\alpha_1}^{k_1} \cdots p_{\alpha_n}^{k_n}$ where $p_{\alpha_1}, \dots, p_{\alpha_n}$ are distinct elements of P and each $k_i \geq 1$.

(4) Let I be an ideal of D and define $a \tau b \Leftrightarrow a - b \in I$. A special case that we will examine later in more detail in Section 5 is the case $D = \mathbf{Z}$ and $I = (n)$, so $a \tau_n b \Leftrightarrow a \equiv b \bmod n$. Here τ_n is multiplicative or associate-preserving only for $n = 2$ and is never divisive.

(5) Let $\star$ be a star-operation on D, and define $a\tau_\star b \Leftrightarrow (a,b)^\star = D$, that is, a and b are $\star$-coprime or $\star$-comaximal. It is easily checked that $\tau_\star$ is both multiplicative and divisive. This example will be studied in Section 4. In the case where $\star = d$ (the d-operation $A \longrightarrow A_d = A$), we have the comaximal factorization of McAdam and Swan [**36**]. Also of interest is the case where $\star$ is the t-operation.

(6) Related to factorizations into v-coprime elements, we have factorizations into relatively prime elements. Define for $a, b \in D^\#$ $a\tau_{[]}b \Leftrightarrow [a,b] = 1$, that is, a and b have no common nonunit factor. While divisive, $\tau_{[]}$ need not be multiplicative. We consider this relation in more detail in Section 4.

(7) In $D[X]$, define $f\partial g \Leftrightarrow \deg f = \deg g$. Clearly ∂ is neither multiplicative nor divisive, but is associate-preserving. We consider ∂ in more detail in Section 3.

(8) Suppose that $\leq$ is a transitive order on $D^\#$. Define $a\tau b \Leftrightarrow a \leq b$. Then an ordered τ-factorization $a = \lambda a_1 \cdots a_n$ is just a factorization where $\lambda \in U(D)$, each $a_i \in D^\#$, and $a_1 \leq \cdots \leq a_n$. For example, for $D[X]$ and $f \leq g \Leftrightarrow \deg f \leq \deg g$, an ordered τ-factorization is just a factorization into polynomials of ascending degree.

From now on we will consider only τ-factorizations. Thus we will assume that τ is symmetric. We invite the reader to formulate the definitions and results given for τ-factorizations in the context of ordered τ-factorizations. Given a factorization, we often want to further factor certain terms or want to combine terms. In general neither action preserves τ-factorizations (see Example 3.2). Our first proposition shows that if τ is divisive (respectively, multiplicative) then the refinement of a τ-factorization by τ-factoring a term (respectively, combining terms in a τ-factorization) again gives a τ-factorization. This good behavior was the main reason for introducing the notions of multiplicative and divisive relations.

Proposition 2.2. *Let D be an integral domain, and let τ be a relation on $D^\#$.* (1) *Suppose that τ is divisive. Let $a, b, b' \in D^\#$ where $b \sim b'$. Then $a\tau b \Leftrightarrow a\tau b'$. So τ is associate-preserving. Thus $a = \lambda a_1 \cdots a_n$ is a τ-factorization of a if and only if $a_1 \cdots (\lambda a_i) \cdots a_n$ is a τ-factorization of a. Hence, when τ is divisive, or more generally associate-preserving, we can dispense with the unit λ.*

(2) *Suppose that* τ *is divisive. Let* $a = a_1 \cdots a_n$ *be a* τ*-factorization of* a*, and let* $a_i = b_1 \cdots b_m$ *be a* τ*-factorization of* a_i*. Then* $a = a_1 \cdots a_{i-1} b_1 \cdots b_m a_{i+1} \cdots a_n$ *is a* τ*-factorization of* a*, called a* τ*-refinement of* a*. Thus when* τ *is divisive, a* τ*-refinement of a* τ*-factorization is a* τ*-factorization.*

(3) *Suppose that* τ *is multiplicative. Let* $a = \lambda a_1 \cdots a_n$ *be a* τ*-factorization of* a*. Then* $a = \lambda a_1 \cdots a_{i-1}(a_i a_{i+1}) a_{i+2} \cdots a_n$ *is a* τ*-factorization of* a*. More generally, if* $\{1, 2, \ldots, n\} = A_1 \dot{\cup} \cdots \dot{\cup} A_s$ *(disjoint union) with each* A_i *non-empty and* $b_i = \Pi\{a_j | j \in A_i\}$*, then* $a = \lambda b_1 \cdots b_s$ *is a* τ*-factorization of* a*.*

Proof. (1) Let $b \sim b'$, so $b'|b$ and $b|b'$. Then $a \tau b$ and $b'|b$ give $a \tau b'$ and $a \tau b'$ and $b|b'$give $a \tau b$. The second assertion follows.

(2) Let $a = a_1 \cdots a_n$ and $a_i = b_1 \cdots b_m$ be τ-factorizations. (By (1) we can dispense with the leading unit factor.) So $a_\ell \tau a_j$ and $b_\ell \tau b_j$ for $\ell \neq j$. Then in $a = a_1 \cdots a_{i-1} b_1 \cdots b_m a_{i+1} \cdots a_n, a_\ell \tau a_j$ and $b_\ell \tau b_j$ for $\ell \neq j$ and for $\ell \neq i$, $a_\ell \tau a_i$, so τ divisive gives $a_\ell \tau b_j$ since $b_j | a_i$.

(3) We prove the first statement, the second is similar. Let $\lambda a_1 \cdots a_n$ be a τ-factorization where τ is multiplicative. Then $a_j \tau a_i$ and $a_j \tau a_{i+1}$ for $j \neq i$, $i + 1$; since τ is multiplicative, then $a_j \tau a_i a_{i+1}$. Thus $\lambda a_1 \cdots a_{i-1}(a_i a_{i+1}) \cdots a_n$ is a τ-factorization. □

A comment concerning τ-refinements is in order. Let D be an integral domain, $a \in D^\#$, and $a = \lambda a_1 \cdots a_n$ a τ-factorization. Now a τ-refinement of a should be obtained by further τ-factoring one or more a_i. A τ-factorization of a_i has the form $a_i = \mu b_1 \cdots b_s$. But here $\lambda a_1 \cdots a_{i-1} \mu b_1 \cdots b_s a_{i+1} \cdots a_n$ is not a τ-factorization as $\mu \notin D^\#$. However, in the case of interest where τ is divisive, we have $(\lambda\mu) a_1 \cdots a_{i-1} b_1 \cdots b_s a_{i+1} \cdots a_n$ is a τ-factorization. Also, a τ-refinement of a τ-factorization cannot necessarily be obtained by τ-factoring individual τ-factors one by one. For example, in $\mathbf{Q}[X]$ define the relation τ by $X^4 \tau X^4$ and $X^2 \tau X^2$. Then $X^8 = X^4 \cdot X^4$ and $X^4 = X^2 \cdot X^2$ are τ-factorizations and $X^8 = X^2 \cdot X^2 \cdot X^2 \cdot X^2$ is the τ-refinement of $X^8 = X^4 \cdot X^4$ obtained by τ-factoring each X^4. However, since $X^4 \not\tau X^2$, the τ-refinement cannot be obtained by first τ-factoring the first X^4 and then τ-factoring the second X^4 as $X^8 = X^2 \cdot X^2 \cdot X^4$ is not a τ-factorization.

We next discuss the relation $|_\tau$ in more detail. Let D be an integral domain and τ a relation on $D^\#$. Let $a, a', b, b', c \in D^\#$. Certainly

$a|_\tau b \Rightarrow a|b$, but the converse is false. For in $\mathbf{R}[X]$, $X|X(X^2+1)$ but $X \nmid_\partial X(X^2+1)$ where ∂ is the relation given in Example 2.1 (7). We have (1) $a|_\tau a$ and (2) $a|_\tau b$ and $b|_\tau a \Leftrightarrow a \sim b$. If $b \sim b'$, then $a|_\tau b \Leftrightarrow a|_\tau b'$. If $a \sim a'$ and τ is associate-preserving, then $a|_\tau b \Leftrightarrow a'|_\tau b$. However, in general $a \sim a'$ and $a|_\tau b \not\Rightarrow a'|_\tau b$; see Example 2.3. If τ is divisive, then (3) $a|_\tau b$ and $b|_\tau c \Rightarrow a|_\tau c$. However, in general this is also false. For in $\mathbf{R}[X]$, $X|_\partial X^2$ and $X^2|_\partial X^2(X^2+1)$, but $X \nmid_\partial X^2(X^2+1)$. We can define the semigroup of τ-divisibility. Define $a \leq b \Leftrightarrow a|_\tau b$. So (1) $a \leq a$ and (2) $a \leq b$ and $b \leq a \Leftrightarrow a \sim b$. For τ divisive, $a \leq b$ and $b \leq c \Rightarrow a \leq c$ and for $a \sim a'$ and $b \sim b'$, $a \leq b \Leftrightarrow a' \leq b'$. Thus for τ divisive, the monoid $D^\star/U(D)$ is partially ordered by $aU(D) \leq bU(D) \Leftrightarrow a|_\tau b$. However, in general we need not have $aU(D) \leq bU(D) \Rightarrow aU(D)cU(D) \leq bU(D)cU(D)$; that is, $a|_\tau b$ need not imply $ac|_\tau bc$. For in $\mathbf{R}[X]$, $X^2|_\partial X^2(X^2+1)$, but $X^3 \nmid_\partial X^3(X^2+1)$. If τ is both multiplicative and divisive, then $a|_\tau b$ and $b\tau c \Rightarrow ac|_\tau bc$.

For $a \in D^\#$, D an integral domain and τ a relation on $D^\#$, a is *τ-irreducible* or a *τ-atom* if the only τ-factorizations of a are the trivial ones. Note that an associate of a τ-atom is again a τ-atom. Let τ' be another relation on $D^\#$. By a *τ-atomic τ'-factorization* for $a \in D^\#$, we mean a τ'-factorization of a into τ-atoms. We say that D is *τ-τ' atomic* if each $a \in D^\#$ has a τ-atomic τ'-factorization. When $\tau = \tau'$, we simply say a *τ-atomic factorization* or that D is *τ-atomic*. We say that a τ-factorization $a = \lambda a_1 \cdots a_n$ is *τ-unrefinable* or *τ-complete* if it has no proper τ-refinements. Of course, a τ-atomic factorization is τ-complete. By Proposition 2.2 (2) for τ divisive a τ-complete factorization is the same thing as a τ-atomic factorization. This is not true, in general, see Example 3.2. And D is *τ-complete* if every $a \in D^\#$ has a τ-complete factorization. Hence for τ divisive, D is τ-complete if and only if it is τ-atomic.

Suppose that D is τ-atomic. We have the τ-length functions

$$\ell_\tau(a) = \inf\{n \mid a = \lambda a_1 \cdots a_n \text{ is a } \tau\text{-atomic factorization}\}$$
$$L_\tau(a) = \sup\{n \mid a = \lambda a_1 \cdots a_n \text{ is a } \tau\text{-atomic factorization}\}$$

for $a \in D^\#$. We could define $\ell_\tau(a) = L_\tau(a) = 0$ for $a \in U(D)$. The *τ-elasticity* of a is $\rho_\tau(a) = L_\tau(a)/\ell_\tau(a)$ ($\rho_\tau(a) = 1$ for $a \in U(D)$) and the *τ-elasticity* of D is $\rho_\tau(D) = \sup\{\tau(a)|a \in D^\#\}$. More generally, using τ-atomic τ'-factorizations we could define τ-τ'-length functions and τ-τ'-elasticity. We invite the reader to extend results on elasticity from [**5, 19**] to this more general setting.

Let τ be a relation on $D^\#$, D an integral domain. Then $a \in D^\#$ is *τ-prime* if whenever $a|\lambda a_1 \cdots a_n$ where $\lambda a_1 \cdots a_n$ is a τ-factorization, then $a|a_i$ for some i. An ideal P of D ($P \neq D$) is *τ-prime* if whenever $\lambda a_1 \cdots a_n$ is a τ-factorization lying in P, then some $a_i \in P$. So (a) is a τ-prime ideal if and only if a is τ-prime. An associate of a τ-prime element is again τ-prime. We call $a \in D^\#$ a *$|_\tau$-prime* if whenever $a|_\tau \lambda a_1 \cdots a_n$ where $\lambda a_1 \cdots a_n$ is a τ-factorization, then $a|_\tau a_i$ for some i. If τ is associate-preserving, then an associate of a $|_\tau$-prime element is again $|_\tau$-prime. However, Example 2.3 below shows that in general an associate of a $|_\tau$-prime element need not be $|_\tau$-prime. Note that τ-primes and $|_\tau$-primes are a special case of what we might call a *τ_1-τ_2-τ_3-prime* (where each τ_i is a relation on $D^\#$): whenever $a|_{\tau_2} \lambda a_1 \cdots a_n$ where $\lambda a_1 \cdots a_n$ is a τ_1-factorization, then $a|_{\tau_3} a_i$ for some i. For example, a τ-prime is a τ-τ'-τ'-prime where $\tau' = D^\# \times D^\#$ (Example 2.1 (1)) and a $|_\tau$-prime is a τ-τ-τ-prime.

Example 2.3. An associate of a $|_\tau$-prime element need not be $|_\tau$-prime. Define the relation τ on $\mathbf{Z}^\#$ by $-4\tau 9$, $9\tau - 4$, and $6\tau 6$. Now $4|_\tau a \Rightarrow a = \pm 4$; so 4 is $|_\tau$-prime. But $-4|_\tau 36 = 6 \cdot 6$ since $36 = (-1)(-4) \cdot 9$, but $-4 \not|_\tau 6$. So -4 is not $|_\tau$-prime. Also, note that while $-4|_\tau 36$, $4 \not|_\tau 36$.

Let D be an integral domain, let τ be a relation on $D^\#$, and let $a \in D^\#$. Clearly if a is irreducible (respectively, prime), then a is τ-irreducible (respectively, τ-prime) and if a is τ-prime or $|_\tau$-prime, then a is τ-irreducible. If τ is multiplicative and divisive, then a τ-prime implies a is $|_\tau$-prime (see Proposition 2.4 below). But in general a prime or τ-prime element need not be $|_\tau$-prime, see Example 3.2. Note that in the definitions of τ-irreducible, τ-prime, and $|_\tau$-prime we did not restrict ourselves to the case of τ-factorizations $\lambda a_1 \cdots a_n$ of length $n = 2$ as is usual. We next show that if τ is multiplicative, we can restrict ourselves to the case $n = 2$. We then give an example showing that in general this is not the case.

Proposition 2.4. *Let D be an integral domain, and let τ be a relation on $D^\#$.* (1) *Suppose that τ is multiplicative, and let $a \in D^\#$. Then a is τ-irreducible* (*respectively, τ-prime, $|_\tau$-prime*) *if and only if*

a has no τ-factorization $a = \lambda a_1 a_2$ (respectively, for a τ-factorization $\lambda a_1 a_2$, $a|\lambda a_1 a_2 \Rightarrow a|a_1$ or $a|a_2$, $a|_\tau \lambda a_1 a_2 \Rightarrow a|_\tau a_1$ or $a|_\tau a_2$).

(2) If τ is both multiplicative and divisive, then a τ-prime element is $|_\tau$-prime.

Proof. (1) Clearly ($\Rightarrow$) holds in each case. Let $\lambda\, a_1 \cdots a_n$ be a τ-factorization. By Proposition 2.2, τ multiplicative gives that $\lambda\, a_1(a_2 \cdots a_n)$ is also a τ-factorization. Thus a nontrivial τ-factorization $a = \lambda a_1 \cdots a_n$ would yield a non-trivial τ-factorization $\lambda a_1(a_2 \cdots a_n)$ of length two. The proofs for the two other cases are similar.

(2) Suppose that τ is both multiplicative and divisive. Assume that $a \in D^{\#}$ is τ-prime. We show that a is $|_\tau$-prime. Suppose that $a|_\tau bc$ where $b\tau c$. So $ra = bc$ where $r\tau a$. Now $a|bc$, so since a is τ-prime, $a|b$, say. Then $sa = b$ for some $s \in D$, so $ra = bc = sac \Rightarrow r = sc$. Now $r\tau a$, $s|r$, and τ is divisive, so $s\tau a$ (the case $s \in U(D)$ is trivial). Thus $a|_\tau b$. $\square$

Example 2.5. Let $n \geq 2$, and let $p_1, \ldots, p_{n+1}$ be the first $n+1$ prime numbers. Define the relation τ on $\mathbf{Z}^{\#}$ by $a\tau b \Leftrightarrow |a| = p_i$ and $|b| = p_j$ for distinct p_i, p_j, $1 \leq i, j \leq n+1$. Note that τ is divisive, but not multiplicative. A nontrivial τ-factorization has the form $(\pm 1)(\pm p_{i_1}) \cdots (\pm p_{i_s})$ where $s \geq 2$ and $p_{i_1}, \ldots, p_{i_s}$ are distinct. Let $a = p_1 \cdots p_{n+1}$. Then a is not τ-irreducible, but a has no τ-factorization of length k where $2 \leq k \leq n$. Since a is not τ-irreducible, a is not τ-prime nor $|_\tau$-prime. Note that for any τ-factorization $b = \lambda b_1 \cdots b_k$, $2 \leq k \leq n$, $a \nmid b$ and $a \nmid_\tau b$. Thus, vacuously, if $\lambda a_1 \cdots a_k$ is a τ-factorization where $1 \leq k \leq n$ and $a|\lambda a_1 \cdots a_k$ (respectively, $a|_\tau \lambda a_1 \cdots a_k$), then $a|a_i$ (respectively, $a|_\tau a_i$) for some i. A more natural example involving the relation $\tau_{[]}$ is given at the end of Section 4.

Let D be a fixed integral domain. For relations τ_1, τ_2 on $D^{\#}$, define $\tau_1 \leq \tau_2 \Leftrightarrow \tau_1 \subseteq \tau_2$, that is, $a\tau_1 b \Rightarrow a\tau_2 b$. Let $\mathbf{R}$ be the set of relations on $D^{\#}$. So $\mathbf{R}$ is partially ordered by $\leq$. Note that $\varnothing$ (Example 2.1 (2)) is the least element of $\mathbf{R}$ and the usual factorization given by $\tau = D^{\#} \times D^{\#}$ (Example 2.1 (1)) is the greatest element. Suppose that τ_1, τ_2 are relations on $D^{\#}$ with $\tau_1 \leq \tau_2$. Then a τ_1-factorization of $a \in D^{\#}$ is also a τ_2-factorization of a. Thus if a is a τ_2-atom (respectively, τ_2-prime), then a is a τ_1-atom (respectively, τ_1-prime). Hence we have the previously mentioned fact that an atom (respectively, prime) of D is a τ-atom (respectively, τ-prime). However, a $|_{\tau_2}$-prime need not be a $|_{\tau_1}$-prime. For example, X^2+1 is a prime of $\mathbf{R}[X]$, that is, a $|_\tau$-prime for $\tau = \mathbf{R}[X]^\# \times \mathbf{R}[X]^\#$, but is not a $|_{\tau_\partial}$-prime by Example 3.2. Suppose that $\star_1$ and $\star_2$ are two star-operations on D with $\star_1 \leq \star_2$, that is, $A^{\star_1} \subseteq A^{\star_2}$ for all $A \in F(D)$. Then $a\tau_{\star_1}b \Leftrightarrow (a,b)^{\star_1} = D \Rightarrow (a,b)^{\star_2} = D \Leftrightarrow a\tau_{\star_2}b$. So $\tau_{\star_1} \leq \tau_{\star_2}$.

There is a natural extension of the notion of a UFD to τ-factorizations. Let D be an integral domain and τ a relation on $D^\#$. We say that D is a *τ-UFD* if (1) D is τ-atomic and (2) if $\lambda a_1 \cdots a_n = \mu b_1 \cdots b_m$ are two τ-atomic factorizations, then $n = m$ and after re-ordering, if necessary, $a_i \sim b_i$ for each i. We leave it to the reader to define a τ-τ'-UFD using τ-atomic τ'-factorizations.

Proposition 2.6. *Let D be an integral domain and τ a relation on $D^\#$. Suppose that $\lambda p_1 \cdots p_n = \mu q_1 \cdots q_m$ are two τ-factorizations where the p_i are $|_\tau$-prime* and the *q_i are τ-atoms. Then $n = m$ and after re-ordering, if necessary, $p_i \sim q_i$.*

Proof. Now p_1 is $|_\tau$-prime, so $p_1|_\tau q_i$ for some i. After re-ordering, if necessary, we can take $i = 1$. So $\varepsilon p_1 = q_1$ for some $\varepsilon \in U(D)$. Thus $\lambda p_1 p_2 \cdots p_n = \mu(\varepsilon p_1) q_2 \cdots q_n$ and hence $\lambda p_2 \cdots p_n = (\mu\varepsilon) q_2 \cdots q_m$ where the last two factorizations are again τ-factorizations. By induction $n-1 = m-1$; so $n = m$, and after re-ordering, if necessary, $p_i \sim q_i$. $\square$

Note that Proposition 2.6 is not true if we replace $|_\tau$-prime by τ-prime. For consider τ_2 on $\mathbf{Z}$. Then for odd primes p_1, p_2; 2, $2p_1$, and $2p_2$ are τ_2-primes and $2p_1p_2$ is a τ_2-atom. Now $2p_1 \cdot 2p_2 = 2 \cdot 2p_1p_2$ but $2p_1 \not\sim 2$ and $2p_1 \not\sim 2p_1p_2$. However, if τ is both multiplicative and divisive, then by Proposition 2.4 (2), a τ-prime is $|_\tau$-prime. So in this case we can replace $|_\tau$-prime by τ-prime in Proposition 2.6. Also, in Proposition 2.6 if τ is divisive and the q_i's are τ-primes, then we can replace $|_\tau$-prime by τ-prime. For if $p_1|q_i$ and $q_i|p_j$, then $p_1|p_j$. So by Lemma 2.10, $p_j \sim p_1$. Hence $p_1 \sim q_i$ and the proof proceeds as in the $|_\tau$-case.

Theorem 2.7. *Let D be an integral domain and τ a relation on $D^{\#}$.*

(1) *Suppose that every element of $D^{\#}$ has a τ-factorization into $|_\tau$-primes. Then D is a τ-UFD. Moreover, $a \in D^{\#}$ is τ-irreducible if and only if a is an associate of a $|_\tau$-prime.*

(2) *Suppose that τ is divisive and that D is a τ-UFD. Then a τ-irreducible element of D is $|_\tau$-prime (and of course the converse always holds).*

(3) *For τ divisive, the following are equivalent:*

(a) *D is a τ-UFD,*

(b) *every element of $D^{\#}$ has a τ-factorization into $|_\tau$-primes, and*

(c) *D is τ-atomic and every τ-irreducible element of D is $|_\tau$-prime.*

Proof. (1) Suppose that every element of $D^{\#}$ has a τ-factorization into $|_\tau$-primes. Since a $|_\tau$-prime is τ-irreducible, D is τ-atomic. By Proposition 2.6, D is a τ-UFD. Suppose that $a \in D$ is τ-irreducible. Then a being a τ-product of $|_\tau$-primes gives that a is an associate of a $|_\tau$-prime.

(2) Suppose that τ is divisive. Let D be a τ-UFD, and let $a \in D$ be τ-irreducible. We show that a is $|_\tau$-prime. Suppose that $a|_\tau a_1 \cdots a_n$, say $c_1 \cdots c_m a = a_1 \cdots a_n$ where both are τ-factorizations. Now τ-factor each c_i, a_i into τ-atoms: $c_i = \Pi c_{ij}$, $a_i = \Pi a_{ij}$. So $\Pi c_{1j} \cdots \Pi c_{mj} a = \Pi a_{1j} \cdots \Pi a_{nj}$. Moreover, since τ is divisive, by Proposition 2.2 (which also says that we can dispense with the unit factors) these are both τ-factorizations. Thus since D is a τ-UFD, $a \sim a_{ij}$ for some i, j. Hence $a|_\tau a_i$. So a is $|_\tau$-prime.

(3) This follows from (1) and (2). $\square$

In Theorem 2.7 (3) if we replace $|_\tau$-prime by τ-prime we have (a) $\Leftarrow$ (b) $\Leftrightarrow$ (c). We do not know if (a) $\Rightarrow$ (b).

Example 2.8. (1) A domain D in which every element of $D^{\#}$ is a τ-product of associates of $|_\tau$-primes, but D is not a τ-UFD. Let τ be defined on $\mathbf{Z}^{\#}$ as in Example 2.3. As is the case for $a = 4$, each $a > 1$, $a \neq 6, 9, 36$, is $|_\tau$-prime. While 6 and 9 are not $|_\tau$-prime, -6 and -9 are $|_\tau$-primes and 36 is a τ-product of associates of $|_\tau$-primes. However, $\mathbf{Z}$ is not a τ-UFD since $36 = 6 \cdot 6 = (-1)(-4)(9)$.

(2) A τ-UFD D in which an element of $D^{\#}$ is not a τ-product of τ-primes (respectively, $|_\tau$-primes). Define τ on $\mathbf{Z}^{\#}$ by $\pm 2\tau \pm 3$, $\pm 5\tau \pm 7$, $\pm 6\tau \pm 35$ and $\pm 10\tau \pm 21$. The only nontrivial τ-products are $(\pm 1)(\pm 2)(\pm 3)$, $(\pm 1)(\pm 5)(\pm 7)$, $(\pm 1)(\pm 6)(\pm 35)$ and $(\pm 1)(\pm 10)(\pm 21)$. So the only $a \in \mathbf{Z}^{\#}$ that are not τ-atoms are $\pm 6, \pm 35$ and ± 210. Clearly $\mathbf{Z}$ is τ-atomic. In fact, $\mathbf{Z}$ is a τ-UFD. For up to associates, the only case where we have two different non-trivial τ-factorizations is $210 = 10 \cdot 21 = 6 \cdot 35$. While the first factorization is τ-atomic, the second one is τ-complete, but not τ-atomic. Note that the τ-atoms ± 10 and ± 21 are neither τ-primes nor $|_\tau$-primes and thus cannot be τ-products of τ-primes (respectively, $|_\tau$-primes). While $\mathbf{Z}$ has unique τ-factorization into τ-atoms, it does not have unique τ-factorization into τ-complete factorizations.

At the end of Section 5, it is shown that for τ_0 on $\mathbf{Z}^{\#}$ defined by $a\tau_0 b \Leftrightarrow a = b$, $\mathbf{Z}$ is a τ_0-UFD in which every element of $\mathbf{Z}^{\#}$ is a τ_0-product of $|_{\tau_0}$-primes, but not every element of $\mathbf{Z}^{\#}$ is a τ_0-product of τ_0-primes.

In [**6**] a number of factorization properties of an integral domain D weaker than unique factorization were studied. Following Cohn [**25**] we say that D is *atomic* if each element of $D^{\#}$ is a product of a finite number of irreducible elements (atoms) of D. We say that D *satisfies the ascending chain condition on principal ideals* (*ACCP*) if there does not exist an infinite strictly ascending chain of principal ideals of D. The integral domain D is a *bounded factorization domain* (*BFD*) if D is atomic and for each nonzero element of D there is bound on the length of factorizations into irreducible elements, or equivalently, each nonzero nonunit has a bound on the length of factorizations into nonunits. A domain D is a *half-factorial domain* (*HFD*) if D is atomic and each factorization of a nonzero nonunit of D into a product of irreducible elements has the same length. This concept was introduced by Zaks [**41, 42**]. The domain D is an *idf-domain* (for irreducible-divisor-finite) if each nonzero element of D has at most a finite number of nonassociate irreducible divisors. They were introduced by Grams and Warner [**32**]. Atomic idf-domains are precisely the domains in which each nonzero nonunit has only a finite number of nonassociate divisors and hence, only a finite number of factorizations up to order and associates. Such a domain is called a *finite factorization domain* (FFD). They are considered in more detail in [**13**]. In general

$$\begin{tikzcd}
& \text{HFD} \arrow[dr] & & & \\
\text{UFD} \arrow[ur] \arrow[r] \arrow[dr] & \text{FFD} \arrow[r] \arrow[d] & \text{BFD} \arrow[r] & \text{ACCP} \arrow[r] & \text{atomic.} \\
& \text{idf-domain} & & &
\end{tikzcd}$$

FIGURE 1.

Examples were given in [**6**] to show that no other implications were possible. We can also define the following τ-factorization properties weaker than τ-unique factorization. Let D be an integral domain and τ a relation on $D^{\#}$. We have already defined τ-atomic. We say that D satisfies *τ-ACCP* if for each infinite sequence $\{a_n\}_{n=1}^{\infty}$ of elements of $D^{\#}$ with $a_{n+1}|_\tau\, a_n$ for each $n \geq 1$, there is an N (depending on the sequence) with $a_{k+1} \sim a_k$ for each $k \geq N$. The domain D is a *τ-half-factoral domain* (*τ-HFD*) if D is τ-atomic and whenever $\lambda a_1 \cdots a_n = \mu b_1 \cdots b_m$ are two τ-atomic factorizations, then $n = m$. Of course, D is a τ-HFD if and only if D is τ-atomic and $\rho_\tau(D) = 1$.

We say that D is a *τ-bounded factorization domain* (*τ-BFD*) if D is τ-atomic and for each $a \in D^{\#}$, there is a natural number $N(a)$ so that if $a = \lambda a_1 \cdots a_n$ is a τ-atomic factorization of a, then $n \leq N(a)$, that is, $L_\tau(a)$ is finite. Note that for τ divisive, D is a τ-BFD if and only if for each $a \in D^{\#}$, there is a natural number $N(a)$ so that for any τ-factorization $a = \lambda a_1 \cdots a_n$, $n \leq N(a)$. This follows from Proposition 2.2 (2) which gives that for τ divisive a maximal length τ-factorization is a τ-atomic factorization. Thus for τ divisive, a BFD is a τ-BFD.

We say that D is a *τ-idf-domain* if each $a \in D^{\#}$ has at most finitely many nonassociate τ-factors that are τ-atoms. D is a *τ-finite-factorization domain* (*τ-FFD*) if D is τ-atomic and each $a \in D^{\#}$ has only finitely many τ-factorizations (up to order and associates) into τ-irreducibles. Clearly a τ-FFD is a τ-BFD. Suppose the τ is divisive. Then a modification of the proof of [**6**, Theorem 5] gives that the following are equivalent: (1) D is a τ-FFD, (2) D is a τ-atomic τ-idf-domain, (3) each $a \in D^{\#}$ has only finitely many τ-factorizations up to order and associates.

Theorem 2.9. *Let D be an integral domain and τ a divisive relation on $D^{\#}$. Then D a τ-BFD implies D satisfies τ-ACCP and if D satisfies τ-ACCP, then D is τ-atomic.*

Proof. (1) Suppose that D is a τ-BFD. Suppose that D doesn't satisfy τ-ACCP. So there is an infinite sequence $\{a_n\}_{n=1}^{\infty}$ of elements of $D^{\#}$ with $a_{n+1}|_{\tau}a_n$ but $a_{n+1} \nsim a_n$ for each $n \geq 1$. Let $a_n = r_{n+11} \cdots r_{n+1s_{n+1}}a_{n+1}$ be a τ-factorization of a_n. So $a_1 = r_{21} \cdots r_{2s_2}a_2 = r_{21} \cdots r_{2s_2}r_{31} \cdots r_{3s_3}a_3 = \cdots$, where each factorization is a τ-factorization since τ is divisive. Moreover, again since τ is divisive, each of these τ-factorizations can be τ-refined to a τ-atomic factorization. So $L_{\tau}(a_1) \geq s_2 + s_3 + \cdots + s_n + 1 \geq n$ for each natural number n. But this contradicts that D is a τ-BFD.

(2) Suppose that D satisfies τ-ACCP, but that D is not τ-atomic. So some $a \in D^{\#}$ does not have a τ-factorization into τ-atoms. Thus $a = a_1 \cdots a_n$, a τ-factorization where $n > 1$. Now some a_i, say a_1, must not be a τ-product of τ-atoms (here we use divisive). Here $a_1|_{\tau}a$ and $a_1 \nsim a$. Put $b_1 = a_1$. Again, a_1 cannot be a τ-atom; so a_1 has a τ-factorization $a_1 = a_{21} \cdots a_{2n_2}$ where $n_2 \geq 2$. One of the τ-factors, say a_{21}, cannot be a τ-product of τ-atoms. Here $a_{21}|_{\tau}a_1 = b_1$ and $a_{21} \nsim a_1$. Put $b_2 = a_{21}$. Continuing we get a sequence $\{b_n\}_{n=1}^{\infty}$ of elements of $D^{\#}$ with $b_{n+1}|_{\tau}b_n$ but $b_{n+1} \nsim b_n$ for each $n \geq 1$. But this is a contradiction. $\square$

Let D be an integral domain and τ a relation on $D^{\#}$. Note that if D satisfies ACCP, then D satisfies τ-ACCP. Thus for τ divisive, ACCP $\Rightarrow$ τ-atomic. So for τ divisive, a UFD, FFD, HFD and BFD are τ-atomic. Thus a FFD (respectively, BFD) is a τ-FFD (respectively, τ-BFD) for τ divisive.

We next show that for τ divisive, a UFD is a τ-UFD. We need the following lemma which states that for τ divisive, a τ-atomic factorization is a mix of atomic factorizations and coprime factorizations.

Lemma 2.10. *Let D be an integral domain, and let τ be a divisive relation on $D^{\#}$. Let $a_1 \cdots a_n$ be a τ-atomic factorization. Then for $i \neq j$, either $[a_i, a_j] = 1$ or $a_i \sim a_j$ are atoms.*

Proof. Suppose that $[a_i, a_j] \neq 1$; so there is an $a \in D^{\#}$ with $a|a_i$ and $a|a_j$. Write $a_i = a'_i a$ and $a_j = a'_j a$. Now $a'_i a \tau a'_j a$; so since τ is divisive, either a'_i is a unit or $a'_i \tau a$. But if $a'_i \tau a$, then $a_i = a'_i a$ is a nontrivial τ-factorization of the τ-atom a_i; a contradiction. Hence a'_i

is a unit. Likewise a'_j is a unit. So $a_i \sim a \sim a_j$. Suppose that a is not an atom. So $a = bc$ where $b, c \in D^{\#}$. But then $b|a_i$ and $b|a_j$; so as before $a_i \sim b \sim a_j$. Thus $a \sim b$; a contradiction. Hence $a_i \sim a_j$ are irreducible. □

Theorem 2.11. *Let D be a UFD and τ a divisive relation on $D^{\#}$. Then D is a τ-UFD.*

Proof. We have already remarked that for τ divisive a UFD is τ-atomic.

Suppose that $a \in D^{\#}$ has a τ-atomic factorization $a = a_1 \cdots a_n$ where $[a_i, a_j] \neq 1$ for some $i \neq j$. We show that up to order and associates this is the unique τ-atomic factorization for a. By Lemma 2.10 $a_i \sim p \sim a_j$ for some prime p. Suppose that p^t is the largest power of p dividing a. Re-ordering, we can assume that $a_1 \sim a_2 \sim \cdots \sim a_t \sim p$. Suppose that $b_1 \cdots b_m$ is another τ-atomic factorization of a. If p divides two different b_i's, then we can re-order so that $b_1 \sim b_2 \sim \cdots \sim b_t \sim p$. Thus after a "unit adjustment" $a_{t+1} \cdots a_n = \lambda b_{t+1} \cdots b_m$ are two τ-atomic factorizations. By induction $n-t = m-t$; so $n = m$, and after re-ordering $a_k \sim b_k$ for $t+1 \leq k \leq n$. Thus we are reduced to the case where say $p^t|b_1$. Let $b_1 = p^t b'_1$ and let $b'_1 = b'_{11} \cdots b'_{1s}$ be a τ-atomic factorization of b'_1. So canceling p^t we get $a_{t+1} \cdots a_n = \lambda b'_{11} \cdots b'_{1s} b_2 \cdots b_m$ where both factorizations are τ-atomic factorizations. We can assume that $m > 1$. By induction $b_m \sim a_k$ for some k with $i+1 \leq k \leq n$. Re-ordering, $a_n \sim b_m$. So $a_1 \cdots a_{n-1} = \mu b_1 \cdots b_{m-1}$ for some unit μ. Again by induction $n-1 = m-1$, so $n = m$ and after re-ordering $a_\ell \sim b_\ell$ for $1 \leq \ell \leq m-1$.

Thus we are reduced to the case where every τ-atomic factorization of a, $a = a_1 \cdots a_n$, has $[a_i, a_j] = 1$ for $i \neq j$. Let p_1 be a prime with $p_1|a$; say $p_1|a_1$. We can assume that $p_1|b_1$. So if $p_1^{t_1}$ is the largest power of p_1 dividing a, then $p_1^{t_1}|a_1$ and $p_1^{t_1}|b_1$. Suppose that $p_1^{t_1} \cdots p_s^{t_s}$ $(t_i > 0)$ is the product of prime powers dividing a_1 and b_1. So $a_1 = p_1^{t_1} \cdots p_s^{t_s} a'_1$ and $b_1 = p_1^{t_1} \cdots p_s^{t_s} b'_1$ where $[a'_1, b'_1] = 1$. Let $a'_1 = a'_{11} \cdots a'_{1\ell}, b'_1 = b'_{11} \cdots b'_{1k}$ be τ-atomic factorizations of a'_1 and b'_1. Canceling $p_1^{t_1} \cdots p_s^{t_s}$ gives $a'_{11} \cdots a'_{1\ell} a_2 \cdots a_n = b'_{11} \cdots b'_{1k} b_2 \cdots b_m$ are two τ-atomic factorizations. By induction on the length of an atomic factorization, we can assume that these two τ-atomic factorizations are unique up to order and associates. If some $a_i \sim b_j$, then we can cancel $a_i \sim b_j$ from $a_1 \cdots a_n = b_1 \cdots b_m$ and use induction to get uniqueness.

Thus we can assume that for each $i \geq 2$, $a_i \sim b'_{1j}$ for some j and $b_i \sim a'_{ij'}$ for some j'. So $a_2 \cdots a_n$ is a factor of b_1 and $b_2 \cdots b_m$ is a factor of a_1. Write $a_1 = p_1^{t_1} \cdots p_s^{t_s} b_2 \cdots b_m A$ and $b_1 = p_1^{t_1} \cdots p_s^{t_s} a_2 \cdots a_n B$. Now $p_1^{t_1} \cdots p_s^{t_s} b_1 b_2 \cdots b_m A = a_1 b_1 = p_1^{t_1} \cdots p_s^{t_s} a_1 \cdots a_n B$. Canceling gives $A = B$. But $[a'_1, b'_1] = 1$ gives $[A, B] = 1$; so A is a unit. So $a_1 = A(p_1^{t_1} \cdots p_s^{t_s}) b_2 \cdots b_m$ is a τ-factorization of the τ-atom a_1; a contradiction. □

So we have the following diagram where $*$ indicates τ is divisive. Example 3.2 shows that $\mathbf{R}[X]$ with the ∂ relation (which is not divisive) is a UFD, FFD, BFD and is atomic, but is not a ∂-UFD, ∂-FFD, ∂-BFD, nor ∂-atomic. Also, $\mathbf{R}[X]$ satisfies ∂-ACCP but is not ∂-atomic. The relation τ_2 (Example 2.1 (4)) is both associate-preserving and multiplicative, but $\mathbf{Z}$, while being a UFD, is not a τ_2-UFD (see Section 5). We do not know whether atomic $\Rightarrow$ τ-atomic for τ divisive.

We have based our factorization theory on τ-atomic factorizations. We could have instead used τ-complete factorizations. We leave it to the reader to define τ-complete length functions and τ-complete elasticity. Let D be an integral domain. Define D to be a *τ-complete HFD* (respectively, *τ-complete UFD*) if (1) D is τ-complete and (2) if $a = \lambda a_1 \cdots a_n = \mu b_1 \cdots b_m$ are two τ-complete factorizations of $a \in D^{\#}$, then $n = m$ (respectively, and after re-ordering, if necessary, $a_i \sim b_i$ for $i = 1, \ldots, n$). We say that D is a *τ-complete FFD* (respectively, *τ-complete BFD*) if for each $a \in D^{\#}$ there are only

$$
\begin{tikzcd}
 & \tau\text{-IIFD} \arrow[dr, Rightarrow] & & & \\
\tau\text{-UFD} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \tau\text{-BFD} \arrow[r, Rightarrow, "*"'] & \tau\text{-ACCP} \arrow[r, Rightarrow, "*"'] & \tau\text{-atomic} \\
 & \tau\text{-FFD} \arrow[ur, Rightarrow] & & & \\
\text{UFD} \arrow[uu, Rightarrow, "*"'] \arrow[r, Rightarrow] & \text{FFD} \arrow[u, Rightarrow, "*"'] \arrow[r, Rightarrow] & \text{BFD} \arrow[uu, Rightarrow, "*"'] \arrow[r, Rightarrow] & \text{ACCP} \arrow[uu, Rightarrow] \arrow[r, Rightarrow] & \text{atomic}
\end{tikzcd}
$$

FIGURE 2.

$$\begin{tikzcd} & \tau\text{-complete HFD} \arrow[dr, Rightarrow] & & & \\ \tau\text{-complete UFD} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \tau\text{-complete BFD} \arrow[r, Rightarrow] & \tau\text{-ACCP} \arrow[r, Rightarrow] & \tau\text{-complete} \\ & \tau\text{-complete FFD} \arrow[ur, Rightarrow] & & & \end{tikzcd}$$

FIGURE 3.

finitely many τ-factorizations for a up to units, order, and associates (respectively, there is a natural number $N(a)$ so that for each τ-complete factorization $a = \lambda a_1 \cdots a_n$, $n \leq N(a)$). We have the following diagram which unlike the case for τ-atomic holds for any relation τ. For τ divisive, this reduces to Figure 2.

Example 2.8 (2) shows that even in a τ-UFD, a τ-factorization cannot necessarily be τ-refined to a τ-atomic factorization. Let us say that an integral domain D is *τ-atomizable* (respectively, *τ-completeable*) if each τ-factorization of D can be τ-refined to a τ-atomic (respectively, τ-complete) factorization. We have the following implications:

$$\begin{array}{ccccc} & & \tau\text{-atomizable} \Rightarrow & & \tau\text{-atomic} \\ & & \Downarrow & & \Downarrow \\ \tau\text{-ACCP} \Rightarrow & & \tau\text{-completeable} \Rightarrow & & \tau\text{-complete.} \end{array}$$

Note that for τ divisive, τ-complete $\Rightarrow$ τ-atomizable, but τ-complete need not imply τ-ACCP since an atomic domain need not satisfy ACCP. We have already remarked that Example 2.8 (2) shows that τ-atomic $\not\Rightarrow$ τ-atomizable. Example 3.2 shows that τ-completeable $\not\Rightarrow$ τ-atomizable and that τ-complete $\not\Rightarrow$ τ-atomic. We next show that τ-complete $\not\Rightarrow$ τ-completeable. Let S be the additive abelian monoid $\langle \{1/2^n\}_{n=0}^{\infty} \rangle$, $\mathbf{Q}[X;S]$ the monoid ring and $D = \mathbf{Q}[X;S]_N$ where $N = \{f \in \mathbf{Q}[X;S] | f \text{ has } 0 \text{ constant term}\}$. So every element of $D^{\#}$ has the form λX^a where $\lambda \in U(D)$ and $a \in S - \{0\}$. Define τ on $D^{\#}$ by $2X^{a/2}\tau(1/2)X^{a/2}$ for each $a \in S - \{0\}$ and $X^{1/2^n}\tau X^{1/2^n}$

for each $n \geq 1$. Now $\lambda X^a = \lambda \cdot 2X^{a/2} \cdot (1/2)X^{a/2}$ is a τ-complete factorization of λX^a, so D is τ-complete. However, $X = X^{1/2} \cdot X^{1/2}$ is a τ-factorization of X that can not be τ-refined to a τ-complete factorization of X. So D is not τ-completeable.

3. The ∂ relation on $D[X]$. Let D be an integral domain with quotient field K. Define the relation ∂ on $(D[X])^{\#}$ by $f\partial g \Leftrightarrow \deg f = \deg g$. Thus for $f \in (D[X])^{\#}$, a ∂-factorization is a factorization $f = \lambda f_1 \cdots f_n$ where λ is a unit and $\deg f_i = \deg f_j$, $1 \leq i, j \leq n$. So for $a \in D^{\#}$, a ∂-factorization of a is just a usual factorization of a in D. Note that ∂ is neither multiplicative ($X\partial X$ and $X\partial X$, but $X \not\partial X^2$) nor divisive ($X^2\partial X^2$ and $X|X^2$, but $X \not\partial X^2$). However, ∂ is associate-preserving.

Theorem 3.1. *Let D be an integral domain with quotient field K. Then the following are equivalent.* (1) *$D[X]$ is a ∂-UFD.*

(2) *$D[X]$ is ∂-atomic.*

(3) *$D = K$ is algebraically closed.*

Proof. (1) $\Rightarrow$ (2). A ∂-UFD is ∂-atomic by definition. (2) $\Rightarrow$ (3). Suppose $D[X]$ is ∂-atomic. Then D is atomic. Suppose that D is not a field, so there is an atom $a \in D$. Then the polynomial $X^2 + a$ is irreducible in $D[X]$. Consider $g = (X^2+a)X^2$. Then $g = (X^2+a) \cdot X^2$ is a ∂-factorization, so g is not ∂-irreducible. But since $X^2 + a$ is ∂-irreducible while X^2 is not, g does not have a ∂-atomic factorization. This contradiction gives that $D = K$ is a field. Suppose that $f \in K[X]$ is irreducible where $\deg f = n > 1$. Then as above $g = fX^n$ does not have a ∂-atomic factorization. Thus $\deg f = 1$; and hence K is algebraically closed. (3) $\Rightarrow$ (1). Since D is algebraically closed, every nonconstant polynomial of $D[X]$ is a product of linear polynomials. Thus a ∂-atom of $D[X]$ is just a linear polynomial and hence a ∂-atomic factorization in $D[X]$ is the same thing as a factorization into atoms. But $D[X]$ is a UFD and hence is a ∂-UFD. □

For $n > 1$, the element $g = fX^n$ in the previous proof is a ∂-factorization that cannot be ∂-refined to a ∂-atomic factorization. Let D be an integral domain, τ a relation on $D^{\#}$, and $a \in D^{\#}$. Recall that a τ-factorization of a, $a = \lambda a_1 \cdots a_n$, is *$\tau$-unrefinable* if this τ-factorization has no proper τ-refinements.

Example 3.2. In $\mathbf{R}[X]$, $(X^2+1)X^2 = (X^2+1)\cdot X^2$ is a ∂-factorization that is ∂-unrefinable, but is not a ∂-factorization into ∂-atoms. Note that while $X \cdot X$ is a ∂-factorization of X^2, $(X^2+1)\cdot X \cdot X$ is not a ∂-factorization of $(X^2+1)X^2$. So a ∂-refinement of a ∂-factorization need not be a ∂-factorization. Likewise $X^3 = X \cdot X \cdot X$ is a ∂-factorization, while $X^3 = (X \cdot X) \cdot X$ is not. Here $\mathbf{R}[X]$ satisfies ACCP and hence ∂-ACCP and is atomic; but $\mathbf{R}[X]$ is not ∂-atomic. Also, $\mathbf{R}[X]$ is a UFD, HFD, FFD, and BFD, but is not a ∂-UFD, ∂-HFD, ∂-FFD, nor ∂-BFD. Note that while $\mathbf{R}[X]$ has only finitely many ∂-factorizations (up to order and associates) and has a bound on the length of ∂-factorizations, $\mathbf{R}[X]$ is neither a ∂-FFD nor ∂-BFD. The two ∂-factorizations $(X(X^2+1)) \cdot X^3 = (X^2+1) \cdot X^2 \cdot X^2$ show that even in a PID, τ-unrefinable factorizations can have different lengths and that a prime (and hence τ-prime) element need not be $|_\tau$-prime. For $X^2+1 |_\partial (X(X^2+1)) \cdot X^3$ but $X^2+1 \nmid_\partial (X(X^2+1))$ and $X^2+1 \nmid_\partial X^3$.

Let D be an integral domain. Cohn [**25**] defined $c \in D^\#$ to be *primal* if $c|a_1a_2 \Rightarrow c = c_1c_2$ where $c_i|a_i$ and then defined D to be a *Schreier domain* if D is integrally closed and each element of $D^\#$ is primal. We will use the following characterization of Schreier domains given by McAdam and Rush [**35**]: an integral domain D with quotient field K is a Schreier domain if and only if for $0 \neq f \in D[X]$ with $f = \alpha\beta$ where $\alpha, \beta \in K[X]$ with $\deg \alpha$, $\deg \beta \geq 1$, then $f = gh$ where $g, h \in D[X]$ with $\deg g$, $\deg h \geq 1$. Moreover, in this case we can take $\deg g = \deg \alpha$ and $\deg h = \deg \beta$. Recall that a polynomial $f \in D[X]$ with $\deg f \geq 1$ is said to be *indecomposable* if f cannot be factored as $f = gh$ where $\deg g$, $\deg h \geq 1$. The equivalence (1) $\Leftrightarrow$ (2) of the next theorem is also given in [**18**].

Theorem 3.3. *Let D be an integral domain with quotient field K. Then the following conditions are equivalent.* (1) *D is Schreier and K is algebraically closed.*

(2) *Every indecomposable polynomial $f \in D[X]$ has degree one, or equivalently, every polynomial $f \in D[X]$ with $\deg f \geq 1$ is a product of linear polynomials.*

(3) *Every $f \in D[X]$ with $\deg f \geq 1$ has a ∂-atomic factorization.*

Proof. (1) $\Rightarrow$ (2). Let $f \in D[X]$ with $\deg f > 1$. Since K is algebraically closed, in $K[X]$ we can write $f = gh$ where $\deg g$,

$\deg h \geq 1$. But D is Schreier, so $f = g_1 h_1$ with $g_1, h_1 \in D[X]$ where $\deg g_1 = \deg g \geq 1$ and $\deg h_1 = \deg h \geq 1$. So f is not indecomposable. (2) $\Rightarrow$ (1). Every irreducible polynomial of $K[X]$ has degree one; thus K is algebraically closed. Let $f \in D[X]$ with $n = \deg f > 1$. Suppose that $f = gh$ in $K[X]$ where $\deg g$, $\deg h \geq 1$. Now in $D[X]$, $f = f_1 \cdots f_n$ where each f_i is linear. Let $g' = f_1 \cdots f_s$ where $s = \deg g$ and $h' = f_{s+1} \cdots f_n$ where $\deg h = n - s$. So $f = g'h'$ in $D[X]$ where $\deg g' = \deg g$ and $\deg h' = \deg h$. Thus D is Schreier. (2) $\Rightarrow$ (3). Clear. (3) $\Rightarrow$ (2). Suppose that $f \in D[X]$ is indecomposable with $\deg f = n > 1$. Consider $g = fX^n$. Let $g = fX^n = f_1 \cdots f_s$ be a ∂-atomic factorization for g; so $s > 1$ since g is not ∂-irreducible. Put $f_i = f_i' X^{t_i}$, $t_i \geq 0$, where $X \nmid f_i'$. So $fX^n = f_1' \cdots f_s' X^{t_1 + \cdots + t_s}$. Hence $t_1 + \cdots + t_s = n$ and $f = f_1' \cdots f_s'$. Since f is indecomposable, each f_i' except one, say f_1', is in D. But then $\deg f_1' = n$, so $\deg f_1 = n$ and hence $s = 2$. Then $f_2 = f_2' X^n$ is ∂-irreducible, a contradiction. Thus $n = 1$. $\square$

4. $\star$-comaximal factorizations. Let D be an integral domain with quotient field K, and let $\star$ be a finite character star-operation on D. Then $\star$ induces a relation $\tau_\star$ on $D^\#$ by $a\tau_\star b \Leftrightarrow (a,b)^\star = D$; that is, a and b are *$\star$-comaximal*. (Since we are applying $\star$ to finitely generated ideals, there is no loss of generality in assuming that $\star$ has finite character.) If $a\tau_\star b$ and $a'|a$ and $b'|b$, then $D = (a,b)^\star \subseteq (a',b')^\star \subseteq D^\star$; so $(a',b')^\star = D$. Hence $\tau_\star$ is divisive. Also, if $a\tau_\star b$ and $a\tau_\star c$, then $(a,b)^\star = D = (a,c)^\star$. So $D = ((a,b)^\star(a,c)^\star)^\star = ((a,b)(a,c))^\star \subseteq (a,bc)^\star \subseteq D^\star$, and hence $(a,bc)^\star = D$. So $\tau_\star$ is also multiplicative.

Consider the case of $\star = d$ (the d-operation is the star-operation $A \to A_d = A$). Here $a\tau_d b \Leftrightarrow (a,b) = D$, that is, a and b are comaximal. So this gives the comaximal factorizations of McAdam and Swan [**36**]. Let us recall their definitions and some of their results.

An element $b \in D^\#$ is *pseudo-irreducible* (respectively, *pseudo-prime*) if $b = cd$ (respectively, $b|cd$) with c, d comaximal implies c or d is a unit (respectively, $b|c$ or $b|d$). Note that b is pseudo-prime if and only if $D/(b)$ is indecomposable [**36**, Lemma 3.1]. For $b \in D^\#$, $b = b_1 \cdots b_m$ is a (*complete*) *comaximal factorization* of b if the b_i are pairwise comaximal nonunit (pseudo-irreducible) elements. Evidently a comaximal factorization $b = b_1 \cdots b_m$ is complete if and only if it has no proper refinements that are also comaximal factorizations of b. Then D is a *comaximal factorization domain* (*CFD*) if every nonzero nonunit of D has a complete comaximal factorization and D is a *unique comaximal factorization domain* (*UCFD*) if D is a CFD in which complete comaximal factorizations are unique up to order and associates.

They showed [**36**, Lemma 1.1] that an integral domain D is a CFD if either (i) each nonzero nonunit of D has only finitely many minimal primes or (ii) each nonzero nonunit of D is contained in only finitely many maximal ideals. Here (i) insures that a Noetherian domain is a CFD. We will show that if D satisfies ACCP, then D is a CFD.

A key concept in their work is the notion of an S-ideal. A nonzero ideal I of D is an *S-ideal* if $I = (a, c) = (a^2, c)$ for some $a, c \in I$. The relation to comaximal factorizations is the observation that $(a, c) = (a^2, c)$ if and only if there is an element $b \in D$ with $(a, b) = D$ and $c|ab$ [**36**, Lemma 1.2]. They proved that S-ideals are invertible and that any two-generated invertible ideal is isomorphic to an S-ideal [**36**, Lemma 1.5]. They gave the following characterization of UCFD's (we have added (3)).

Theorem 4.1 [**36**, Theorem 1.7]. *For an integral domain D the following conditions are equivalent.* (1) *D is a* UCFD.

(2) *D is a* CFD *and every pseudo-irreducible element of D is pseudo-prime.*

(3) *Every nonzero nonunit of D has a comaximal factorization into pseudo-prime elements.*

(4) *D is a* CFD *and every two-generated invertible ideal of D is principal.*

(5) *D is a* CFD *and every S-ideal of D is principal.*

It should be noted that in the previous theorem we cannot add $Pic(D) = 0$ as the following example from [**36**, Section 4] shows. Let A_n be the subring of $B_n = \mathbf{R}[X_0, \cdots, X_n]/(X_0^2 + \cdots + X_n^2 - 1)$, the ring of real-valued polynomial functions on the n-sphere S^n, consisting of all even functions. Then for $n \geq 2$, A_n is a regular domain that is a UCFD, but $Pic(A_n) \neq 0$.

Following suggestions from Zafrullah, the first author [**4**] generalized comaximal factorizations to $\star$-comaximal factorizations where $\star$ is a finite character star-operation on D. He defined $a \in D^{\#}$ to be $\star$-*pseudo-irreducible* (respectively, $\star$-*pseudo-prime*) if for $b, c \in D$ with $(b,c)^\star = D$, $a = bc$ (respectively, $a|bc$) implies b or c is a unit (respectively, $a|b$ or $a|c$). A factorization $b = b_1 \cdots b_m$ into nonunits is a (*complete*) $\star$-*comaximal factorization* if for $i \neq j$, $(b_i, b_j)^\star = D$ (and each b_i is $\star$-pseudo-irreducible). Finally, D is a $\star$-CFD if every nonzero nonunit of D has a complete $\star$-comaximal factorization and a $\star$-CFD is a $\star$-UCFD if each complete $\star$-comaximal factorization is unique up to order and associates. Let us put these definitions into the context of τ-factorization. So as before we define $a\tau_\star b \Leftrightarrow (a,b)^\star = D$. So $\tau_\star$ is both multiplicative and divisive. A $\star$-comaximal factorization is a $\tau_\star$-factorization. Since $\tau_\star$ is multiplicative, we only need consider $\tau_\star$-factorizations of length two in defining $\tau_\star$-irreducible and $\tau_\star$-prime (Proposition 2.4). Hence $a \in D^{\#}$ is $\star$-pseudo-irreducible (respectively, $\star$-pseudo-prime) if and only if a is $\tau_\star$-irreducible (respectively, $\tau_\star$-prime). Since $\tau_\star$ is divisive, a complete $\star$-comaximal factorization is the same thing as a $\tau_\star$-atomic factorization. So D is a $\star$-CFD (respectively, $\star$-UCFD) if and only if D is $\tau_\star$-atomic (respectively, a $\tau_\star$-UFD). However, in this section we adopt the terminology and notation of Section 2 with one change; we will abbreviate $\tau_\star$ by $\star$. For example, we will speak of $\star$-atoms (respectively, $\star$-atomic factorizations) instead of $\tau_\star$-atoms or $\star$-pseudo-irreducible elements (respectively, $\tau_\star$-atomic factorizations or complete $\star$-comaximal factorizations).

Thus by the paragraph after Theorem 2.9, if D satisfies ACCP, then D is $\star$-atomic. We do not know whether atomic implies $\star$-atomic, even for $\star = d$. We next generalize the previously mentioned result of McAdam and Swan that D is a CFD if either (1) each element of $D^{\#}$ is contained in only finitely many maximal ideals or (2) for each element x of $D^{\#}$, there are only finitely many prime ideals minimal over (x).

Theorem 4.2. *Let D be an integral domain and $\star$ a finite character star-operation on D. Let $\mathcal{S}$ be a set of prime $\star$-ideals of D. For $x \in D^{\#}$, let $V(x) = \{P \in \mathcal{S} \mid x \in P\}$ and let $\min V(x)$ be the set of minimal elements of $V(x)$. Suppose that (1) for each $x \in D^{\#}$, $0 < |\min V(x)| < \infty$ and (2) for $0 \neq (x) \subsetneq (y) \subseteq P \in \min V(y)$, there is a $Q \in \min V(x)$ with $Q \subseteq P$. Then D is $\star$-atomic.*

Proof. Let $x \in D^{\#}$ and $\min V(x) = \{P_1, \ldots, P_n\}$. Suppose that x is not $\star$-irreducible; so $x = ab$ where $a, b \in D^{\#}$ with $(a,b)^\star = D$.

Since $(a,b)^\star = D$, each P_i contains exactly one of a or b. Reordering, suppose $a \in P_1, \dots, P_s$ and $b \in P_{s+1}, \dots, P_n$. Note that $1 \leq s < n$. For suppose $a \in P_1, \dots, P_n$; so $b \notin P_1, \dots, P_n$. Let $P \in \min V(b)$. Now for some i, $ab \in P_i \subseteq P$. But then $a, b \in P$, a contradiction. A similar argument gives that $\min V(a) = \{P_1, \dots, P_s\}$ and $\min V(b) = \{P_{s+1}, \dots, P_n\}$. The proof proceeds by induction on $n = |\min V(x)|$. The case $n = 1$ follows from the above remarks. So by induction we have $\star$-atomic factorizations $a = a_1 \cdots a_t$ and $b = b_1 \cdots b_{t'}$. But then $x = ab = a_1 \cdots a_t b_1 \cdots b_{t'}$ is a $\star$-atomic factorization for x since $(a,b)^\star = D$. □

Corollary 4.3. *Let $\star$ be a finite character star-operation on the integral domain D. Then D is $\star$-atomic if either* (1) *each nonzero nonunit of D is contained in only finitely many maximal $\star$-ideals or* (2) *for each nonzero nonunit $x \in D$, there are only finitely many prime ideals minimal over (x).*

Proof. For (1) take $\mathcal{S}$ to be the set $\star$-max(D) of maximal $\star$-ideals and for (2) take $\mathcal{S}$ to be the set of prime ideals of D minimal over a principal ideal (note that such an ideal is a $\star$-ideal). □

Using the equivalence (1) $\Leftrightarrow$ (4) of Theorem 4.1, McAdam and Swan [**36**, Corollary 1.8] proved that a UFD is a UCFD. This result is actually a special case of Theorem 2.11 as seen by the next theorem.

Theorem 4.4. *Let D be an integral domain and $\star$ a finite character star-operation on D.* (1) *If D satisfies* ACCP, *then D is $\star$-atomic.*

(2) *If D is a* UFD, *then D is* $\star$-UFD.

Proof. (1) We have already remarked that since $\tau_\star$ is divisive, this follows from the paragraph after Theorem 2.9. (2) Again since $\tau_\star$ is divisive, Theorem 2.11 gives that D is a $\star$-UFD. □

Corollary 4.5. *Let D be an integral domain.*

(1) *If D satisfies* ACCP, *then D is a* CFD.

(2) *If D is a* UFD, *then D is a* UCFD.

Let D be a UFD and $\star$ the v-operation, or equivalently, the t-operation on D. By the previous theorem D is a t-UFD. Here t-

$\max(D) = X^{(1)}(D)$. A t-irreducible element has the form λp^n where $(p) \in X^{(1)}(D)$, $\lambda \in U(D)$, and $n \geq 1$; and a t-atomic factorization has the form $\lambda p_1^{n_1} \cdots p_s^{n_s}$ where $\lambda \in U(D)$, $p_1, \ldots, p_s$ are nonassociate principal primes, $s \geq 1$, and each $n_i \geq 1$. Here D is a t-UFD and each t-irreducible element is contained in a unique maximal t-ideal. Our next goal is to characterize the $\star$-UFD's with this property. But first we need to review the associated star-operation $\star_w$ which was introduced in [**9**].

Let $\star$ be a finite character star-operation on the integral domain D with quotient field K. For $A \in F(D)$, define $A^{\star_w} = \{x \in K | xI \subseteq A$, I is a finitely generated ideal of D with $I^\star = D\}$. Then $A \to A^{\star_w}$ is a finite character star-operation on D with $\star_w \leq \star$. Moreover, $A^{\star_w} = \bigcap\{AD_P | P \in \star\text{-}\max(D)\}$, $\star\text{-}\max(D) = \star_w\text{-}\max(D)$, and $A \in F(D)$ is $\star$-invertible if and only if it is $\star_w$-invertible. Thus $C\ell_\star(D) = C\ell_{\star_w}(D)$ and an ideal I of D has the property that $I^\star = D \Leftrightarrow I^{\star_w} = D$. For $a, b \in D^\#$, we have $(a,b)^\star = D \Leftrightarrow (a,b)^{\star_w} = D$ and so $\tau_\star = \tau_{\star_w}$. Thus all the notions involved with $\tau_\star$-factorization and $\tau_{\star_w}$-factorization coincide.

Theorem 4.6. *Let D be an integral domain, and let $\star$ be a finite character star-operation on D. Then the following are equivalent.*

(1) *D is $\star$-atomic and each $\star$-irreducible element is contained in a unique maximal $\star$-ideal.*

(2) *D is a $\star$-*UFD *and each $\star$-irreducible element is contained in a unique maximal $\star$-ideal.*

(3) *Each $x \in D^\#$ has a ($\star$,$\star$-atomic) factorization $x = x_1 \cdots x_n$ where each x_i is contained in a unique maximal $\star$-ideal.*

(4) *$\star\text{-}\max(D)$ is independent of finite character and $C\ell_\star(D) = 0$.*

(5) *For each $x \in D^\#$ and each $P \in \star\text{-}\max(D)$, $xD_P \cap D$ is principal and is contained in at most one maximal $\star$-ideal.*

(6) *The natural map $G(D) \to \Pi_{P \in \star-\max(D)} G(D_P)$ has image $\oplus_{P \in \star\text{-}\max(D)} G(D_P)$ where $G(D)$ is the group of divisibility of D.*

Moreover, in case any of (1)–(6) *hold, then up to order and associates a $\star$-atomic factorization of $x \in D^\#$ has the form $x = x_1 \cdots x_n$ where $P_1, \ldots, P_n$ are the maximal $\star$-ideals containing x and $x_i D = xD_{P_i} \cap D$.*

Proof. From the comments of the preceding paragraph there is no harm in assuming that $\star = \star_w$. Let $\mathcal{F} = \star\text{-max}(D)$. Then in the terminology of [**17**], $\mathcal{F}$ is a defining family of D (i.e., $D = \bigcap_{P\in\mathcal{F}} D_P$) of incomparable primes and $\star = \star_{\mathcal{F}}$ where $A^{\star_{\mathcal{F}}} = \bigcap\{A_P \mid P \in \mathcal{F}\}$. It is immediate from [**17**, Corollary 3.5] that (4)–(6) are equivalent and (3) $\Rightarrow$ (4). (4) $\Rightarrow$ (3). Again by [**17**, Corollary 3.5] each $x \in D^{\#}$ has a factorization $x = y_1 \cdots y_n$ where each y_i is contained in a unique maximal $\star$-ideal M_i. Suppose that $M_i = M_j$ for $i \neq j$. Then M_i is the unique maximal $\star$-ideal containing $y_i y_j$. Thus by combining the y_i's contained in the same maximal $\star$-ideal, we can write $x = x_1 \cdots x_n$ where x_i is contained in the unique maximal $\star$-ideal M_i and $M_i \neq M_j$ for $i \neq j$. But then $(x_i, x_j)^\star = D$, that is, $x = x_1 \cdots x_n$ is a $\star$-factorization which is even a $\star$-atomic factorization since each x_i is $\star$-irreducible. For if $x_i = yz$ where $y, z \in D^{\#}$, then clearly M_i is the only maximal $\star$-ideal containing y or z, so $(y, z)^\star \neq D$. (1)$\Rightarrow$(3). Clear. (3)$\Rightarrow$(1). The proof of (4) $\Rightarrow$ (3) shows that if x is contained in a unique maximal $\star$-ideal, then x is $\star$-irreducible. Thus D is $\star$-atomic. Suppose that $x \in D^{\#}$ is $\star$-irreducible. Since x has a $\star$-atomic factorization $x = x_1 \cdots x_n$ where each x_i is contained in a unique maximal $\star$-ideal M_i, x $\star$-irreducible gives $n = 1$. So x is contained in a unique maximal $\star$-ideal. (2) $\Rightarrow$ (1). Clear. (1) $\Rightarrow$ (2). By hypothesis, D is $\star$-atomic. Let $x \in D^{\#}$, and let $x = x_1 \cdots x_n$ be a $\star$-atomic factorization. By hypothesis x_i is contained in a unique maximal $\star$-ideal M_i. And $(x_i, x_j)^\star = D$ for $i \neq j$ gives $M_i \neq M_j$ for $i \neq j$. Thus $xD_{M_i} \cap D = x_i D$. Since (1) $\Rightarrow$ (5), $x_i D_{M_i} \cap D$ is principal, say, $x_i D_{M_i} \cap D = (y)$ and certainly M_i is the unique maximal $\star$-ideal containing y. But then $x_i D \subseteq yD$, so $x_i = ry$ for some $r \in D$. Since x_i and y are contained in the unique maximal $\star$-ideal M_i, either r is a unit or $r \in M_i$. But $r \in M_i$ gives $rD_{M_i} y D_{M_i} = x_i D_{M_i} = y D_{M_i}$, a contradiction. So r is a unit. So $xD_{M_i} \cap D = x_i D_{M_i} \cap D = x_i D$. Thus the $\star$-atomic factorization $x = x_1 \cdots x_n$ is unique up to order and associates. This also proves the moreover statement. $\square$

Corollary 4.7. *For an integral domain the following statements are equivalent.* (1) *D is a* CFD *and each pseudo-irreducible element is contained in a unique maximal ideal.*

(2) *D is a* UCFD *and each pseudo-irreducible element is contained in a unique maximal ideal.*

(3) *Each $x \in D^{\#}$ has a* (*comaximal, complete comaximal*) *factor-*

ization $x = x_1 \cdots x_n$ where each x_i is contained in a unique maximal ideal.

(4) *D is h-local and* $\operatorname{Pic}(D) = 0$.

(5) *For $x \in D^{\#}$ and M a maximal ideal, $xD_M \cap D$, is principal and is contained in at most one maximal ideal.*

(6) *The natural map $G(D) \to \Pi_{M \in \max(D)} G(D_M)$ has image $\oplus_{M \in \max(D)} G(D_M)$.*

Proof. Here we have $\star = d$. The proof is immediate once we note that $C\ell_d(D) = \operatorname{Pic}(D)$ and d-max(D) = max(D) is independent of finite character precisely when D is h-local. $\square$

Corollary 4.8. *Let D be a $\star$-*UFD *with the property that each $\star$-irreducible element is contained in a unique maximal $\star$-ideal. For $x \in D^{\#}$, the following are equivalent*: (1) *x is $\star$-prime,* (2) *x is $|_\star$-prime, and* (3) *x is $\star$-irreducible.*

Proof. (1) $\Rightarrow$ (2). Since $\tau_\star$ is both multiplicative and divisive, Proposition 2.4 (2) gives that a $\star$-prime element is $|_\star$-prime for any integral domain D. (2) $\Rightarrow$ (3). This holds for any integral domain and relation τ. (3) $\Rightarrow$ (1). Suppose that D is a $\star$-UFD with the property that each $\star$-irreducible element is contained in a unique maximal $\star$-ideal. Let $x \in D$ be $\star$-irreducible; so x is contained in a unique maximal $\star$-ideal M. Suppose that $x | x_1 \cdots x_n$ where $x_1 \cdots x_n$ is a $\star$-atomic factorization. Then each x_i is contained in a unique maximal $\star$-ideal M_i and $M_i \neq M_j$ for $i \neq j$. Since $x_1 \cdots x_n \in M$, some x_i, say x_1, must lie in M. Hence $M_1 = M$. But then $x_1 D = x_1 D_M \cap D \subseteq xD_M \cap D = xD$; so $x | x_1$. Thus x is $\star$-prime. $\square$

Recall that an element $x \in D^{\#}$, D an integral domain, is *t-pure* if x is contained in a unique maximal t-ideal. So Theorem 4.6 for the case $\star = t$ gives the result mentioned in the introduction that each element of $D^{\#}$ is a product of t-pure elements if and only if the intersection $D = \bigcap_{P \in t\text{-}\max(D)} D_P$ is locally finite and independent and $C\ell_t(D) = 0$.

Of course in a UCFD a pseudo-irreducible element need not be contained in a unique maximal ideal; for example, X in $K[X, Y]$, K a field, is contained in infinitely many maximal ideals. However, we next show that in the one-dimensional case this cannot happen.

Theorem 4.9. *Let D be a one-dimensional integral domain and $\star$ a finite character star-operation on D. The the following are equivalent.*

(1) *D is a* $\star$-UFD.

(2) *D is a* UCFD.

(3) *D is weakly factorial.*

(4) *Every element of $D^{\#}$ is contained in only finitely many maximal ideals and* $\operatorname{Pic}(D) = 0$.

Proof. (1) $\Leftrightarrow$ (2). Since $\dim D = 1$, $\max(D) = \star\text{-}\max(D)$; so $\star_w = d$. Thus $\tau_\star = \tau_{\star_w} = \tau_d$. (3) $\Leftrightarrow$ (4). [**10**, page 149]. (4) $\Rightarrow$ (2). Corollary 4.7 or Theorem 4.1. (2) $\Rightarrow$ (3). Let $d \in D^{\#}$ be pseudo-prime. Then $D/(d)$ is indecomposable and thus $D/\sqrt{(d)}$ is also indecomposable. But then $D/\sqrt{(d)}$ being zero-dimensional and reduced is von Neumann regular. Thus $D/\sqrt{(d)}$ is an indecomposable von Neumann regular ring and hence is a field. So $\sqrt{(d)}$ is a maximal ideal and hence (d) is a primary ideal. Since every element of $D^{\#}$ is a product of pseudo-prime elements, every element of $D^{\#}$ is a product of primary elements. So D is weakly factorial. $\square$

However, in a one-dimensional CFD a pseudo-irreducible element may be contained in infinitely many maximal ideals. In [**31**], Grams gives an example of an almost Dedekind domain D with ACCP which is not Dedekind. By Corollary 4.5, D is a CFD. Since D is not Dedekind, some element of $D^{\#}$, and hence some pseudo-irreducible element of D, must be contained in infinitely many maximal ideals. It is interesting to note that the Nagata ring $D(X)$ is not a CFD. For suppose that $D(X)$ is a CFD. Then $D(X)$ being Bezout is actually a UCFD. But then by Theorem 4.9 each element of $D(X)^{\#}$ is contained in only finitely many maximal ideals, a contradiction.

We next give a $\star$-factorization version of Theorem 4.1. By a $S^\star$-ideal we mean a $\star$-ideal I with $I = (a, c)^\star = (a, c^2)^\star$ for some $a, c \in D$.

Theorem 4.10. *Let D be an integral domain, and let $\star$ be a finite character star-operation on D. Consider the following seven conditions.*

(1) *D is $\star$-atomic and every two-generated $\star$-invertible $\star$-ideal of I of D is principal.*

(2) *D is $\star$-atomic and each $S^{\star_w}$-ideal of D is principal.*

(3) *D is $\star$-atomic and every $\star$-atom is $\star$-prime.*

(4) *Each $a \in D^{\#}$ is a $\star$-product of $\star$-primes.*

(5) *D is a $\star$*UFD.

(6) *Each $a \in D^{\#}$ is a $\star$-product of $|_\star$-primes.*

(7) *D is $\star$-atomic and every $\star$-atom is $|_\star$-prime.*

Then $(1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4) \Rightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7)$.

Proof. $(1) \Rightarrow (2)$. Let I be a $\star_w$-ideal with $I = (a,c)^{\star_w} = (a,c^2)^{\star_w}$. Then for $P \in \star\text{-max}(D)$, $(a,c)D_P = ID_P = (a,c^2)D_P$. Thus either c is a unit in D_P or $cD_P \subseteq aD_P$. So ID_P is principal. Hence I is $\star_w$-invertible and so is $\star$-invertible. So $I^\star = (a,c)^\star$ is principal. But I $\star_w$-invertible gives $I = I_v$. Hence $I = I^\star$ is principal. $(2) \Rightarrow (3)$. Let c be a $\star$-atom. Suppose that $c \mid ab$ where $(a,b)^\star = D$. So $a \in a(a,b)^{\star_w} = (a^2,ab)^{\star_w} \subseteq (a^2,c)^{\star_w}$ and hence $(a^2,c)^{\star_w} = (a,c)^{\star_w}$. So by hypothesis $(a,c)^{\star_w}$ is principal and hence $(a,c)^\star$ is principal. Now $(a,b)^\star = D$, so $((a,c),(b,c))^\star = D$. Hence $(a,c)^\star \cap (b,c)^\star = ((a,c)(b,c))^\star$. So $(c) \subseteq (a,c)^\star \cap (b,c)^\star = ((a,c)(b,c))^\star \subseteq (c)$. Thus $((a,c)(b,c))^\star = (c)$. Since $(a,c)^\star$ is principal, so is $(b,c)^\star$. Say $(a,c)^\star = (d)$ and $(b,c)^\star = (e)$. Then $(c) = (d)(e)$; so $c = \lambda de$ for some unit λ. Now $(d,e)^\star = ((a,b)^\star,(b,c)^\star)^\star = D$; so c a $\star$-atom gives d or e is a unit. If d is a unit, $(c) = (e) = (b,c)^\star$, so $c \mid b$. If e is a unit, then $(c) = (d) = (a,c)^\star$, so $c \mid a$. Hence c is a $\star$-prime. $(3) \Rightarrow (4)$. Clear. $(4) \Rightarrow (3)$. A $\star$-prime is a $\star$-atom. $(4) \Rightarrow (5)$. Theorem 2.7 (1). $(5) \Leftrightarrow (6) \Leftrightarrow (7)$. Here $\tau_\star$ is divisive, so Theorem 2.7 (3) applies. ◻

We suspect, but have been unable to prove, that (1)–(7) of Theorem 4.10 are equivalent (as is the case for $\star = d$ by Theorem 4.1). Note that (1)–(7) are indeed equivalent if any one of the following holds: (a) each $\star$-atom of D is contained in a unique maximal $\star$-ideal (Theorem 4.6), (b) $\dim D = 1$ (Theorem 4.9), or (c) D has only finitely many maximal $\star$-ideals (Corollary 4.11).

Corollary 4.11. *Let D be an integral domain and $\star$ a finite character star-operation on D. Suppose that D has only finitely many maximal $\star$-ideals $M_1, \dots, M_n$. Then D is a $\star$-UFD. Moreover, for each $x \in D^{\#}$, $\ell_\star(x) \leq n$, and there is an $x \in D^{\#}$ with $\ell_\star(x) = n$.*

Proof. By Corollary 4.3, D is $\star$-atomic. Since $\star$-max(D) is finite, $C\ell_\star(D) = 0$ by [**16**, Remark 2.3]. So by Theorem 4.10, D is a $\star$-UFD. Suppose that $x \in D^\#$ is in exactly m maximal $\star$-ideals of D. Then $\ell_\star(x) \leq m \leq n$ by the proof of Theorem 4.2. For each i, $1 \leq i \leq n$, choose $x_i \in M_i - \bigcup_{1 \neq i} M_j$. Consider $x = x_1 \cdots x_n$. For $i \neq j$, $(x_i, x_j)^\star = D$ and since M_i is the unique maximal $\star$-ideal containing x_i, x_i is $\star$-irreducible. So $\ell_\star(x) = n$. □

One can ask about the converse of the second statement of the previous corollary. Namely, if D is an integral domain and $\star$ is a finite character star-operation on D with $\ell_\star(x) \leq n$ for all $x \in D^\#$, must D have at most (or equivalently, exactly) n maximal $\star$-ideals? For $\star = d$ and $n = 1$, this is true. For suppose that D has at least two maximal ideals, say M_1 and M_2. Then $M_1 + M_2 = D$, so we have $x_i \in M_i$ with $m_1 + m_2 = 1$. But then $x = m_1 m_2$ has $\ell_d(x) \geq 2$.

In attempting to prove that (1)–(7) of Theorem 4.10 are equivalent, we discovered the following two theorems.

Theorem 4.12. *Let τ be a divisive relation on the integral domain D. Let $a \in D$ be τ-prime and $b \in D$ a τ-atom. Then either $a\tau b$ or ab is a τ-atom.*

Proof. Suppose that $a \not\tau b$. We show that ab is a τ-atom. Deny. So $ab = c_1 \cdots c_n$ where $c_i \tau c_j$ for $i \neq j$ and $n \geq 2$. Now a is τ-prime and $a \mid ab = c_1 \cdots c_n$, so $a \mid c_i$ for some i, say $a \mid c_1$. Write $c_1 = ra$. So $ab = rac_2 \cdots c_n$ and hence $b = rc_2 \cdots c_n$. Suppose that r is not a unit. Then $r \mid c_1$ and $c_1 \tau c_j$ $(j \neq 1)$ gives $r \tau c_j$ since τ is divisive. Thus $b = rc_2 \cdots c_n$ is a non-trivial τ-factorization of the τ-atom b; a contradiction. Hence r must be a unit. So again $rc_2 \cdots c_n$ is a non-trivial τ-factorization of b unless $n = 2$. Hence $n = 2$, $b = rc_2$ and $a = r^{-1}c$. So $c_1 \tau c_2$ and $c_1 \sim a$, $c_2 \sim b$ give $a\tau b$; a contradiction. □

Theorem 4.13. *Suppose that D is a τ-*UFD *where τ is divisive. Then the following conditions are equivalent.* (1) *If a and b are τ-atoms of D with $a \not\tau b$, then ab is a τ-atom.*

(2) *If $c \in D^\#$ has a factorization $c = c_1 \cdots c_n$ into τ-atoms, then $\{1, 2, \ldots, n\} = S_1 \cup \cdots \cup S_m$ (disjoint union) so that each $b_i = \Pi\{c_j \mid c_j \in S_i\}$ is a τ-atom and $c = b_1 \cdots b_m$ is a τ-atomic factorization of c.*

(3) *Each τ-atom of D is τ-prime.*

Proof. (1) $\Rightarrow$ (2). If $c_i \tau c_j$ for each $i \neq j$, we are done. So assume, say, $c_1 \not\tau c_2$. Then $c_1 c_2$ is a τ-atom and $c = (c_1 c_2) c_3 \cdots c_n$ where $c_1 c_2, c_3, \ldots, c_n$ are τ-atoms. The result follows by induction. (2) $\Rightarrow$ (3). Let $a \in D^{\#}$ be a τ-atom. Suppose that $a | a_1 \cdots a_n$ where $a_1 \cdots a_n$ is a τ-factorization. We need that $a \mid a_i$ for some i. There is no harm in assuming that each a_i is a τ-atom (here we use that τ is divisive). Suppose that $ra = a_1 \cdots a_n$. Let $r = r_1 \cdots r_m$ be a τ-atomic factorization. Then grouping the terms of $r_1 \cdots r_m a$ as in (2) we get a τ-atomic factorization of $a_1 \cdots a_n$ with one factor having the form $r'a$ where r' is a product of some of the r_i's or is 1. By unique τ-factorization $r'a \sim a_i$ for some i. But then $a | a_i$. (3) $\Rightarrow$ (1). Theorem 4.12. $\square$

Carlitz [**21**] showed that a ring of algebraic integers D is a HFD if and only if $|C\ell(D)| \leq 2$. Now the implication $\Leftarrow$ holds for any Dedekind domain. But the implication $\Rightarrow$ does not hold for a general Dedekind domain; Carlitz used the facts that $|C\ell(D)| < \infty$ and that each class of $C\ell(D)$ contains a prime. We recast the implication $\Leftarrow$ as follows. Let D be an integral domain with the property that for each $a \in D^{\#}$, (a) is a d-product of prime ideals (that is, D is a π-domain). Suppose that $|C\ell_d(D)| \leq 2$ (that is, $|\mathrm{Pic}\,(D)| \leq 2$). Then D is a HFD. We next give a star-operation version of this result.

Theorem 4.14. *Let D be an integral domain, and let $\star$ be a finite character star-operation on D. Suppose that each nonzero proper principal ideal of D is a $\star$-product of prime ideals. If $|C\ell_\star(D)| \leq 2$, then D is a $\star$-*HFD.

Proof. Suppose that $a \in D^{\#}$. Then $(a) = (P_1 \cdots P_n)^\star$ for some prime ideals $P_1, \ldots, P_n$ of D. Since $\star$ has finite character, $(a) = (P_1 \cdots P_n)_t$. So every nonzero proper principal ideal of D is a t-product of prime ideals. So by [**34**] or [**11**, Corollary 3.2], D is a Krull domain. Hence D satisfies ACCP and so is $\star$-atomic by Theorem 4.4. If $C\ell_\star(D) = 0$, then D is a $\star$-UFD by Theorem 4.10 and hence is a $\star$-HFD. So assume that $|C\ell_\star(D)| = 2$.

Let $a \in D^{\#}$; so $(a) = (P_1^{s_1} \cdots P_n^{s_n})^\star$ where $P_1, \ldots, P_n$ are distinct prime ideals with each $P_i^\star \neq D$ and $s_i \geq 1$. Now P_i is $\star$-invertible and hence $\star_w$-invertible. Thus $P_iD_{P_i}$ is a principal prime ideal in the Krull domain D_{P_i}. Hence $P_i \in X^{(1)}(D)$. Thus $(a) = (P_1^{s_1} \cdots P_n^{s_n})_t$ and hence this is the unique representation of (a) as a t-product of height-one primes. Thus $(a) = (P_1^{s_1} \cdots P_n^{s_n})^\star$ is the unique representation of (a) as a $\star$-product of prime ideals where each $P_i^\star \neq D$. Since $|C\ell_\star(D)| = 2$, either P_i is principal, so each $P_i^{s_i}$ is principal, or P_i is not principal and hence $(P_i^{s_i})^\star$ is principal $\Leftrightarrow s_i$ is even.

Observe that a $\star$-factorization of $a \in D^\#$ corresponds to a partition $\mathcal{C}$ of $\{P_1^{s_1}, \dots, P_n^{s_n}\}$ with the property that for each $C \in \mathcal{C}$, $(\Pi\{P_i^{s_i} | P_i^{s_i} \in \mathcal{C}\})^\star$ is principal. Moreover, $\mathcal{C}$ corresponds to a $\star$-atomic factorization of a if and only if no $C \in \mathcal{C}$ can be split as a disjoint union of nonempty subsets $C = C_1 \cup C_2$ where for $j = 1, 2$, $(\Pi\{P_i^{s_i} | P_i^{s_i} \in \mathcal{C}_j\})^\star$ is principal. So suppose the $\mathcal{C}$ corresponds to a $\star$-atomic factorization of a. Now $\{P_i^{s_i}\}$ is a class of $\mathcal{C} \Leftrightarrow (P_i^{s_i})^\star$ is principal $\Leftrightarrow P_i$ is principal or P_i is not principal but s_i is even. Suppose that $C = \{P_{i_1}^{s_{i_1}}, \dots, P_{i_k}^{s_{i_k}}\}$ is a class of $\mathcal{C}$ with $k = |C| \geq 3$. Observe that each $(P_{i_j}^{s_{i_j}})^\star$ is not principal, so each s_{i_j} is odd. Since $s_{i_1} + \cdots + s_{i_k}$ is even and s_{i_1} and s_{i_2} are odd, $s_{i_3} + \cdots + s_{i_k}$ is also even. So $(P_{i_1}^{s_{i_1}} P_{i_2}^{s_{i_2}})^\star$ and $(P_{i_3}^{s_{i_3}} \cdots P_{i_k}^{s_{i_k}})^\star$ are both principal. But then we can split C as $\{P_{i_1}^{s_{i_1}} P_{i_2}^{s_{i_2}}\} \cup \{P_{i_3}^{s_{i_3}} \cdots P_{i_k}^{s_{i_k}}\}$, a contradiction. So each nonprincipal class of $\mathcal{C}$ has cardinality two. So $|\mathcal{C}| = \ell + (n - \ell)/2 = (n + \ell)/2$ where $\ell = |\{P_i^{n_i}| \mid (P_i^{n_i})^\star$ is principal$\}|$. Hence any $\star$-atomic factorization of a has length $(n + \ell)/2$. Thus D is a $\star$-HFD. $\square$

We end this section by considering the relation $\tau_{[\,]}$ which is related to τ_t. Let D be an integral domain. We denote the gcd of $a, b \in D$ by $[a, b]$. Define the relation $\tau_{[\,]}$ on $D^\#$ by $a\tau_{[\,]}b \Leftrightarrow [a, b] = 1$. Note that $[a, b] = 1 \Leftrightarrow (a, b)$ is not contained in a proper principal ideal. Also, $(a, b)_v = D \Rightarrow [a, b] = 1$, but not conversely. Certainly the relation $\tau_{[\,]}$ is divisive. However, $\tau_{[\,]}$ is multiplicative precisely when for $a, b, c \in D^\#$, $[a, b] = [a, c] = 1 \Rightarrow [a, bc] = 1$, the so called PP-property. The PP-property has been studied in regards to Gauss' lemma, see [**3**] for details and references. Briefly, D satisfies the GL-property (Gauss' lemma: $f = a_0 + a_1X + \cdots + a_nX^n$, $g = b_0 + b_1X + \cdots + b_mX^m \in D[X]$ with $[a_0, \dots, a_n] = [b_0, \dots, b_m] = 1 \Rightarrow$ the coefficients of fg have gcd 1) $\Rightarrow D$ satisfies the PP-property $\Rightarrow D$ satisfies the AP-property (atoms of D are prime). Hence a GCD domain satisfies the PP-property and an atomic domain satisfying the PP-property is a UFD.

We will abbreviate $\tau_{[\,]}$ by $[\,]$, for example, we will speak of a $[\,]$-factorization or say that D is a $[\,]$-UFD rather than speaking of a $\tau_{[\,]}$-factorization or saying that D is a $\tau_{[\,]}$-UFD. Thus a $[\,]$-factorization is just a factorization $a = \lambda a_1 \cdots a_n$ where $[a_i, a_j] = 1$ for $i \neq j$. Since $\tau_{[\,]}$ is divisive, if D satisfies ACCP, then D is $[\,]$-atomic. Also a UFD is a $[\,]$-UFD. Here a $[\,]$-factorization has the form $\lambda p_1^{k_1} \cdots p_s^{k_s}$ where $\lambda \in U(D)$, $p_1, \dots, p_s$ are nonassociate principal primes, and each $k_i \geq 1$. An integral domain D has every element of $D^{\#}$ a $[\,]$-atom if and only if (D, M) is quasilocal and for each finitely generated ideal $I \subseteq M$, there is an $m \in M$ with $I \subseteq (m) \subseteq M$. Hence in a valuation domain V every element is a $[\,]$-atom (and hence V is a $[\,]$-UFD) and for a Noetherian domain the converse is true. Note the relationship to "Kummer's property" [**30**]: For $x, y \in D$, either $(x, y) = D$ or (x, y) is contained in a proper principal ideal.

The relation $\tau_{[\,]}$ is another example of a relation τ where it is not sufficient to consider only τ-factorizations $a = \lambda a_1 \cdots a_n$ of length $n = 2$ when defining τ-irreducibles, τ-primes, and $|_\tau$-primes. Let $K \subseteq L$ be fields, and let $D = K + XL[[X]]$. Since D satisfies ACCP, D is $[\,]$-atomic. Up to associates, the nonzero nonunits (respectively, atoms) of D have the form uX^n where $u \in L^*$ and $n \geq 1$ (respectively, $n = 1$). Let $u, v \in L^*$. Then uX and vX are associates $\Leftrightarrow uv^{-1} \in K$. If $n > 1$, $uX^n = (vX)(uv^{-1}X^{n-1})$. Hence $[uX^n, vX^m] = 1 \Leftrightarrow n = m = 1$ and $uv^{-1} \notin K$. Thus if $f \in D^{\#}$ has a $[\,]$-factorization of length 2, then $\operatorname{ord} f = 2$. For $K = L$, $D = L[[X]]$ is a DVR, D is a $[\,]$-UFD, and each element of $D^{\#}$ is a $[\,]$-atom. So suppose $K \subsetneq L$. Let $u \in L - K$. Then $u+1 \notin K$ and $(u+1)u^{-1} \notin K$. So $u(u+1)X^3 = X(uX)((u+1)X)$ is a $[\,]$-factorization. Thus $u(u+1)X^3$ is not a $[\,]$-atom. However, $u(u+1)X^3$ has no $[\,]$-factorization of length 2. Also, if $X^3 | fg$ for $f, g \in D^{\#}$, then $\operatorname{ord} fg \geq 3$, so fg has no $[\,]$-factorization of length 2. Thus vacuously, if $X^3 | fg$ (respectively, $X^3 |_{[\,]} fg$) where fg is a $[\,]$-factorization, then $X^3 | f$ or $X^3 | g$ (respectively, $X^3 |_{[\,]} f$ or $X^3 |_{[\,]} g$); but X^3 is not $[\,]$-prime (respectively, $|_{[\,]}$-prime). Finally, observe that D is not a $[\,]$-UFD. For if $u \in L - K$, then we have three $[\,]$-atomic factorizations of uX^2: $uX^2 = uX \cdot X = u^2X \cdot u^{-1}X = u(u+1)X \cdot (u+1)^{-1}X$. Now the first two $[\,]$-atomic factorizations are distinct unless $u^2 \in K$, while the first and third are distinct unless $u(u+1) \in K$. But if both $u^2 \in K$ and $u(u+1) \in K$, then $u = u(u+1) - u^2 \in K$, a contradiction.

5. The congruence modulo n τ-relation. As defined in Section 2, the "congruence modulo n" relation is given by $a\tau_n b \Leftrightarrow a \equiv b \bmod n$ for $a, b \in \mathbf{Z}^{\#}$, $n \in \mathbf{N}^{\#} = \{2, 3, 4, \dots\}$. The second author would like to thank Suzanne Hamon for several discussions of this example.

Note that this relation is never divisive. Recall that a divisive relation must be associate-preserving; if this relation preserved associates, then we would have $k \equiv -k \bmod n$ for every $k \in \mathbf{Z}$. This would imply $2k \equiv 0 \bmod n$ for all $k \in \mathbf{Z}$, which is clearly only possible for $n = 2$. Thus none of the other τ_n is associate-preserving, and so τ_n is not divisive for $n > 2$. While τ_2 is clearly associate-preserving, the counterexample $6\tau_2 4$ but $3 \not\tau_2 2$ demonstrates that this τ, too, is not divisive.

If $n \neq 2$, then $2 \equiv 2 \bmod n$, but $2 \cdot 2 \not\equiv 2 \bmod n$. Thus the relation τ_n is not multiplicative for $n > 2$. However, it is multiplicative in the case $n = 2$; $a\tau_2 b$ is equivalent to saying "a and b are both even" or "a and b are both odd."

$\mathbf{Z}$ need not be τ_n-atomic; in fact for odd $n \geq 7$, $4r$, where r is a prime congruent to $4 \bmod n$, is an element with a τ_n-complete factorization $(4 \cdot r)$ but not a τ_n-atomic factorization (as $4 = 2 \cdot 2$ is not a τ_n-atom, and $2 \cdot 2 \cdot r$, $2(2r)$ are not τ_n-factorizations). Whether $\mathbf{Z}$ is τ_n-atomic or not, note that for any $n \in \mathbf{N}^{\#}$, the bound on the length of standard factorizations in the UFD $\mathbf{Z}$ gives a bound on the length of τ_n-factorizations. Thus $\mathbf{Z}$ is a τ_n-complete BFD and hence is τ_n-complete. However, for every n, we can find an element with two distinct τ_n-factorizations (so $\mathbf{Z}$ is never a τ_n-UFD under this relation). To see this, consider $n^2 p_1 p_2$, where the p_i are (standard) primes congruent to $1 \bmod n$. We can easily verify that np_i and np_1p_2 are τ_n-atoms if n itself is a τ_n-atom, and that $(np_1)(np_2) = (np_1p_2)(n)$ gives two distinct τ_n-atomic factorizations of the same element. When n is not a τ_n-atom, then n can be factored as $n = \lambda a_1 \cdots a_k$ for λ a unit and $a_i \equiv a_j$ for all $i, j \in 1, \dots, k$. Then $\lambda(a_1p_1)(a_2p_2)\cdots a_k = (a_1p_1p_2)a_2 \cdots a_k$ yields our counterexample. Now $\mathbf{Z}$ also need not be a τ_n-HFD; for example, if $n = 30$, $216 = 6 \cdot 6 \cdot 6 = -1 \cdot 18 \cdot -12$ yield two τ_n-atomic factorizations of different lengths. Similar counterexamples can be manufactured for many composite n; indeed, in the cases $n > 6$, we have yet to find an n for which τ_n-atomic factorizations must have unique length.

Given a specific $n \in \mathbf{N}^{\#}$, it is possible to compute the $|_{\tau_n}$-prime, τ_n-prime, and τ_n-atomic elements of $\mathbf{Z}$. The elements to be considered will vary in each case, but always depend on the (usual) primary decomposition of n.

To fully illustrate the relation τ_n, let us consider the most basic case: $a\tau_2 b \Leftrightarrow a \equiv b \bmod 2$. (Note that, throughout this section, λ and μ indicate units of $\mathbf{Z}$.) In this case, a τ_2-product is simply a product which consists of either all odd or all even numbers. Therefore, if $n \in \mathbf{Z}$ is odd, its τ_2-atomic factorization is its prime factorization (note that the primary factorization of n is not a τ_2-atomic factorization, as for general m, primary elements of $\mathbf{Z}$ are τ_m-atoms $\Leftrightarrow$ they are prime). If the standard primary factorization of n is given by $n = 2^{k_0}{p_1}^{k_1}\cdots{p_m}^{k_m}$, where the $p_i \equiv 1 \bmod 2$, then each τ_2-factor of n must be a multiple of 2. However, we may completely τ_2-factor $4j = 2(2j)$ for odd j (and similarly for larger powers of 2), so the τ_2-atoms are the (usual) primes and integers of the form $2p_1\cdots p_n$ where the p_i are (not necessarily unique) odd (standard) primes. Thus $\ell_{\tau_2}(n) = L_{\tau_2}(n) = k_0$ for $n = 2^{k_0}{p_1}^{k_1}\cdots{p_m}^{k_m}$ and $\ell_{\tau_2}(n) = L_{\tau_2}(n) = \sum_{i=1}^{m} k_i$ for $n = {p_1}^{k_1}\cdots{p_m}^{k_m}$, where the p_i are odd primes. Therefore $\mathbf{Z}$ is a τ_2-HFD and τ_2-FFD, although it is clearly not a τ_2-UFD.

The τ_2 relation also gives examples of τ_2-primes which are not standard primes and of standard primes which are not $|_{\tau_2}$-primes. Standard primes are of course τ_n-primes for any n. Also consider $a = 2p$, for p a standard odd prime. If $2p|\lambda b_1\cdots b_m$, with $b_i\tau_2 b_j$ for $i \neq j$, then 2 must divide some b_i. Therefore, all of the b_i are even, and so there must be some b_i divisible by both 2 and p. Thus $a = 2p$ is a τ_2-prime for this relation. However, $a = 2p_1\cdots p_m$ is not a τ_2-prime if $m \geq 2$ (as $a|2p_1 \cdot 2p_2\cdots p_m$, but divides neither factor). Since products of odd primes are τ_2-reducible, these cannot be τ_2-primes unless the products have length 1. Thus the only τ_2-primes are the standard primes and $a = 2p$ for p an odd prime. If p is an odd prime and $p|_{\tau_2} b_1\cdots b_m$, then the b_i must all be odd. Since p is prime and p, b_i are odd, then $p|_{\tau_2} b_i$ for some $1 \leq i \leq m$. However, note that $2|_{\tau_2} 6\cdot 10$, but 2 τ_2-divides neither factor; thus 2 is a standard prime which is not a $|_{\tau_2}$-prime (as is -2). Similarly, for p an odd prime, $2p|_{\tau_2}(6p)(2q)$, where $q \neq p$ is another odd prime, but $2p$ τ_2-divides neither factor. Therefore $2p$ is a τ_2-prime which is not a $|_{\tau_2}$-prime. Thus the only $|_{\tau_2}$-primes are the odd primes of standard factorization, illustrating that, as asserted in Section 2, a a standard prime need not imply a is a $|_\tau$-prime for non-divisive τ.

From Section 2, recall that τ_n-primes and $|_{\tau_n}$-primes must be τ_n-atoms. For small n, we may compute the τ_n-primes and $|_{\tau_n}$-primes (full details are given in [**26**]). For $n > 6$, we may categorize the $|_{\tau_n}$-primes and τ_n-primes.

Theorem 5.1 [**27**]. *For $n > 6$, the relation $a\tau_n b \Leftrightarrow a \equiv b \bmod n$ yields no $|_{\tau_n}$-prime elements in $\mathbf{Z}$.*

The $|_{\tau_n}$-primes yielded by the smaller n are easily computed (full details are given in [**26**]). The relation τ_5 has no $|_{\tau_5}$-primes. The odd standard primes are the $|_{\tau_4}$-primes; the standard primes different from 3 and -3 are the $|_{\tau_3}$-primes; all primes congruent to ± 1 mod 6 are the $|_{\tau_6}$-primes, and so therefore all standard primes except $\pm 2, \pm 3$ are $|_{\tau_6}$ primes.

We may also easily classify the τ_n-prime elements for any τ_n:

Theorem 5.2 [**27**]. *Fix $n \in \mathbf{N}^{\#}$. Then $a \in \mathbf{Z}$ is τ_n-prime if and only if $a = \lambda {p_1}^{e_1} \cdots {p_k}^{e_k} q$, where $e_i = 1$ or 0, p_i are distinct prime divisors of n, and q is a (standard) prime which does not divide n (or q is a unit).*

To prove that such a are necessarily τ_n-prime, we must show that if a prime p divides a τ_n-product, then each τ_n-factor is a multiple of p:

Lemma 5.3 [**27**]. *Choose $n \in \mathbf{N}^{\#}$; suppose p is a prime divisor of n and $p | \lambda b_1 \cdots b_m$, where $b_i \equiv b_j \bmod n$ for $i \neq j$. Then $p | b_i$ for all $i \in 1, \dots, m$.*

The preceding results depend on the knowledge that a is τ_n-irreducible. Since $a = \lambda {p_1}^{e_1} \cdots {p_k}^{e_k} q$, it's not difficult to check by hand whether a given a is τ_n-irreducible (at worst, consider each of the partitions of the prime factors; compare their products to see if a τ_n-factoring is possible). Indeed, for a given $z \in \mathbf{Z}^{\#}$, considering each of the factorizations given by the (finite number) of partitions of the (not necessarily distinct) prime factors of z yields a decidable procedure for determining whether z is τ_n-irreducible. We invite the reader to create an effective program which determines whether a given a is τ_n-irreducible (for congruence mod a specified n) by exhausting possible partitions. To close, we describe several classes of elements which must be τ_n-irreducible for a fixed but arbitrary n (as well as some obvious types of elements that must be τ_n-reducible).

To construct all of these cases, we use the mod$n$ multiplication table. Instead of considering the congruence classes as represented by the elements $0, \ldots, n-1$, we instead write the table considering the equivalence classes as represented by the integers between $-(n-1)/2$ and $(n-1)/2$ for n odd, or as the integers between $-(n-2)/2$ and $n/2$ for n even. Since factorizations of integers may always be augmented by units, we need only consider products of integers between 0 and $n/2$ (or 0 and $(n-1)/2$, if n odd). Of course then prime elements of $\mathbf{Z}$ are either factors of n or are relatively prime to n. If $p|n$ for p prime, then p is the only prime element of $\mathbf{Z}$ in its congruence class; congruence classes represented by multiples of p will contain no standard primes. All other congruence classes do contain (infinitely many) primes (19 and -71 are examples of "primes congruent to ± 4" mod 15). Since we may always insert units into a factorization as necessary (for example, writing $-4 \cdot 4 = -1 \cdot -4 \cdot -4$), we will refer to primes as being congruent to the absolute value of their class representative (for example, we will treat 71 as a "prime congruent to 4 mod 15," even though strictly speaking it is congruent to -4).

Theorem 5.4 [**27**]. *If $b \in \mathbf{Z}$ is a τ_n-irreducible, $b \not\equiv \pm 1 \bmod n$, then $\mu b p_1 \cdots p_n$ is a τ_n-irreducible for any (not necessarily distinct) primes $p_i \equiv \pm 1 \bmod n$.*

This theorem allows us to consider only the primes not congruent to 1 when determining the τ_n-irreducibility of a given $z \in \mathbf{Z}$. This theorem allows us to inductively build up more τ_n-irreducibles.

We next give some classes of elements whose τ_n-atomicity may be determined. The list is by no means exhaustive, but is intended to give the reader some concrete practice in manipulating τ_n-products. Full details of the following cases may be found in [**27**].

Example 5.5. (1) Let $n = {s_1}^{a_1} \cdots {s_i}^{a_i}$, where s_i are primes and $a_i \geq 1$; then $b = {s_1}^{a_1} \cdots {s_i}^{a_i}\, q_1 \cdots q_j$ is τ_n-irreducible if $a_i = 1$ for at least one i and either j is 0 or the q_l are primes coprime to n.

(2) If q, r are primes from different congruence classes $\bmod n$, then μqr is a τ_n-irreducible.

(3) If $c = \mu q_1 \cdots q_j s_1 \cdots s_k$, where $s_i \equiv \pm 1 \bmod n$ for each i, $q_i \equiv \mu q_l \bmod n$ for some unit μ and each $i, l \in 1, \ldots, j$, then c is τ_n-reducible.

(4) Write $n = {s_1}^{a_1} \cdots {s_k}^{a_k}$, where s_i are primes; then $d = \mu {s_1}^{b_1} \cdots {s_k}^{b_k} c$ is τ_n-reducible whenever $b_i \geq 2a_i$ for all $i \in 1, \ldots, k$ and $c \in \mathbf{Z}$.

We end this section, and hence the paper, by considering the relation τ_n for the degenerate cases $n = 0, 1$. First, suppose that $n = 1$. So $a\tau_1 b \Leftrightarrow a \equiv b \bmod 1$. But since this is always true, we have $\tau_1 = \mathbf{Z}^\# \times \mathbf{Z}^\#$ and hence we get the usual factorization in $\mathbf{Z}$.

The case $n = 0$ is more interesting. Here $a\tau_0 b \Leftrightarrow a \equiv b \bmod 0 \Leftrightarrow a = b$. So τ_0 is not associate-preserving, and hence not divisive, and is not multiplicative. A τ_0-factorization has the form $(\pm 1)b \cdots b$ where $b \in \mathbf{Z}^\#$. So $a|_{\tau_0} b \Leftrightarrow b = (\pm 1)a^n$ for some $n \geq 1$. And $a \in \mathbf{Z}^\#$ is a τ_0-atom unless $a = (\pm 1)b^n$ for some $b \in \mathbf{N}^\#$ and $n \geq 2$. Note that $a = (\pm 1)b \cdots b$ (n b's) is a τ_0-atomic factorization unless $b = (\pm 1)c^m$ for some $c \in \mathbf{N}^\#$ and $m \geq 2$. But then $a = (\pm 1)c \cdots c$ (mn c's) is a τ_0-factorization of a. This shows that each τ_0-factorization can be refined to a τ_0-atomic factorization and hence $\mathbf{Z}$ is τ_0-atomic. In fact, $\mathbf{Z}$ is even a τ_0-UFD. For suppose that $a = (\pm 1)b \cdots b = (\pm 1)c \cdots c$ are two τ_0-atomic factorizations of a. We can easily reduce to the case where a, b, c are positive. Thus we have $b^n = c^m$ where $b, c \geq 2$ and b and c are not themselves proper powers. But then factoring b and c into prime powers and employing unique factorization in $\mathbf{Z}$ yields that $n = m$ and hence $b = c$.

Since $a \in \mathbf{Z}^\#$ is a τ_0-atom unless $a = (\pm)b^n$ for some $b \in \mathbf{N}^\#$ and $n \geq 2$, the τ_0-atoms of $\mathbf{Z}$ have the form $a = (\pm 1)p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ where $p_1, \ldots, p_s$ are distinct primes, each $\alpha_i \geq 1$, and $[\alpha_1, \ldots, \alpha_s] = 1$. The τ_0-primes are $(\pm 1)p_1 \cdots p_s$ where $p_1, \ldots, p_s$ are distinct primes. (For if say $\alpha_1 > 1$, then $p_1^{\alpha_1} \cdots p_s^{\alpha_s} | (p_1 \cdots p_s)^k$ where $k = \max\{\alpha_1, \ldots, \alpha_s\}$, but $p_1^{\alpha_1} \cdots p_s^{\alpha_s} \nmid p_1 \cdots p_s$.) So even though $\mathbf{Z}$ is a τ_0-UFD, a τ_0-irreducible element need not be τ_0-prime and an element of $\mathbf{Z}^\#$ need not be a τ_0-product of τ_0-primes. However, the $|_{\tau_0}$-primes of $\mathbf{Z}^{\#}$ are just the τ_0-irreducibles. Certainly if a is $|_{\tau_0}$-prime, then a is τ_0-irreducible. Conversely, suppose that a is τ_0-irreducible and suppose that $a|_{\tau_0}(\pm 1)b\cdots b$, a τ_0-factorization. So $(\pm 1)a^{\ell} = (\pm)b\cdots b$ for some $\ell \geq 1$. Factor b into τ_0-irreducibles, say $b = (\pm 1)c\cdots c$. Then $(\pm 1)a\cdots a = (\pm 1)c\cdots c$ are two τ_0-atomic factorizations. Hence $a \sim c$. So $(\pm 1)a^m = b$, i.e., $a|_{\tau_0}b$. So a is $|_{\tau_0}$-prime. Note that every element of $\mathbf{Z}^{\#}$ is a τ_0-product of $|_{\tau_0}$-primes.

Several of the questions raised in this paper have been answered by the first author's student Jason Juett in "Two counterexamples in abstract factorization," which will appear in this journal. He has conducted a thorough study of τ-UFDs. He has given a much simpler proof of Theorem 2.11 showing that if $\tau_1 \leq \tau_2$ are divisive and D is a τ_2-UFD, then D is a τ_1-UFD and has shown that if we replace $|_{\tau}$-prime by τ-prime in Theorem 2.7 (3), then (a) $\not\Rightarrow$ (b). He has also given an example of an atomic domain that is not d-atomic (i.e., not a CFD). This answers in the negative the question concerning Figure 2 whether atomic $\Rightarrow$ τ-atomic for τ divisive and the question preceding Theorem 4.2 as to whether an atomic domain is $*$-atomic.

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Department of Mathematics, The University of Iowa, Iowa City, IA 52242
Email address: dan-anderson@uiowa.edu

Department of Mathematics, North Central College, Naperville, IL 60540; Current address: 1452 West Cortez, Apt. 1R, Chicago, IL 60642
Email address: andrea.frazier@gmail.com