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CRYSTALLOGRAPHIC GROUPS AND THEIR MATHEMATICS

DANIEL R. FARKAS*

Introduction. Symmetry, whether found or created, seems to have been a major interest of human beings for thousands of years. When Galois' insights were finally understood by the mathematical community, a precise and fruitful interpretation of symmetry began to play a central role in modern mathematics. One analyzes a gadget by looking for its symmetry; one describes the symmetry by studying transformations natural to the class of gadgets to which this particular one belongs and by isolating those transformations which preserve this symmetry.

The relationship between the every-day notion of symmetry and the theory of groups is lucidly presented in Weyl's classic, *Symmetry* [**4**]. Indeed, any introductory text on abstract algebra is remiss if it does not discuss the dihedral group (usually as the group of symmetries of a regular polygon). Many books on recreational mathematics go one step further and present the classification of "wallpaper designs", the plane crystallographic groups. We can, in fact, thank M. C. Escher for renewed attention to these matters ([**27**], [**36**], [**32**]). Yet it is very difficult for the more mature student to find an exposition of the general theory of n-dimensional crystallographic groups.

The features of an ideal mathematical crystal are easy to describe. One has a pattern that fills up n-space. The pattern can be reconstructed from a small, bounded piece by rigidly moving the piece around space. This is done so that the resulting pattern is "evenly spaced". The challenge is to translate such intuitive notions to actual mathematics.

It should be no surprise that once the groups of symmetries of a crystal were carefully studied, far-flung applications were found. In the past few years, algebraic topologists and differential geometers ([**15**], [**11**], [**10**]) have found these groups useful. Properties are discovered and rediscovered. A scorecard is needed. While these notes are not meant to provide a survey of all theorems using the words "crystallographic group", we hope that the foundations of the subject are pretty much here. We recommend Milnor's article in [**28**] as an introduction to this introduction. (Some of the material in these notes can also be found in

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[**46**], although there is some confusion there as to whether all crystallographic groups are split.) The reader who chafes at the leisurely pace of these notes may wish to read L. Auslander's paper [**3**] first and then just skim this work for added information. Auslander's article inspired much of the thinking in our first five chapters.

Granted that an elementary, coherent treatment of the mathematics of crystallographic groups is desirable, what viewpoints and biases are found here? First of all, we have not tried to present propositions about crystallographic groups as special cases of theorems about discrete subgroups of Lie groups or as corollaries to some more general theory. Most of the arguments are peculiar to crystallographic groups. One advantage is that nearly all results can be proved from first principles, avoiding appeals to complex mathematical machinery. On the other hand, the general style of writing is meant to position this theory within abstract algebra. With due apologies to the crystallographers, we have found that traditional notation obscures some very simple ideas. Consequently our approach is co-ordinate and origin free, as much as makes sense.

For some reason, there is no available account of the "category" of crystallographic groups. It is certainly not well known that normal subgroups of crystallographic groups are crystallographic. Nor are homomorphic images of these groups ever discussed. These notes, having a modern algebraic orientation, address such questions.

There are several technical innovations. The one that we mention here might actually be considered a regression. Recent workers, especially topologists, are enamored of cohomological methods. The attitude taken here is that any result using group cohomology can be proved as well using the notion of "splitting group". We have voted to sacrifice glamour for simplicity. The reader should compare our proofs of Theorems 18 and 22 with the originals, using spectral sequences.

Finally, we should mention what is not contained in these notes. The two-and three-dimensional crystallographic groups are not derived. Lists of these groups and discussions of their classification already exist in the literature ([**17**], [**40**]). The representation theory (character theory) of the crystallographic groups is not included, because the techniques in this area are not intrinsic. For instance, the method used to construct irreducible unitary characters consists of adapting induction theorems for finite groups [**7**]. With much regret, we are unable to present the theorem of Farrell-Hsiang [**15**]; finitely generated projective modules over the integral group ring of a torsion free crystallographic group are stably free. A short, elementary proof of that remarkable theorem has eluded this mathematician for some time.

This essay evolved from lectures given in a seminar including geologists and chemists as well as mathematicians. The questions and criticisms of the participants have much to do with the mixture of abstraction and concreteness in the exposition. The author is particularly indebted to Ed Green for his careful reading and suggestions.

CONTENTS

I. Affine Space. Crystallography is no more than the study of certain permutation groups on points of Euclidean space. We begin by investigating the nicest permutations of $\mathbf{R}^n$: translations and linear transformations.

$\mathbf{R}^n$ can be viewed in two different ways. Algebraically it is frequently depicted as a vector space. We prefer to think of $\mathbf{R}^n$ as a collection of points E together with the set of translations, Trans(E), our old friends the vectors regarded as permutations on E. Trans(E) has two crucial properties:

(i) Trans(E) is an n-dimensional real vector space whose addition coincides with compostion; and

(ii) If P and Q are in E, there is a unique $t \varepsilon$ Trans(E) such that $t(P) = Q$.
E together with its group of translations is referred to as *affine space*.

This description of space does not specify a basis; it does not even pick out an origin. We can clarify this situation by fixing a particular point O in E. According to the unique transitivity, the evaluation map Trans$(E) \to E$ given by $t \mapsto t(O)$ is a one-to-one correspondence. Consequently this map endows E with the structure of a vector space having origin O, which we denote E_O.

If P and Q are two points in E, we obtain an isomorphism between E_P and E_Q via

$$E_P \xleftarrow{\cong} \mathrm{Trans}(E) \xrightarrow{\cong} E_Q.$$

Here if $t \varepsilon$ Trans(E), then the map $E_P \to E_Q$ can be described as sending $t(P)$ to $t(Q)$. Let $\mathbf{PQ} \ \varepsilon$ Trans(E) denote the unique translation sending P to Q. Then

$$\mathbf{PQ}\ (t(P)) = (\mathbf{PQ} + t)(P) = (t + \mathbf{PQ})(P) = t(Q)$$

for all points $t(P)$. This means that $\mathbf{PQ}$: $E_P \to E_Q$ is the original vector space isomorphism.

We need to choose an origin in order to speak about linear transformations. For a fixed O in E we let GL(E_O) comprise the invertible linear transformations from E_O to itself. Clearly the vector space isomorphism $\mathbf{PQ}$: $E_P \to E_Q$ induces a group isomorphism

$$\mathbf{PQ}(\cdot)\mathbf{QP}\colon \mathrm{GL}(E_P) \to \mathrm{GL}(E_Q).$$

We define the *affine group*, Aff(E), to be the group of permutations of E generated by Trans(E) and GL(E_O). According to the previous few sentences, this definition is independent of the choice of O. More can be said.

THEOREM 1. Trans$(E) \lhd$ Aff(E) *and* Trans$(E) \cap$ GL$(E_O) = \{1\}$.

PROOF. The argument consists, mainly, of straightening out all of our identifications. First notice that if $\xi \in$ Trans(E) and $P \in E_O$, then

$$\xi(O) + P = \xi(O) + \mathbf{OP}(O) = (\xi + \mathbf{OP})(O) = \xi(\mathbf{OP}(O)) = \xi(P).$$

Suppose $a \in$ GL(E_O) and $t \in$ Trans(E). Pick any point $P \in E_O$. According to the beginning of the argument $ata^{-1}(P) = a(t(O) + a^{-1}(P))$. Since a is linear, $a(t(O) + a^{-1}(P)) = at(O) + P$. Let s be the unique translation sending O to $at(O)$. Using the first paragraph again, we find $at(O) + P = s(P)$. In other words, ata^{-1} is a translation. Because translations commute, it follows readily that the translation subgroup is normal in the affine group.

Finally, unique transitivity implies the only translation with a fixed point is the identity translation. A linear transformation fixes O and so can be in Trans(E) only if it is 1.

This theorem may be interpreted as giving a description of Aff(E) as an internal semidirect product $\mathrm{Aff}(E) = \mathrm{Trans}(E) \rtimes \mathrm{GL}(E_O)$. To exploit the product, we introduce some notation. Let $\pi_O\colon \mathrm{Aff}(E) \to \mathrm{GL}(E_O)$ denote the obvious projection. If f is an affine permutation of E, define ad $f \in \mathrm{GL}(\mathrm{Trans}(E))$ by $(\mathrm{ad} f)(t) = ftf^{-1}$. If $x \in \mathrm{Aff}(E)$, the diagram below commutes.

$$\text{(AD)} \qquad \begin{array}{ccc} \mathrm{Trans}(E) & \xrightarrow{ev\ at\ O} & E_O \\ {\scriptstyle \mathrm{ad}\, x}\big\downarrow & & \big\downarrow{\scriptstyle \pi_O(x)} \\ \mathrm{Trans}(E) & \xrightarrow{ev\ at\ O} & E_O \end{array}$$

Indeed, if $x = s\pi_O(x) \in \mathrm{Trans}(E) \rtimes \mathrm{GL}(E_O)$, then

$$xtx^{-1} = s(\pi_O(x)t\pi_O(x)^{-1})s^{-1} = \pi_O(x)t\pi_O(x)^{-1}$$

is the unique translation sending O to $\pi_O(x)t(O)$. This has a surprising consequence. If P and Q are two points in E and if $x \in \mathrm{Aff}(E)$, then the linear invariants of $\pi_P(x)$ and $\pi_Q(x)$ are the same.

To give the reader an idea of the use of this line of argument, notice that the diagram (AD) immediately yields Ker(ad) = Trans(E). We single out one linear invariant for later use. If $x \in \mathrm{Aff}(E)$, then its *axial rank* $\rho(x)$ is $\dim\{t \in \mathrm{Trans}\ (E) \mid (\mathrm{ad}\ x)(t) = t\}$. This rank can be regarded as a measure of how far x is from a translation. For if $\rho(x)$ equals the dimension of the entire space, then ad(x) acts like the identity and so x is a translation.

The semidirect product representation can be made explicit by fixing a basis for E_O. The map $\mathrm{Trans}(E) \rtimes \mathrm{GL}(E_O) \to \mathrm{GL}(E_O \oplus \mathbf{R})$ given by

$$t \cdot a \mapsto \left[\begin{array}{c|c} mat\ (a) & t(O) \\ \hline 0 \ldots 0 & 1 \end{array}\right]$$

is an injection. The action of Aff(E) on E can be recovered via the copy of $E_O \subseteqq E_O \oplus \mathbf{R}$ given by $w \mapsto (w, 1)$ and by using the ordinary action of matrices on vectors.

The first theorem suggests an alternate, origin-free definition of the affine group; a permutation f of E is a member of Aff(E) if and only if ad f maps Trans(E) continuously onto Trans(E). (We will be purposely vague here about the relevant topology; as the reader will see in a moment, we only require "continuity with respect to scalar multiplication".) Observe that one direction of the characterization is essentially the normality of the subgroup of translations in Aff(E). On the other hand, suppose $f\colon E \to E$ satisfies the continuous normalizing condition. If t and s are translations,

$$(\operatorname{ad} f)(s + t) = fstf^{-1} = (fsf^{-1})(ftf^{-1}) = (\operatorname{ad} f)(s) + (\operatorname{ad} f)(t).$$

Continuity allows us to jump from the additivity of ad f to its linearity, i.e., ad f can be represented by an element in GL(Trans(E)). If we pick an origin $O \in E$, then we can complete the diagram below with an $a \in \mathrm{GL}(E_O)$.

$$\begin{array}{ccc} \mathrm{Trans}(E) & \xrightarrow{ev\ at\ O} & E_O \\ {\scriptstyle \operatorname{ad} f}\downarrow & & \downarrow{\scriptstyle a} \\ \mathrm{Trans}(E) & \xrightarrow{ev\ at\ O} & E_O \end{array}$$

Let $s = \mathbf{Of(O)}$. We assert that $f = sa$. Let P be any point in E.

$$f(P) = f\mathbf{OP}(O) = f\mathbf{OP}f^{-1}s(O) = sf\mathbf{OP}f^{-1}(O) = sa(P).$$

Hence $f = sa$.

Having developed some mental pictures of Aff(E), we now turn to the algebra of affine spaces. How does one create new affine spaces out of old? What are the interesting subsets and images of an affine space?

The "subobjects" in the theory of affine spaces are called *flats*. They are subsets of E with the form $W(P)$ where W is a subspace of Trans(E) and $P \in E$. Notice that if $Q \in W(P)$, then the translation sending P to Q is in W. An easy calculation now shows that $W(P) = W(Q)$. If we set $F = W(P)$, then we can always identify Trans(F) with the subspace W. How are Aff(E) and Aff(F) related? Suppose that x is an element of Aff(E) such that $x(F) = F$. We will have established that $x|F$ is in Aff(F) if we show that ad(x)(Trans(F)) = Trans(F). Let $O \in F$ and $t \in$ Trans(F). Then xtx^{-1} is a translation sending O to a point in F, and so is in Trans(F). Therefore ad(x)(Trans(F)) $\subseteqq$ Trans(F); the opposite inclusion follows from considering x^{-1} in place of x.

Invariant flats can arise in several ways. First notice that the unique transitivity property of Trans(E) implies that $(\bigcap_I W_i)(P) = \bigcap_I (W_i(P))$ for any family, $\{W_i | \, i \in I\}$, of subspaces of Trans(E). Consequently it makes sense to talk about the smallest flat containing a set of points in E. Suppose Γ is a subgroup of Aff(E). In the case that Γ describes the symmetry of a crystal, our experience tells us that this crystal should fill up the whole space. Space is a flat with lots of symmetry! If $P \in E$, let $W(P)$ be the smallest flat containing the orbit of P under Γ. At the least we expect this flat to inherit the "symmetry" of the orbit itself. This may have two interpretations.

For one thing, W is invariant under ad Γ. To see this, we first claim that (ad x) $W(P)$ contains $y(P)$ for all x, y in Γ. Choose $w \in W$ such that $x^{-1}P = wP$ and $v \in W$ such that $x^{-1}yP = vP$. Then

$$x(vw^{-1})x^{-1}P = xvw^{-1}wP = xvP = yP.$$

Since (ad x)$W(P)$ contains the full orbit of P, $W \subseteqq$ (ad x)W. Similarly, $W \subseteqq$ (ad x^{-1})W = (ad x)^{-1}W, so (ad x)$W \subseteqq W$. In summary, if F is the flat generated by an orbit or Γ, then Trans(F) is ad Γ-invariant.

Consider the general situation in which F is any flat in E and $z \in$ Aff(E) has the property that ad(z) (Trans(F)) = Trans(F). We would like to conclude that $z(F) = F$. Unfortunately, there are counterexamples. Suppose E is the ordinary co-ordinate plane, t is translation one unit (horizontally) to the right and z is reflection in the x-axis. Clearly $ztz^{-1} = t$. Let F be the flat through (0, 1) with translation group $\mathbf{R}t$. Although ad(z)$\mathbf{R}t = \mathbf{R}t$, this flat is not invariant under reflection. However, suppose there is always a point P in F such that $z(P) \in F$. Any point Q in F can be written $Q = s(P)$ where $s \in$ Trans(F). $z(Q) = zs(P) =$ ad(z)(s)$z(P)$, so $z(Q) \in F$. Repeating the argument with z^{-1} instead of z, we see that $z(F) = F$. In the case that Γ is a subgroup of Aff(E) and F is the smallest flat containing the orbit of the point P, we automatically know that $x(P) \in F$ for all $x \in \Gamma$. Consequently, the flat generated by an orbit of Γ is invariant under every element in Γ.

It is not so obvious what we should choose as the appropriate definition of "morphism". Suppose E and F are affine spaces and α: $E \to F$ is a function. One hope is that an x in Aff(E) corresponds to an x^* in Aff(F) somehow compatible with α, say

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[d, "x"'] & F \arrow[d, "x^*"] \\ E \arrow[r, "\alpha"] & F \end{tikzcd}$$

To avoid ambiguity, we limit ourselves to the situation in which α is surjective. In that case, observe that a unique permutation x^* can be produced for x precisely when $\alpha(P) = \alpha(Q) \Leftrightarrow \alpha x(P) = \alpha x(Q)$. With this criterion, it is easy to see that those elements of Aff(E) which can fit into the commutative square comprise a subgroup, which we write $C_{\mathrm{Aff}(E)}(\alpha)$. If x and y are in $C_{\mathrm{Aff}(E)}(\alpha)$, then

$$x^*y^* \circ \alpha = x^* \circ \alpha \circ y = \alpha \circ xy = (xy)^* \circ \alpha.$$

In summary, α induces a homomorphism which we now denote $\hat{\alpha}$ from $C_{\mathrm{Aff}(E)}(\alpha)$ to the permutations of F.

An example of particular interest arises from a linear transformation f: Trans(E) $\to$ Trans(E). Pick an origin $O \in E$ and and let F be the flat

$(\operatorname{Im} f)(O)$. We regard F as the affine space whose set of translations is $\operatorname{Im} f$. Evaluation at O defines a map f_O: $E \to F$ via

$$\begin{array}{ccc} \operatorname{Trans}(E) & \xrightarrow{ev\ at\ O} & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_O} \\ \operatorname{Im} f & \xrightarrow{ev\ at\ O} & F. \end{array}$$

(Notice that f_O: $E_O \to F_O$ is linear.) We claim that $C_{\operatorname{Aff}(E)}(f_O) = \{x \in \operatorname{Aff}(E) \mid \operatorname{ad}(x)\operatorname{Ker} f = \operatorname{Ker} f\}$.

Suppose $x \in C_{\operatorname{Aff}(E)}(f_O)$ and $v \in \operatorname{Ker} f$. We have $f(v)(O) = O = f(0)(O)$, so $f_O(v(O) = f_O(O)$. Hence $f_O(xv(O)) = f_O(x(O))$. Let $s = \mathbf{Ox(O)}$. Then $f_O((\operatorname{ad}(x)v + s)(O)) = f_O(s(O))$. That is, $f(\operatorname{ad}(x)v + s) = f(s)$. We conclude that $\operatorname{ad}(x)v \in \operatorname{Ker} f$. Because $\operatorname{ad}(x)$ is injective, $\operatorname{ad}(x)\operatorname{Ker} f = \operatorname{Ker} f$.

Conversely, suppose that $\operatorname{ad}(x) \operatorname{Ker} f = \operatorname{Ker} f$. Then the map $\operatorname{Im} f \to \operatorname{Trans}(E)$ given by $f(\xi) \mapsto f \circ \operatorname{ad}(x)(\xi)$ is a well-defined linear transformation. Similarly, this transformation is injective. Therefore it can be extended to a map in $\operatorname{GL}(\operatorname{Trans}(E))$. Since ad: $\operatorname{Aff}(E) \to \operatorname{GL}(\operatorname{Trans}(E))$ is surjective, there is a $y \in \operatorname{Aff}(E)$ such that $\operatorname{ad}(y) \circ f = f \circ \operatorname{ad}(x)$. There is a degree of latitude in the choice of y in that $\operatorname{ad}(y)$ is not affected when composing y with a translation. Thus we may assume $y(O) = x(O)$. Let P be an arbitrary point in E; set $t = \mathbf{OP}$ and $s = \mathbf{Ox(O)}$.

$$\begin{aligned} f_O(xP) &= f_O(xt(O)) \\ &= f_O((\operatorname{ad}(x)t + s)(O)) \\ &= [f(\operatorname{ad}(x)t + s)](O) \\ &= [(f \circ \operatorname{ad}(x))(t) + f(s)](O) \\ &= [(\operatorname{ad} y \circ f)(t) + f(s)](O) \\ &= (f(s) - s) \circ [(\operatorname{ad} y \circ f)(t) + s](O) \\ &= [(f(s) - s)y] f_O(P). \end{aligned}$$

Since x fits into the diagram

$$\begin{array}{ccc} E & \xrightarrow{f_O} & F \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle (f(s)-s)y} \\ E & \xrightarrow{f_O} & F \end{array}$$

we find that $x \in C_{\operatorname{Aff}(E)}(f_O)$. A proof of the claim is completed.

We record the calculation in the previous paragraph as the *affine morphism formula*; if f_O: $E \to F$ is a surjective linear transformation induced from an endomorphism f on $\operatorname{Trans}(E)$, then $\hat{f}_O(x) = (f(s) - s)y$ is an affine permutation where $s = \mathbf{Ox(O)}$, $\operatorname{ad}(y) \circ f = f \circ \operatorname{ad}(x)$, and $y(O) = x(O)$.

REFERENCES. The ancestor of most abstract treatments of affine geometry is E. Artin's *Geometric Algebra* **[1]**. A more elementary source is **[47]**.

II. Euclidean Space. Trans(E), a finite dimensional real vector space, can be equipped with an inner product $\langle \cdot, \cdot \rangle$. This induces an inner product $\langle \cdot, \cdot \rangle_O$ on each E_O via the evaluation map. We would like to restrict our attention to those maps in Aff(E) which preserve distance between points. These "rigid motions" will be our source of symmetries. So let us fix an origin O and suppose $x \in$ Aff(E) has the property that $|P - Q|_O = |x(P) - x(Q)|_O$ for all P and Q in E_O. Notice that $x(P) = (x\mathbf{OP}x^{-1} + \mathbf{Ox(O)})(O)$. Using this equality, we can lift the original distance equality to an equality in Trans(E).

$$|\mathbf{OP} - \mathbf{OQ}| = |x\mathbf{OP}x^{-1} + \mathbf{Ox(O)} - x\mathbf{OQ}x^{-1} - \mathbf{Ox(O)}|$$

Since the translation $\mathbf{OP} - \mathbf{OQ}$ can be arbitrary, we have proved that the original equality is equivalent to $|t| = |(ad\ x)(t)|$ for all $t \in$ Trans(E). Any norm-preserving linear map on a real inner product space preserves the inner product. Hence $x \in$ Aff(E) preserves the E_O-distance function if and only if ad $x \in \mathbb{O}$(Trans(E)), the orthogonal group. In particular, x will now preserve the distance function associated with other origins.

We say $x \in$ Aff(E) is an *isometry* if ad $x \in \mathbb{O}$(Trans(E)) and we denote the group of all isometries by Isom(E). Since ad is the identity map when restricted to translations, we see that Trans(E) $\lhd$ Isom(E). The diagram (AD) in §1 shows that $x \in$ Isom(E) if and only if $\pi_P(x) \in \mathbb{O}(E_P)$ for each (or any) $P \in E$. Hence Isom(E) = Trans(E) $\rtimes$ $\mathbb{O}(E_P)$.

Let us review some facts about orthogonal transformations. If V is an inner product space and $A \in \mathbb{O}(V)$, set $\Phi(A) = \{v \in V \mid Av = v\}$. It is not difficult to show that $\Phi(A)^\perp$ is invariant under A. Observe that $A - 1$ acts like the zero transformation on $\Phi(A)$ but must be invertible on $\Phi(A)^\perp$. Hence $\Phi(A)^\perp = (A - 1)V$. As an application, if $v \in V$, then there exists $w \in V$ such that $v - (A - 1)w \in \Phi(A)$.

Recall that if G is any group and H is a subgroup of G, then the centralizer of $g \in G$ in H is $C_H(g) = \{h \in H \mid hg = gh\}$. We now have two notations for the same concept. For if $x \in$ Isom(E), then $C_{\mathrm{Trans}(E)}(x) = \{t \in \mathrm{Trans}(E) \mid xtx^{-1} = t\} = \Phi(\mathrm{ad}\ x)$.

THEOREM 2 (Axis Theorem). *If* $x \in$ Isom(E), *then* $xC_{\mathrm{Trans}(E)}(x)$ *has an element with a fixed point.*

PROOF. According to the semidirect product decomposition, there exists an $s \in$ Trans(E) such that sx has a fixed point Q. Choose $w \in$ Trans(E) with $s - (\mathrm{ad}\ x - 1)\, w \in \Phi(\mathrm{ad}\ x)$.

$$\begin{aligned}[(\mathrm{ad}\ x - 1)\ w - s](wQ) &= [xwx^{-1} - w - s + w](Q)\\ &= (xwx^{-1})(xQ)\\ &= x(wQ).\end{aligned}$$

Hence $[s - (\mathrm{ad}\ x - 1)\ w]x = x[s - (\mathrm{ad}\ x - 1)w]$ has a fixed point.

This theorem has a geometric interpretation. Suppose $t \in C_{\mathrm{Trans}(E)}(x)$ and $tx(P) = P$. Then $x = t^{-1}(tx)$ is the decomposition of x with respect to the origin P. The translational part, t^{-1}, of x is invariant under $\pi_P(x)$ in that $\pi_P(x)(t^{-1}P) = (txt^{-1})(P) = x(P) = t^{-1}tx(P) = t^{-1}P$. In other words, an isometry can always be decomposed so that its translational part is along the axis of its orthogonal part.

COROLLARY. Fix $x \in \mathrm{Isom}(E)$ *and let* $C = C_{\mathrm{Isom}(E)}(x)$. *Then there is a point in E whose orbit under C is contained in a flat of dimension* $\rho(x)$.

PROOF. Suppose $xy = yx$ for $y \in \mathrm{Isom}(E)$. By the theorem we may choose an origin P so that $x = ta$ and $y = sb$ in $\mathrm{Trans}(E) \rtimes ⓛ\ (E_P)$, and $t \in \Phi(\mathrm{ad}\ x)$. Then $xy = t\ \mathrm{ad}(x)(s)ab$ and $yx = s\ \mathrm{ad}(y)(t)ba$. These are both decompositions with respect to the origin P. We must have $(\mathrm{ad}\ x - 1)s = (\mathrm{ad}\ y - 1)\ t$. Since ad x and ad y commute, $(\mathrm{ad}\ y - 1)t \in \Phi(\mathrm{ad}\ x)$. Hence $(\mathrm{ad}\ x - 1)\ s \in \Phi(\mathrm{ad}\ x) \cap \Phi(\mathrm{ad}\ x)^{\perp}$. We conclude that $s \in \Phi(\mathrm{ad}\ x)$.

Therefore $y(P) = s(P) \in \Phi(\mathrm{ad}\ x)(P)$. We are done once we remember $\rho(x) = \dim \Phi(\mathrm{ad}\ x)$.

As to morphisms, the general discussion of the previous section carries over verbatim once we replace Aff(E) with Isom (E). We shall need a special case of the example. Suppose V is a subspace of Trans(E) and p: Trans $(E) \to V$ is an orthogonal projection. If $x \in \mathrm{Isom}(E)$, then V is invariant under ad(x) if and only if $V^{\perp}$ is invariant. Notice that $V^{\perp} = \mathrm{Ker}\, p$. Moreover, in this case $\mathrm{ad}(x) \circ p = p \circ \mathrm{ad}(x)$. If an origin O is chosen, then with a slight abuse of notation $\hat{p}$: $\{x \in \mathrm{Isom}(E) \mid \mathrm{ad}(x)V = V\} \to \mathrm{Isom}(V(O))$ is given by the affine morphism formula $\hat{p}(x) = (p(s) - s)x$ restricted to $V(O)$ where $s = \mathbf{Ox(O)}$. We reall refer to $\hat{p}$ as the projection of the domain with respect to V and the new equality as the *Euclidean morphism formula.*

Trans(E) has a norm topology derived from its inner product. If $t_n \to t$ in Trans(E), then $t_n(O) \to t(O)$ in E_O with its topology derived from the induced inner product. Since $s(O) + P = s(P)$ for any translation s and point P, we see that $|t_n(P) - t(P)|_O = |t_n(O) - t(O)|_O$. Consequently $t_n \to t$ in the topology of uniform convergence for E_O. Reversing this argument we see that if $t_n(P) \to t(P)$ for some point P in E_O, then $t_n \to t$ in Trans(E). In summary, the toplogies of pointwise convergence and uniform convergence of Trans (E) on E_O agree with the intrinsic topology.

A careful look at these calculations shows that if $P_n \to P$ in E_O, then $P_n \to P$ in E_Q for any other $Q \in E$. Indeed, let v_n (respectively v) be the unique translation sending O to P_n (resp. O to P). If $P_n \to P$ in E_O, then $v_n \to v$ in Trans(E). Therefore, $v_n(O) \to v(O)$ in E_Q. From now on we may as well speak about the topology on E, and remember that it agrees with the norm topology on each E_Q.

It now makes sense to adopt as the topology for Isom(E), the topology of point-wise convergence on E. Pick an origin O and consider t, $t_n \in$ Trans(E), and a, $a_n \in \mathbb{O}(E_O)$ such that $t_n a_n \to ta$. In particular $t_n a_n(O) \to ta(O)$, so $t_n \to t$ in Trans (E). If $P \in E$, let s_n(respectively s) be the unique translation sending O to $a_n(P)$ (resp. $a(P)$). Then $t_n a_n(P) \to ta(P)$ implies $(t_n + s_n)\ (O) \to (t + s)(O)$. Hence $t_n + s_n \to t + s$ in Trans(E). Subtracting, we find that $s_n \to s$ and therefore that $s_n(O) \to s(O)$. That is, $a_n(P) \to a(P)$ for each $P \in E_O$. This means that $a_n \to a$ pointwise. A repetition of much the same argument allows us to conclude that the topology of point-wise convergence on Isom(E) coincides with the product topology on the set Trans(E) $\rtimes$ $\mathbb{O}(E_O)$. Stated in another useful way, $x_n \to x$ in Isom(E) if and only if $\mathrm{ad}(x_n) \to \mathrm{ad}(x)$ and $x_n(Q) \to x(Q)$ for your favorite $Q \in E$.

It is well known that the topology of point-wise convergence in GL(E_O) is the same as the topology induced from the operator norm, $\|\cdot\|$. Recall that $\|a\| = \sup\{|a(w)|/|w| \mid w \neq 0\}$. (Here the norm on the vectors is the one arising from the inner product on E_O.) We shall study this norm in great detail in a later section. One of the crucial properties of $\mathbb{O}(E_O)$ with the norm (or point-wise convergence) topology is that it is compact. This is most easily seen by choosing an orthonormal basis for E_O and writing out a matrix by its columns as an element of $E_O \times E_O \cdots \times E_O$. Then $\mathbb{O}(E_O)$ is a closed subset of $S(E_O) \times S(E_O) \cdots \times S(E_O)$ where $S(E_O)$ is the unit sphere in E_O.

THEOREM 3. Isom(E) *with the topology described above is a topological group.*

PROOF. What we mean to prove is that if f, f_n, g, and g_n are in Isom(E) with $f_n \to f$ and $g_n \to g$, then $f_n g_n \to fg$ and $f_n^{-1} \to f^{-1}$.

We first prove that if $P_n \to P$, then $f_n(P_n) \to f(P)$. Fix an origin O and write $f_n = t_n a_n$ and $f = ta$ in Trans(E) $\rtimes$ $\mathbb{O}(E_O)$. Then

$$|a_n(P_n) - a(P)|_O \leqq \|a_n\|\ |P_n - P|_O + \|a_n - a\|\ |P|_O.$$

According to remarks above, $\|a_n - a\| \to 0$. Hence $a_n(P_n) \to a(P)$. By using the trick (a few paragraphs back) of turning $a_n(P_n)$ into the "target" of a translation, we see that $t_n a_n(P_n) \to ta(P)$.

It is now clear that $f_n \to f$ and $g_n \to g$ imply $f_n g_n \to fg$. Again, $f_n \to f$

implies $t_n \to t$ and $a_n \to a$. We leave it as an exercise to check that $t_n^{-1} \to t^{-1}$ and $a_n^{-1} \to a^{-1}$. By the first part of the proof, $a_n^{-1}t_n^{-1} \to a^{-1}t^{-1}$, i.e., $f_n^{-1} \to f^{-1}$.

REFERENCES. Halmos' text **[21]** remains a good reference for the properties of orthogonal matrices as well as basic information about norms and topologies on the space of linear transformations.

Theorem 2 is an elaboration of a remark of L. Auslander at the beginning of his Lemma 5 **[3]**. It is found implicitly in the usual classification of plane isometries when odd isometries without fixed points are shown to be glide reflections.

III. Crystallographic Groups. We are finally prepared to introduce the star. A subgroup Γ of $\mathrm{Isom}(E)$ is a *crystallographic group* provided it is discrete and $\mathrm{Isom}(E)/\Gamma$ is compact.

(When we say that Γ is discrete, we mean that if y_n and y are in Γ such that $y_n \to y$, then the sequence of y_n is eventually constant. Clearly it is always enough to test this for $y = 1$. Furthermore, Γ is automatically closed in $\mathrm{Isom}(E)$. For if y_n is in Γ and $x \in \mathrm{Isom}(E)$ such that $y_n \to x$, then $y_n y_{n+1}^{-1} \to 1$. Hence y_n is constant for large n; x is already in Γ.)

Each part of this definition must be reconciled with our intuition about crystals. "Discrete" should mean that the crystal pattern never "bunches up". That is, there must be no accumulation points in the orbit of a point. Suppose Γ is discrete but the orbit of $O \in E$ has an accumulation point. Then we can find $x_i \in \Gamma$ with the points $x_i(O)$ distinct but Cauchy. Write $x_i = t_i a_i \in \mathrm{Trans}(E) \rtimes \mathbb{O}(E_O)$. Since the orthogonal group is compact, we can find a subsequence of the a_i which converges. Since $x_i(O) = t_i(O)$, the sequence of t_i is Cauchy and so converges. Therefore a subsequence of distinct x_i converges in $\mathrm{Isom}(E)$, contradicting discreteness.

Actually, the converse is true. If Γ acts on E so that no orbit has an accumulation point, then Γ is discrete. Suppose $x_j \to 1$ in Γ. If $P \in E$, then $x_j(P) \to P$. That means $x_j(P) = P$ for all sufficiently large j. But then such x_j must lie in $\mathbb{O}(E_P)$. Being linear, x_j is determined by its action on finitely many points. On the other hand, at a given point x_j has the same value for large j. Therefore x_j is eventually constant. Of course this constant must be 1.

A crystal should have a bounded pattern that is repeated until it fills up space. This repeating cell can be described mathematically as a *covering domain*, a subset $D \subseteqq E$ such that for each $P \in E$ there is an $x \in \Gamma$ with $x(P) \in D$. We require that Γ act on E with a compact covering domain. Note that the definition of the quotient topology implies that $\mathrm{Isom}(E)/\Gamma$ is compact precisely when there is a compact subset of $\mathrm{Isom}(E)$ with the property that any isometry can be left multiplied into the set by a member of Γ. (We could say that Γ has a compact covering domain for its action by multiplication on $\mathrm{Isom}(E)$). We claim that Γ acts with a compact covering domain in one case precisely when it does in the other.

A trivial calculation shows that if $C \subseteq \mathrm{Isom}(E)$ is a compact covering domain for the left multiplication action of Γ and $P \in E$ is any point, then $C(P)$ is a compact covering domain for Γ on E. Conversely, if D is a compact covering domain for Γ on E, fix an origin $O \in D$ and set $C = \{x \in \mathrm{Isom}(E) | x(O) \in D\}$. C is clearly a covering domain for Γ on $\mathrm{Isom}(E)$. Furthermore C is compact because $C \cap \mathrm{Trans}(E)$ is homeomorphic to D and $C = (C \cap \mathrm{Trans}(E)) \rtimes \mathbb{O}(E_O)$.

We turn to the "category" of crystallographic groups. As we shall see later, any finite group can appear as a subgroup of a crystallographic group while no finite group can act transitively enough to fill up space. Although arbitrary subgroups of crystallographic groups are not crystallographic, large subgroups are.

THEOREM 4. *Suppose $\Lambda \subseteq \Gamma$ are two subgroups of* $\mathrm{Isom}(E)$ *with $|\Gamma : \Lambda| < \infty$. Then Γ is crystallographic if and only if Λ is crystallographic.*

PROOF. Let $y_1, \ldots, y_d$ be left coset representatives for Λ in Γ.

($\Rightarrow$) If Γ is discrete, certainly Λ is. Suppose D is a compact covering domain for Γ acting on E. Then $\bigcup_j y_j^{-1} D$ is compact. If $P \in E$, there exists $x \in \Gamma$ such that $x(P) \in D$. But $x = y_i x'$ for some $x' \in \Lambda$, so $x'(P) \in y_i^{-1} D$.

($\Leftarrow$) Suppose $x_j \in \Gamma$ are distinct elements such that $x_j \to 1$. By dropping to a subsequence we may assume that $x_j = y_\alpha x_j'$ for a fixed α and with $x_j' \in \Lambda$. Then $x_j' \to y_\alpha^{-1}$, a statement about convergence in Λ. But Λ is discrete. Therefore a subsequence of the x_j are constantly 1, a contradiction. Notice that a covering domain for Λ is automatically a covering domain for Γ.

We will prove much later that a normal subgroup of a crystallographic group remains crystallographic in an appropriate sense.

One might expect that the projection of a crystallographic group is a crystallographic group. This is incorrect. In our example E is the plane and $\mathrm{Trans}(E)$ is $\mathbf{R}^2$ with the usual inner product. Pick an origin O once and for all. If $t = (r_1, r_2) \in \mathrm{Trans}(E)$, then we adopt the convention of letting (r_1, r_2) denote the point $t(O)$ as well. We leave it to the reader to show that $\mathbf{Z}^2 \subseteq \mathbf{R}^2$ is a crystallographic subgroup of $\mathrm{Isom}(E)$. Since this group consists entirely of translations, every subspace of $\mathbf{R}^2$ is $\mathrm{ad}(\mathbf{Z}^2)$-invariant. Consider the orthogonal projection $p: \mathbf{R}^2 \to \mathbf{R}(1, \sqrt{2})$ given by

$$p = \begin{bmatrix} 1/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & 2/3 \end{bmatrix}.$$

The group $\hat{p}(\mathbf{Z}^2)$ acts on the one dimensional space $\mathbf{R}(1, \sqrt{2})$. If $t \in \mathbf{Z}^2$, then the Euclidean morphism formula when applied to a pure translation says that $\hat{p}(t) = p(t)$; $\hat{p}(\mathbf{Z}^2)$ is the group of translations on $\mathbf{R}(1, \sqrt{2})$ generated by $(1/3, \sqrt{2}/3)$ and $(\sqrt{2}/3, 2/3)$. Choose sequences of integers a_n and b_n so that $a_n + b_n \sqrt{2} \to 0$ (Kronecker's Theorem, cf. [**22**]). $a_n(1/3, \sqrt{2}/3) + b_n(\sqrt{2}/3, 2/3)$ tends to zero without ever being eventually constant. Thus the orbit of the origin has an accumulation point. Unfortunately, $\hat{p}(\Gamma)$ is not a discrete group of isometries.

This counterexample motivates the next definition. Suppose Γ is a subgroup of $\mathrm{Isom}(E)$. A subspace $W \subseteq \mathrm{Trans}(E)$ is *Γ-rational* if it is spanned by a subgroup of $\Gamma \cap \mathrm{Trans}(E)$. If $W \subseteq \mathrm{Trans}(E)$ is Γ-rational, then one can find a basis for W with elements in $\Gamma \cap W$. There is a compact ball B (around 0) in W such that $W = B + (\Gamma \cap W)$. (Think of the example $\mathbf{R} = [-1/2, 1/2] + \mathbf{Z}$). Intuitively, W and $\Gamma \cap W$ are the same "up to a compact set".

THEOREM 5. *Suppose Γ is a crystallographic subgroup of* $\mathrm{Isom}(E)$ *and V is a subspace of* $\mathrm{Trans}(E)$ *whose orthogonal complement is* ad *Γ-invariant and Γ-rational. Then the projection of Γ with respect to V is crystallographic.*

PROOF. Let p: $\mathrm{Trans}(E) \to V$ be an orthogonal projection. Fix an origin O in E and set $F = V(O)$, a Euclidean space whose set of translations is V. Recall that the projection $\hat{p}$: $\Gamma \to \mathrm{Isom}(F)$ is defined by

$$\begin{tikzcd} E \arrow[r, "p_O"] \arrow[d, "x"'] & F \arrow[d, "\hat{p}(x)"] \\ E \arrow[r, "p_O"] & F \end{tikzcd}$$

If D is a covering domain for Γ acting on E, then $p_O(D)$ is a covering domain for $\hat{p}(\Gamma)$ acting on F. $p_O(D)$ is certainly compact when D is.

We consider discreteness and suppose x_n is a sequence in Γ such that $\hat{p}(x_n) \to 1$. The strategy is to show that the sequence x_n, or a reasonable facsimile, converges in Γ. Since the orthogonal group is compact, we may drop to a subsequence and assume $\mathrm{ad}(x_n)$ converges. We need to show that $x_n(O)$ converges. First look at $(1 - p_O)(x_n(O))$ in E_O. Because $V^\perp$ is Γ-rational, the projection of $x_n(O)$ onto $V^\perp$ can be pushed into a compact set by a translation in $\Gamma \cap V^\perp$. So again drop to a subsequence and choose $w_n \in \Gamma \cap V^\perp$ such that $(1 - p_O)(w_n x_n(O))$ converges. Notice that $\hat{p}(w_n x_n) = \hat{p}(w_n)\hat{p}(x_n) = \hat{p}(x_n)$ because p annihilates $V^\perp$. Hence $p_O(w_n x_n(O)) = \hat{p}(w_n x_n)(O) = \hat{p}(x_n)(O) \to O$. That is, $\mathrm{ad}(w_n x_n) = \mathrm{ad}(x_n)$ converges and $w_n x_n(O)$ converges. We conclude that $w_n x_n$ converges in Γ. Since Γ is discrete, $w_n x_n$ is eventually constant. But then $\hat{p}(x_n)$ is eventually the constant 1. This shows that any sequence $\hat{p}(x_n) \to 1$ has a subsequence consisting of 1 only, a statement equivalent to discreteness.

Recall that the center of an abstract group is the collection of all elements which commute with every member of the group.

COROLLARY. *If Γ is a crystallographic subgroup of* Isom(E) *and* $\mathscr{Z}$ *denotes its center, then* $\Gamma/\mathscr{Z}$ *can be represented as a crystallographic group whose group of translations is the image of* $\Gamma \cap$ Trans(E) *modulo* $\mathscr{Z}$.

PROOF. Assume F is any flat $V(O)$ and p: Trans$(E) \to V$ is an orthogonal projection. We begin by fixing a translation $w \in V$ and generally calculating $\hat{p}^{-1}(w)$. So suppose $x \in \Gamma$, $s = \mathbf{Ox(O)}$, and $\hat{p}(x) = w$. According to the Euclidean morphism formula, $(p(s) - s)x$ acts like w on $V(O)$;

$$[(p(s) - s)\ \mathrm{ad}(x)(v)s](O) = wv(O) \text{ for all } v \in V.$$

The expression in brackets is a translation. Unique transitivity implies $(p(s) - s) + (\mathrm{ad}(x)(v)) + s = w + v$ for all $v \in V$. That is, $(\mathrm{ad}(x) - 1)(v) = w - p(s)$ for all $v \in V$. Thus $(\mathrm{ad}(x) - 1)\ V$ is a subspace of Trans(E) consisting of the single element $w - p(s)$. This forces $w - p(s) = (\mathrm{ad}(x) - 1)(v) = 0$ for all v. Therefore $\hat{p}^{-1}(w) = \{x \in \Gamma \mid V \subseteqq \Phi(\mathrm{ad}(x))$ and $w = p(\mathbf{Ox(O)})\}$.

Consider the special case of the corollary. Let W be the span of $\mathscr{Z}$ in Trans(E), set $V = W^{\perp}$, and write p: Trans$(E) \to V$ for the corresponding orthogonal projection. W is clearly ad(Γ)-invariant and Γ-rational. If $x \in \Gamma$, then ad(x) fixes every member of the center and so $W \subseteqq \Phi(\mathrm{ad}(x))$. Therefore $V \subseteqq \Phi(\mathrm{ad}(x))$ implies x is a translation on E.

If we regard the identity isometry as the zero translation, we see that our calculation yields Ker $\hat{p} = \{t \in \Gamma \cap \mathrm{Trans}(E) \mid t \in W\} = \mathscr{Z}$. By Theorem 5, $\Gamma/\mathscr{Z}$ is crystallographic. Further suppose t is a translation in Γ. Since $\hat{p}(t) = p(t)$, the image of t is certainly a translation in V. On the other hand, if $\hat{p}(x)$ is a translation in V, then x is a translation. Hence $\hat{p}(\Gamma) \cap V = \hat{p}(\Gamma \cap \mathrm{Trans}(E))$.

Ultimately our goal is to find a completely algebraic description of crystallographic groups. To begin, we analyze those groups which are abelian.

LEMMA. *Suppose Γ is a group of isometries of E which acts with a compact covering domain. Then the smallest flat containing the orbit of a point is always the entire space E.*

PROOF. Pick a point P in E. We decided earlier that if $W(P)$ is the smallest flat containing the orbit of P, then W is invariant under ad(Γ).

We might as well assume that the compact covering domain is a ball in E_P of radius d around P. If $W \neq$ Trans(E), choose $t \in W^{\perp}$ with $|t| > d$. For any $x \in \Gamma$ there is a $w \in W$ such that $x(P) = w(P)$. Then ad$(x)t$ and w are always perpendicular. Hence

$$|xt(P)|_P = |\mathrm{ad}(x)(t) + w| \geqq |\mathrm{ad}(x)(t)| = |t| > d.$$

That is, no element in Γ can move $t(P)$ into the ball, a contradiction.

THEOREM 6. *The center in a group of isometries with compact covering domain consists wholly of translations.*

PROOF. Suppose $\Gamma \subseteqq \mathrm{Isom}(E)$ is the group. If $x \in \Gamma$ is central, then there is a point P whose orbit under $C_{\mathrm{Isom}(E)}(x)$ is contained in a flat of dimension $\rho(x)$. (See the corollary to Theorem 2.) But by the previous lemma, the smallest flat containing the orbit of P under $\Gamma \subseteqq C(x)$ is already E. Hence $\rho(x)$ is the dimension of the underlying space; x is a pure translation.

THEOREM 7. *A discrete subgroup of* $\mathbf{R}^n$ *has a finite set of* $\mathbf{R}$*-independent generators.*

PROOF. We argue by induction on n for the discrete group A. If $n = 1$, choose $x \in A$ of minimal positive length. Clearly $A = \mathbf{Z}x$.

If $n > 1$, once again choose $x \in A$ of minimal positive length. As above, $A \cap \mathbf{R}x = \mathbf{Z}x$. Hence if $x_2, \ldots, x_d \in A$ are $\mathbf{R}$-independent and generate A modulo $\mathbf{R}x$, then $x_1 = x, x_2, \ldots, x_d$ have the desired properties for A.

We need only prove that the image of A in $\mathbf{R}^n/\mathbf{R}x$ is discrete and apply induction. We must show that if $a_t \in A$ and $a_t \to 0 \pmod{\mathbf{R}x}$, then $a_s \in \mathbf{R}x$ for all large s. So assume there is a sequence $r_t \in \mathbf{R}$ such that $a_t - r_t x \to 0$. By adding integer multiples of x to a_t we may assume that $|r_t| \leqq 1/2$. For large s, $|a_s - r_s x| < (1/2)|x|$, so $|a_s| < |x|$. Therefore $a_s = 0$.

COROLLARY. *If Γ is an abelian crystallographic subgroup of* $\mathrm{Isom}(E)$ *then Γ is a free abelian group whose rank is* $\dim \mathrm{Trans}(E)$ *and whose span is all of* $\mathrm{Trans}(E)$.

Such a group is frequently called a *lattice* in $\mathrm{Trans}(E)$.

REFERENCES. Most of this chapter follows [3]. The discussion of projections of crystallographic groups in Theorem 5 is an attempt to place L. Auslander's reduction (in his proof of Bieberbach's First Theorem) in a broader context.

IV. Finite Groups. In the previous chapter we captured some of the torsion free structure of crystallographic groups by looking at abelian groups. Now we shall look at the other end of the spectrum and study finite subgroups of isometries. We begin by eliminating the "translational aspects" of this study.

THEOREM 8. *If G is a finite subgroup of* $\mathrm{Aff}(E)$, *then there is a point in E simultaneously fixed by all members of G.*

PROOF. Fix any origin O. If P is any point in E_O, set

$$P' = \frac{1}{|G|} \sum_{y \in G} y(P) \in E_O.$$

P' is familiar as the "center of mass" for the orbit of P. Write $x \in G$ as $x = ta \in \mathrm{Trans}(E) \rtimes \mathrm{GL}(E_O)$. Then

$$\begin{aligned} x(P') &= t\left(\frac{1}{|G|} \sum_y ay(P)\right) \\ &= t(O) + \frac{1}{|G|} \sum_y ay(P) \\ &= \frac{1}{|G|} \sum_y (t(O) + ay(P)) \\ &= \frac{1}{|G|} \sum_y tay(P) \\ &= \frac{1}{|G|} \sum_y (xy)(P) = P'. \end{aligned}$$

This theorem states that a finite subgroup of $\mathrm{Aff}(E)$ really sits inside $\mathrm{GL}(E_P)$ for some point P. Before refining this result, we digress for a short while to discuss a lovely example of how geometric intuition can be used to verify algebraic properties of crystallographic groups.

LEMMA. *Suppose Γ is a discrete subgroup of* $\mathrm{Isom}(E)$. *If D is a compact subset of E, then $\{x \in \Gamma \mid D \cap x(D) \neq \varnothing\}$ is finite.*

PROOF. Pick an origin O and let $C = \{x \in \mathrm{Isom}(E) \mid x(O) \in D\}$. When discussing compact covering domains we saw that C was compact. Then $C \cdot C^{-1} \subseteqq \mathrm{Isom}(E)$ is compact. (This set is a continuous image of $C \times C$.) It is easy to check that $x \in C \cdot C^{-1}$ when $D \cap x(D) \neq \varnothing$. But a (closed) discrete subset of a compact set is finite.

THEOREM 9. *A crystallographic group has only finitely many conjugacy classes of finite subgroups.*

PROOF. $\Gamma \subseteqq \mathrm{Isom}(E)$ is the group in question. By virtue of Theorem 8, if H is a finite subgroup of Γ, then the elements of H fix some point P in E. Let D denote a compact covering domain for Γ on E and choose $y \in \Gamma$ such that $y(P) \in D$. For all $h \in H$, $(yhy^{-1})(yP) = yP$. Thus, $D \cap yhy^{-1}D \neq \varnothing$. That is, every finite subgroup of Γ is conjugate to a subgroup inside the finite set $\{x \in \Gamma \mid D \cap x(D) \neq \varnothing\}$.

Crystallographic groups happen to appear in algebraic topology in roughly the following way. There is a useful class of compact Riemannian manifolds whose universal covering space is E and whose fundamental group consists of isometries on E. If the elements of the group are thought of as deck transformations of E, then they must act freely.

In general, a group of isometries *acts freely* on E if the only element with a fixed point is the identity element. The "non-finiteness" proposition below is another application of Theorem 8.

THEOREM 10. *A crystallographic group acts freely if and only if it is torsion free.*

PROOF. Γ is a crystallographic subgroup of Isom(E). If Γ acts freely, then Theorem 8 implies that $\{1\}$ is its only finite subgroup. Conversely, suppose Γ is torsion free. If $x \in \Gamma$ and $P \in E$ are such that $x(P) = P$, then $x \in \Gamma \cap \mathbb{O}(E_P)$. But $\Gamma \cap \mathbb{O}(E_P)$ is a discrete, closed subgroup of a compact group. Since $\Gamma \cap \mathbb{O}(E_P)$ is finite, it must be $\{1\}$.

We return to our finite subgroups of the general linear group. For obvious reasons, it is desirable to move such a group into the orthogonal group.

THEOREM 11. *Let V be a finite dimensional inner product space. If G is a finite subgroup of* GL(V), *then there is an $a \in$ GL(V) such that $aGa^{-1} \subseteqq \mathbb{O}(V)$.*

PROOF. We define a new inner product on V by $\langle v, w\rangle' = \sum_{g\in G}\langle gv, gw\rangle$. If $e_1, \ldots, e_n$ is an orthonormal basis for $\langle \cdot, \cdot \rangle$ and $e_1', \ldots, e_n'$ is an orthonormal basis for $\langle \cdot, \cdot \rangle'$ define $a \in \mathrm{GL}(V)$ by $ae_i' = e_i$. Thus $\langle av, aw\rangle = \langle v, w\rangle'$. If $h \in G$, then

$$\langle aha^{-1}v, aha^{-1}w\rangle = \langle ha^{-1}v, ha^{-1}w\rangle' = \langle a^{-1}v, a^{-1}w\rangle' = \langle v, w\rangle.$$

Some control is needed on the classes of finite groups which can appear as subgroups of isometries on a particular space E. The following theorem may seem more relevant if one thinks of GL(n, **Z**) as the set of elements in GL(Trans(E)) leaving the lattice $\mathbf{Z}^n$ invariant.

THEOREM 12. *If p is an odd prime, then the kernel of the natural map* $\mathrm{GL}(n, \mathbf{Z}) \to \mathrm{GL}(n, \mathbf{Z}/(p))$ *is torsion free.*

PROOF. Let H be the kernel. If the result is false, there is an element $a \in H$ of prime order q. Write $a = I + pb$ where $b \neq 0$.

$$I = a^q = (I + pb)^q = I + qpb + \sum_{j=2}^{q}\binom{q}{j}p^j b^j.$$

Subtracting and then dividing by p we find

$$(*) \qquad qb + \sum_{j=2}^{q}\binom{q}{j}p^{j-1}b^j = 0.$$

Consider the highest power α such that p^α divides every entry of b. The terms to the right of $\sum$ in (*) have a factor of pb^2, so an entry of such a term is divisible by $p^{2\alpha+1}$. Hence $2\alpha + 1 \leqq \alpha$ if $p \neq q$ and $2\alpha \leqq \alpha$ if $p = q$. The first case is clearly impossible. If $p = q$, then $\alpha = 0$; divide (*) by p,

$$b + \binom{p}{2} b^2 + \sum_{j=3}^{p} \binom{p}{j} p^{j-2} b^j = 0.$$

Since $p > 2$, the entries of b are divisible by p, a contradiction.

COROLLARY. *For a fixed n,* $\mathrm{GL}(n, \mathbf{Z})$ *contains only finitely many isomorphism classes of finite groups.*

PROOF. Every finite subgroup of $\mathrm{GL}(n, \mathbf{Z})$ faithfully embeds in the finite group $\mathrm{GL}(n, \mathbf{Z}/(3))$.

REFERENCES. Detail on the role of crystallographic groups as fundamental groups of flat Riemannian manifolds (cf. our Theorem 10) can be found in **[46]**.

Theorem 11 is credited in **[12]** to Loewy and Moore just before the turn of the century.

Theorem 12 is due to Minkowski **[31]**; our proof comes from **[43]**.

V. Bieberbach's Characterization Theorem. The beginning of this chapter is devoted to the first major mathematical theorem of crystallography, Jordan's Theorem. As a prerequisite, we need some basic facts about the operator norm.

Suppose V is a finite dimensional real inner product space. Recall that if $a\colon V \to V$ is a linear transformation, then $\|a\| = \sup \{|a(v)| \mid |v| = 1\}$. The first three properties are standard.

(i) $\|\cdot\|$ is a norm on the vector space of linear transformations.

(ii) $\|1\| = 1$ and $\|ab\| \leqq \|a\| \, \|b\|$ for all transformations a and b.

(iii) $\|ax\| = \|xa\| = \|a\|$ for any transformation a and $x \in \mathbb{O}(V)$. (In particular, $\|x\| = 1$ for all $x \in \mathbb{O}(V)$).

If ε is a positive real number, we set $B(\varepsilon) = \{x \in \mathbb{O}(V) \mid \|x - 1\| < \varepsilon\}$.

(iv) $B(\varepsilon)$ is closed under inverses and conjugation by elements in $\mathbb{O}(V)$. Indeed, if $x \in B(\varepsilon)$, then $\|x^{-1} - 1\| \leqq \|x^{-1}\| \, \|1 - x\| = \|x - 1\|$. If, in addition, $y \in \mathbb{O}(V)$, then $\|yxy^{-1} - 1\| = \|y(x - 1)y^{-1}\| = \|x - 1\|$.

Fix an orthonormal basis $e_1, \ldots, e_n$ for V and let (α_{ij}) be the matrix for the transformation $a\colon V \to V$ with respect to this basis.

(v) $|\mathrm{tr}\, a| \leqq n \, \|a\|$.

Notice that $|\alpha_{11}| \leqq (|\alpha_{11}|^2 + |\alpha_{21}|^2 + \cdots + |\alpha_{n1}|^2)^{1/2}$. Hence $|\alpha_{11}| \leqq |a(e_1)|$. Therefore

$$|\mathrm{tr}\, a| \leqq \sum_{i=1}^{n} |\alpha_{ii}| \leqq \sum_{i=1}^{n} |a(e_i)|.$$

Choosing the largest summand, we see that there is a j such that $|\mathrm{tr}(a)| \leqq n|a(e_j)|$. Since $|e_j| = 1$, the result follows.

(vi) $\|a\| \leqq \sqrt{n} \max |\alpha_{ij}|$.
If $v = \sum v_i e_i$ is such that $|v| = (\sum v_i^2)^{1/2} = 1$, then

$$|\langle av, av\rangle| = |\sum_i(\sum_j \alpha_{ij}v_j)^2| \leqq \sum_i(\sum_j |\alpha_{ij}|\,|v_j|)^2.$$

Setting $\mu = \max |\alpha_{ij}|$, we find

$$|\langle av, av\rangle| \leqq \mu^2 \sum_i \sum_j |v_j|^2 = \mu^2 n.$$

LEMMA A. *If g and h are in $B(1/2)$, then $ghg^{-1}h^{-1} = [g, h] \in B(1/2)$ and the sequence*

$$[g, h], [g, [g, h]], [g, [g, [g, h]]], \ldots$$

tends to 1.

PROOF. We first claim that if $x, y \in \mathbb{O}(V)$, then

$$\|[x, y] - 1\| \leqq 2\, \|x - 1\|\, \|y - 1\|.$$

For

$$\begin{aligned} \|xyx^{-1}y^{-1} - 1\| &= \|(xy - yx)x^{-1}y^{-1}\| \\ &= \|xy - yx\| && \text{by (iii)}, \\ &= \|(x - 1)(y - 1) - (y - 1)(x - 1)\|, \\ &\leqq 2\|x - 1\|\, \|y - 1\| && \text{by (i) and (ii)}. \end{aligned}$$

The first part of the lemma is now immediate. If we apply induction to the formula above we obtain

$$\|\overbrace{[g, [g, \ldots [g,}^{m} h] \cdots] - 1\| \leqq 2^m \|g - 1\|^m \|h - 1\|.$$

Since $2\|g - 1\| < 1$, $(2\|g - 1\|)^m \to 0$.

LEMMA B. *Assume $g, h \in \mathbb{O}(V)$ with $h \in B(1)$. If $[g, [g, h]] = 1$, then $[g, h] = 1$.*

PROOF. We first reduce to the case that none of the eigenvalues of $[g, h]$ are 1. Set $W = \Phi([g, h])^{\perp}$, and let $p: V \to W$ be an orthogonal projection. The lemma is immediate if $W = 0$. Assume otherwise.

If $f: V \to V$ is any linear transformation, let us write $\bar{f} = p \circ f \mid W$, a linear transformation on W. $p \circ g = g \circ p$ since g and $[g, h]$ commute, while $p \circ [g, h] = [g, h] \circ p$ for obvious reasons. As a consequence, $\bar{g} \in \mathrm{GL}(W)$ and $\bar{g}\overline{[g, h]} = \overline{[g, h]}\bar{g}$. However we warn that $\bar{h}$ need not be invertible. We nearly get a commutator formula;

$$\overline{[g, h]}\ \bar{h} = p(ghg^{-1}h^{-1})ph|W = pghg^{-1}|W = g(ph)g^{-1}|W = \bar{g}\bar{h}\bar{g}^{-1}.$$

Arguing inductively we find $(\overline{[g, h]})^k\bar{h} = (\bar{g})^k\bar{h}(\bar{g})^{-k}$. Take traces;

$$(*) \qquad \mathrm{tr}(\overline{[g,h]}^k\bar{h}) = \mathrm{tr}(\bar{h}).$$

It is easy to see that $\|\bar{h} - 1\| < 1$ in the restricted operator norm on W.

Let χ be the characteristic polynomial of $\overline{[g, h]}$. Applying (*) we obtain $0 = \mathrm{tr}(\chi(\overline{[g, h]})\bar{h})) = \mathrm{tr}(\bar{h})\chi(1)$. The point of restricting to W is that 1 is not a eigenvalue of $\overline{[g, h]}$ and so $\chi(1) \neq 0$. Hence $\mathrm{tr}(\bar{h}) = 0$.

Then by property (v), $\dim W = |\mathrm{tr}(\bar{h} - 1)| \leqq (\dim W)\|\bar{h} - 1\| < \dim W$, a contradiction.

LEMMA C. *If Γ is a discrete subgroup of* $\mathrm{Isom}(E)$, *then the elements of* $B(1/2) \cap \mathrm{ad}\,\Gamma$ *commute with each other.*

PROOF. Suppose $x, y \in \Gamma$ are such that $\mathrm{ad}(x), \mathrm{ad}(y) \in B(1/2)$. Write $x = (u, \pi_O(x)$ and $y = (v, \pi_O(y))$ with respect to some origin O.

$$\begin{aligned}[x, y] = ((1 &- \mathrm{ad}(x)\,\mathrm{ad}(y)\,\mathrm{ad}(x)^{-1})\,u \\ &+ \mathrm{ad}\,x(1 - \mathrm{ad}(y)\,\mathrm{ad}(x)^{-1}\,\mathrm{ad}(y)^{-1})v, [\pi_O(x), \pi_O(y)])\end{aligned}$$

By the invariance of $B(1/2)$, the length of the translational part of $[x, y]$ is less than $(|u| + |v|)/2$. Thus if $|u|$ and $|v|$ are less than the real number M, so is the absolute value of the translational part of $[x, y]$.

This argument shows that the sequence

$$[x, y], [x, [x, y]], [x, [x, [x, y]]], \ldots$$

lies in the compact set $\{w \in \mathrm{Trans}(E) |\ |w| \leqq M\} \rtimes \mathbb{O}(E_O)$. Consider a convergent subsequence. By Lemma A, ad applied to the subsequence tends to 1. On the other hand Γ is discrete, so the subsequence itself is eventually stationary. Hence for some m.

$$\underbrace{[\mathrm{ad}(x), [\mathrm{ad}(x), \ldots, [\mathrm{ad}}_{m}(x), \mathrm{ad}(y)]]\cdots] = 1.$$

By Lemmas A and B, $\mathrm{ad}(x)$ and $\mathrm{ad}(y)$ commute.

In Jordan's Theorem below, E is an n-dimensional Euclidean space.

THEOREM 13. *If Γ is a discrete subgroup of* $\mathrm{Isom}(E)$, *there is a normal abelian subgroup $A \subseteqq \mathrm{ad}(\Gamma)$ such that*

$$|\mathrm{ad}(\Gamma): A| \leqq (6\sqrt{n} + 1)^{n^2}.$$

(In particular this holds for all finite subgronps of $\mathbb{O}(\mathbf{R}^n)$.)

PROOF. Let A be the subgroup generated by $B(1/2) \cap \mathrm{ad}(\Gamma)$. According to Lemma C, A is abelian.

Fix an orthonormal basis for Trans(E). If $g \in$ ①(Trans (E)) is represented by a matrix with respect to this basis, each of its entries lies in the interval $[-1, 1]$. This gives us a function ① (Trans(E)) $\to [-1, 1]^{n^2}$ afforded by the entries of the matrix. If we wish to write this hypercube as the union of hypercubes with side $1/3\sqrt{n}$, the number of little hypercubes we shall need on each side is the least integer exceeding $2 \cdot 3\sqrt{n}$. Thus the total number of little hypercubes is at most $(6\sqrt{n} + 1)^{n^2}$.

If we are given $g_1, g_2, \ldots, g_N \in \operatorname{ad}(\Gamma)$ with $N > (6\sqrt{n} + 1)^{n^2}$, then at least two of these elements lie in the same small hypercube. That is, there are g_i and g_j such that none of the entries of $g_i - g_j$ exceed $1/3\sqrt{n}$ in absolute value.

$$\|g_i g_j^{-1} - 1\| = \|g_i - g_j\| \leqq \sqrt{n}\,\frac{1}{3\sqrt{n}} < \frac{1}{2}$$

by property (vi). Thus $g_i g_j^{-1} \in A$. We conclude that there cannot be N distinct cosets of A.

The bridge between geometry and algebra is found in the remarkable theorem which follows, due to Bieberbach. The balance of these notes can be regarded as commentary on this one result.

THEOREM 14. *Let Γ be a crystallographic subgroup of* Isom (E). *Then*

(1). $\Gamma \cap$ Trans(E) *is a finitely generated abelian group of rank* $n =$ dim E *which spans* Trans (E), *and*

(2). ad $\Gamma \cong \Gamma/\Gamma \cap$ Trans(E), *the point group of Γ, is finite.*

PROOF. Let W be the span of $\Gamma \cap$ Trans(E). Observe that ad(Γ) leaves $\Gamma \cap$ Trans(E) invariant and so W is both ad(Γ)-invariant and Γ-rational.

We claim that ad(Γ)$|W$ is a discrete (closed) subgroup of ①(W). For if x_i is a sequence in Γ such that ad$(x_i)\,|\,W \to 1\,|\,W$ and $w \in \Gamma \cap$ Trans (E), then ad$(x_i)w \to w$. But ad$(x_i)w \in \Gamma$ and Γ is discrete. Thus ad$(x_i)\, w = w$ for large i. The claim follows from allowing w to range over a basis for W.

Since ①(W) is compact, ad(Γ)$|W$ is finite. The kernel Γ' of the natural map $\Gamma \to$ ad(Γ)$\,|\,W$ is a subgroup of finite index in Γ with $\Gamma' \cap$ Trans(E) $=$ $\Gamma \cap$ Trans(E). Therefore we may assume that ad(Γ)$\,|\,W = \{1\}$.

The proof would be complete if we knew W was all of Trans(E). To determine how much we have missed, let p: Trans(E) $\to W^{\perp}$ be an orthogonal porjection, fix an origin $0 \in E$ and set $F = W^{\perp}(0)$. According to Theorem 5, $\hat{p}(\Gamma)$ is a crystallographic subgroup of Isom(F). On the other hand, $\hat{p}(\Gamma)$ contains no translations other than the identity element. Indeed, suppose $x \in \Gamma$ is chosen so that $\hat{p}(x)$ is a translation. Then ad $\hat{p}(x) =$ 1. But the Euclidean morphism formula implies ad $\hat{p}(x) =$ ad$(x)|W^{\perp}$; by assumption, ad$(x)|W = 1$. Thus ad$(x) = 1$, i.e., $x \in \Gamma \cap$ Trans(E). Since p annihilates W, $\hat{p}(x) = 1$.

We have really decided that the restriction of ad: $\hat{p}(\Gamma) \to \mathbb{O}(W^{\perp})$ is injective. By Jordan's Theorem, $\hat{p}(\Gamma)$ has an abelian subgroup of finite index. This subgroup is an abelian crystallographic subgroup of Isom(F), a lattice in Trans(F). Of course a lattice contains infinitely many translations unless F is just a single point. This must be the case; $W^{\perp} = 0$.

We immediately see that a crude partition of the crystallographic subgroups of Isom(E) can be made according to their point groups. Notice that ad Γ acts faithfully in a linear fashion on $\Gamma \cap$ Trans(E), a lattice. If n is the dimension of E, then every lattice in Trans(E) is isomorphic to $\mathbf{Z}^n$. In other words, ad Γ can be faithfully represented by $n \times n$ integer matrices. The corollary to Theorem 12 contends that there are only finitely many isomorphism classes of point groups for crystallographic subgroups of Isom(E).

The subgroup $\Gamma \cap$ Trans(E) of Γ can be described geometrically as the subgroup consisting of translations. There is an algebraic characterization as well. If Γ is any group, we can define its *finite conjugate subgroup* $\Delta(\Gamma) = \{x \in \Gamma \mid |\Gamma: C_\Gamma(x)| < \infty\}$. x is in $\Delta(\Gamma)$ precisely when it has finitely many conjugates. It is straightforward to verify that $\Delta(\Gamma)$ is actually a characteristic subgroup of Γ.

We shall need a simple criterion for recognizing the finite conjugate subgroup.

LEMMA. *Assume H is a group and A is a normal, torsion free abelian subgroup with $|H: A| < \infty$. Then $A = \Delta(H)$ if and only if A is self-centralizing. (That is, the only elements of H which centralize A are already in A.)*

PROOF. Suppose $A = \Delta(H)$ and x centralizes A. That means $A \subseteq C_H(x)$. Since $|H: A| < \infty$, we have $x \in \Delta(H)$.

Conversely, assume that A is self-centralizing and $x \in \Delta(H)$. x acts trivially on $A \cap C_H(x)$, which is a subgroup of finite index in A. Hence if $a \in A$, there is a positive integer m such that $a^m \in A \cap C_H(x)$. Thus $(xax^{-1})^m = a^m$. Because A is a torsion free abelian group, $xax^{-1} = a$ so that x centralizes all of A.

Look at the crystallographic subgroup $\Gamma \subseteq$ Isom(E). By Bieberbach's Theorem, $\Gamma \cap$ Trans(E) is a normal, torsion free abelian subgroup of Γ with $\Gamma/\Gamma \cap$ Trans(E) finite. If x in Γ centralizes $\Gamma \cap$ Trans(E), then ad(x) $= 1$. In other words, $x \in \Gamma \cap$ Trans(E). This establishes that $\Delta(\Gamma) = \Gamma \cap$ Trans(E). Now define an abstract group Γ to be a *Bieberbach group* provided $\Delta(\Gamma)$ is a finitely generated torsion free abelian subgroup and $\Gamma/\Delta(\Gamma)$ is finite. We refer to the rank of $\Delta(\Gamma)$ as the *dimension* of Γ. The definition is tailor-made to prove the following theorem.

THEOREM 15. *A crystallographic group is a Bieberbach group.*

We shall prove a converse in the next chapter.

REFERENCES. Our approach to Jordan's Theorem [**23**] borrows heavily from [12], who follows Schur's argument [**38**]. One innovation is the generalization from finite groups to discrete groups of isometries. At the expense of an added paragraph in Lemma C, we are then able to avoid L. Auslander's appeal to Lie group theory [**3**].

The proof of Theorem 14 is essentially that in [**3**]. The original reference is [**4**].

VI. The splitting group. Suppose Γ is a Bieberbach group. For the remainder of this chapter we will use the notation $\Delta = \Delta(\Gamma)$ and $G = \Gamma/\Delta$. Some of the nomenclature for crystallographic groups will be carried over. G will be referred to as the point group of Γ. The map $\mathrm{ad}\colon \Gamma \to \mathrm{Aut}(\Delta)$ is again defined by $\mathrm{ad}(x)(a) = xax^{-1}$; it induces a faithful map which we call $\overline{\mathrm{ad}}\colon G \to \mathrm{Aut}\,(\Delta)$.

How much information must we have about Δ and G to recover Γ? Choose right coset representatives $\{x_g | g \in G\}$ for Δ in Γ so that x_g is mapped to g by the homomorphism $\Gamma \to \Gamma/\Delta$. There is an element $f(g, h)$ in Δ such that $x_g x_h = f(g, h)x_{gh}$. Notice that if a and b are in Δ, then

$$(1) \qquad (ax_g)(bx_h) = [a \cdot \overline{\mathrm{ad}}(g)(b) \cdot f(g,\ h)]x_{gh}.$$

Therefore the information we need is a rank for Δ, the finite point group G, an action of G on Δ, and this correction factor $f\colon G \times G \to \Delta$.

Let Δ^* be a group isomorphic to Δ and embed $\Delta \subseteqq \Delta^*$ as the image of the map $\Delta^* \to \Delta^*$ given by $a \mapsto a^{|G|}$. If one thinks of Δ as a lattice, then Δ^* is obtained by subdividing Δ by a factor of $|G|$. More precisely, if we regard Δ as an additive group, then Δ^* is the subgroup $(1/|G|)\,\Delta$ of $\mathbf{Q} \otimes_{\mathbf{Z}} \Delta$. In any event, $\overline{\mathrm{ad}}\colon G \to \mathrm{Aut}(\Delta)$ extends to a map $\overline{\mathrm{ad}}\colon G \to \mathrm{Aut}\,(\Delta^*)$. Using the original correction factor and the obvious extension of formula (1) to the case when a and b are in Δ^*, we obtain a new group Γ^* with the property that $\Gamma^*/\Delta^* \cong G$. We call Γ^* the *splitting group* for Γ.

We mention some of the features of Γ^*. Because the extension of $\overline{\mathrm{ad}}$ remains faithful, the argument at the end of the previous chapter shows that $\Delta(\Gamma^*) = \Delta^*$. Consequently Γ^* is a Bieberbach group with the same dimension as Γ. By construction Γ is a subgroup of Γ^*. In fact, $|\Gamma^* : \Gamma| = |\Delta^* : \Delta| = |G|^{\dim(\Gamma)}$.

The associativity of the group operation in Γ gives rise to the following identity for f.

$$(2) \qquad f(g, h) \cdot f(gh, k) = \overline{\mathrm{ad}}(g)[f(h, k)] \cdot f(g, hk)$$

for all g, h, and k in G. Take the product of both sides of (2) as k ranges over all of the elements in G and remember that Δ is abelian.

$$f(g, h)^{|G|} \cdot \prod_{k \in G} f(gh, k) = \overline{\mathrm{ad}}(g)[\prod_{k \in G} f(h, k)] \cdot \prod_{k \in G} f(g, hk).$$

If we define $\sigma: G \to \Delta^*$ by $\sigma(d) = (\Pi_{k \in G} f(d, k))^{1/|G|}$, then

$$f(g, h) \cdot \sigma(gh) = \overline{\mathrm{ad}}(g)(\sigma(h)) \cdot \sigma(g). \tag{3}$$

We can pick new coset representatives for Δ^* in Γ^* by setting $y_g = \sigma(g)^{-1}x_g$. Calculating,

$$\begin{aligned} y_g y_h &= [\sigma(g)^{-1} \cdot \overline{ad}(g)(\sigma(h)^{-1}) \cdot f(g, h)]x_{gh} \\ &= \sigma(gh)^{-1}x_{gh} && \text{by (3)} \\ &= y_{gh}. \end{aligned}$$

In other words, Γ^* is isomorphic to a semidirect product, $\Delta^* \rtimes G$!

Let us look at this construction geometrically. Assume that the Bieberbach group Γ happens to sit inside Isom(E) as a crystallographic group. Then $\Delta = \Gamma \cap \mathrm{Trans}(E)$ and the point group of Γ acts on this Δ^* as an extension of the way it acts on Δ. A little thought shows that the construction of Γ^* from Γ can be performed inside Isom(E). Since $|\Gamma^*: \Gamma| < \infty$, Γ^* is a crystallographic subgroup as well. Write $s: G \to \Gamma^*$ for the injection which allows us to write $\Gamma^* = \Delta^* \rtimes s(G)$. By Theorem 8, $s(G)$ has a fixed point P. The semidirect product $\Delta^* \rtimes s(G)$. coincides with the product structure inherited from $\mathrm{Isom}(E) = \mathrm{Trans}(E) \rtimes ①(E_P)$. A crystallographic group which is split in this way is called a *symmorph*.

The splitting group will play an intermediate role in many of the subsequent arguments. It is frequently easy to reduce a theorem to the split case and then prove it for this simpler type of group. As the first example of this strategy, we prove that Bieberbach groups can be realized as crystallographic groups.

THEOREM 16. *Let Γ be an n-dimensional Bieberbach group. Then Γ can be realized as a crystallographic subgroup of* Isom(E) *so that* $\Delta(\Gamma) = \Gamma \cap \mathrm{Trans}(E)$, *where E is n-dimensional Euclidean space.*

PROOF. We first assume Γ is split; $\Gamma = \Delta(\Gamma) \rtimes G$ for some finite subgroup G. Because the rank of Δ is the same as the dimension of Trans(E), there is an injective homomorphism $\phi: \Delta \to \mathrm{Trans}(E)$ which sends Δ to a lattice. The faithful action of G on Δ gives rise to an injection $\phi': G \to \mathrm{GL}(\mathrm{Trans}(E))$. By Theorem 11, there is a conjugate of the finite group $\phi'(G)$ inside ①(Trans (E)). Since the image of a lattice under an invertible transformation is still a lattice, we may assume that ϕ sends Δ to a lattice of translations and $\phi'(G) \subseteq ①(\mathrm{Trans}(E))$. Pick an origin O and let $e: ①(\mathrm{Trans}(E)) \to ①(E_O)$ be the isomorphism defined by the (AD) diagram. Define $\tilde{\phi}: \Delta \rtimes G \to \mathrm{Isom}(E)$ by $\tilde{\phi}(ag) = \phi(a)e\ \phi'(g)$. The split nature of Γ is compatible, via $\tilde{\phi}$, with the splitting of Isom(E) with respect to the origin O. Thus to check that $\tilde{\phi}$ is an injective homomorphism it suffices to verify that the action of G on Δ is consistent with the action of $\mathbb{O}(E_O)$ on Trans(E). This is true by construction.

$\tilde{\phi}(\Delta) = \phi(\Delta)$ is a lattice and so is clearly a crystallographic subgroup of Isom(E). The same holds for $\tilde{\phi}(\Gamma)$ because the index of $\tilde{\phi}(\Delta)$ is $\tilde{\phi}(\Gamma)$ is finite (Theorem 4).

If Γ is not already split, split it. There is an injection $\tilde{\phi}$: $\Gamma^* \to$ Isom(E) such that $\tilde{\phi}(\Gamma^*)$ is a crystallographic subgroup of Isom(E) and $\tilde{\phi}(\Gamma^*) \cap$ Trans$(E) = \tilde{\phi}(\Delta^*)$. By Theorem 4, the subgroup $\tilde{\phi}(\Gamma)$ is crystallographic.

$$\begin{aligned}\tilde{\phi}(\Gamma) \cap \mathrm{Trans}(E) &= \tilde{\phi}(\Gamma) \cap \tilde{\phi}(\Gamma^*) \cap \mathrm{Trans}(E)\\ &= \tilde{\phi}(\Gamma) \cap \tilde{\phi}(\Delta^*) = \tilde{\phi}(\Delta).\end{aligned}$$

As our second application of the splitting group, we prove that a normal subgroup of a crystallographic group is crystallographic. First, we require the algebraic version of this theorem.

LEMMA. *Suppose Γ is a Bieberbach group and $\Lambda \lhd \Gamma$. Then $\Delta(\Lambda) = \Lambda \cap \Delta(\Gamma)$. In particular, Λ is a Bieberbach group.*

PROOF. Certainly $\Lambda \cap \Delta(\Gamma)$ is a normal finitely generated torsion free abelian subgroup of Λ with $|\Lambda: \Lambda \cap \Delta(\Gamma)| < \infty$. Thus it suffices to show that this subgroup is self-centralizing.

Let x be an arbitrary element of Γ. Picking a basis for $\mathbf{Q} \otimes_{\mathbf{Z}} \Delta(\Gamma)$, ad$(x)$ can be regarded as an element of finite order in GL$(n, \mathbf{Q})$ for n the dimension of Γ. Consequently it is diagonalizable over $\mathbf{C}$. In particular ad$(x) - 1$ is nilpotent only when x is in $\Delta(\Gamma)$.

Assume $x \in \Lambda$ centralizes $\Lambda \cap \Delta(\Gamma)$. That means $(\mathrm{ad}(x) - 1)(\Lambda \cap \Delta(\Gamma)) = 1$. The normality of Λ in Γ implies that $xtx^{-1}t^{-1} \in \Lambda \cap \Delta(\Gamma)$ for all $t \in \Delta(\Gamma)$. In other notation, $(\mathrm{ad}(x) - 1)\, \Delta(\Gamma) \subseteqq \Lambda \cap \Delta(\Gamma)$. Hence $(\mathrm{ad}(x) - 1)^2 = 0$. We conclude that $x \in \Delta(\Gamma)$.

THEOREM 17. *Suppose Γ is a crystallographic subgroup of* Isom(E) *and $\Lambda \lhd \Gamma$. Then there is a flat F inside E such that* Trans(F) *is the span of $\Lambda \cap$* Trans(E) *and Λ is a crystallographic subgroup of* Isom(F).

PROOF. We can construct the splitting group Λ^* inside Isom(E). This means that there is a point P in E such that

$$\Lambda^* = (\Lambda^* \cap \mathrm{Trans}(E)) \rtimes (\Lambda^* \cap \mathbb{O}(E_P))$$

and $\Lambda^* \cap$ Trans(E) is contained in the span of $\Lambda \cap$ Trans(E). Let F be the flat containing P whose translation group is this span.

By construction, ad(x) (Trans (F)) = Trans (F) for all $x \in \Lambda$. In addition, $x(P) = t(P)$ for some $t \in \Lambda^* \cap$ Trans(E); hence $x(P)$ is in F for all $x \in \Lambda$, An early observation implies that F is invariant under all elements of Λ.

Consequently there is a homomorphism $\Lambda \to \text{Isom}(F)$ wihch restricts each element in Λ to its action on F.

We claim that this homomorphism is injective. If $x \in \Lambda$ acts like the identity on F, then for all $t \in \Lambda \cap \text{Trans}(E)$, $xt(P) = t(P)$. Since $x(P) = P$, the unique transitivity of translations yields $\text{ad}(x)t = t$. That is, x centralizes $\Lambda \cap \Delta(\Gamma)$. By the previous lemma, x is a translation (as are all of the members of $\Delta(\Gamma)$.) On the other hand, $x(P) = P$. Hence $x = 1$.

View Λ as a subgroup of $\text{Isom}(F)$. Now $\Lambda \cap \text{Trans}(E)$ is a subgroup of finite index in Λ, so we will be finished if it is a crystallographic subgroup of $\text{Isom}(F)$. But this is obvious since $\Lambda \cap \text{Trans}(E)$ is a lattice in its span, $\text{Trans}(F)$.

As yet another illustration of the splitting technique, we establish a result that appears in connection with Smale's theory of dynamical systems **[39]**. We say that a *norm* $|\cdot|$ on an abstract group H is a function from H to the non-negative real numbers such that $|x| = 0$ if and only if $x = 1$, $|x| = |x^{-1}|$ for all $x \in H$, and $|xy| \leqq |x| + |y|$ for all $x, y \in H$. An endomorphism $f\colon H \to H$ is *expansive* if there is a real number $s > 1$ such that $|f(x)| \geqq s|x|$ for all $x \in H$. It is said to be *s-expansive* if there is equality for all x. (We discuss one consequence of having an expansive map in the appendix.)

Now suppose Γ is a torsion free Bieberbach group. Pick an origin O in E and embed Γ^* symmorphically as a crystallographic subgroup of $\text{Isom}(E) = \text{Trans}(E) \rtimes \textcircled{1}(E_O)$. If $x \in \Gamma$, define the *geodesic norm* of x to be the Euclidean norm of the translational part of x. We briefly check that the geodesic norm is indeed a norm. Say $x = ag$ is the decomposition of an element in Γ with respect to the origin O. If $|x| = 0$, then $a = 0$. This means $x \in \textcircled{1}(E_O) \cap \Gamma$. However that subgroup is finite. Since we are assuming Γ is torsion free, $x = 1$. Notice that $x^{-1} = \text{ad}(g^{-1})(a^{-1})g^{-1}$. But $\text{ad}(g^{-1})$ is in the orthogonal group; $|x^{-1}|$ is the Euclidean norm of a^{-1}, which is the Euclidean norm of a. A similar calculation handles the triangle inequality.

THEOREM 18. *Suppose Γ is a torsion free Bieberbach group with point group G. If s is a positive integer such that $s \equiv 1 \pmod{|G|}$, then Γ possesses an s-expansive homomorphism with respect to the geodesic norm.*

PROOF. Recall that Γ^* is embedded symmorphically in $\text{Trans}(E) \rtimes \textcircled{1}(E_O)$. If $f\colon \Gamma^* \to \Gamma^*$ is defined by $f(ag) = a^s g$, it is easy to see that f is a homomorphism. We are done if $f(\Gamma) \subseteqq \Gamma$.

If $x = ag \in \Gamma$, then $xf(x)^{-1} = agg^{-1}a^{-s} = a^{1-s}$. But $(\Delta^*)^{|G|} \subseteqq \Delta$. Hence $xf(x)^{-1} \in \Gamma$. Obviously $f(x) \in \Gamma$.

References. The calculations at the beginning of the chapter, arising from the associativity of the group operation, are a standard part of group cohomology, cf. [35].

Theorem 16 was first observed by Zassenhaus [49].

A discussion of expansive maps is found in [39]. Epstein and Shub [13] prove Theorem 18 using spectral sequences.

VII. Classification. In his celebrated list of problems, David Hilbert asked, "Is there in n-dimensional Euclidean space ... only a finite number of essentially different kinds of groups of motions with a [compact] fundamental region?" This problem had been previously solved by Fedorov [16] and Schönfliess [37] for three dimensional groups, with arguments unique to solid geometry. In 1910, Bieberbach answered Hilbert in the affirmative.

In settling Hilbert's problem we must first come to grips with a suitable difinition of "equivalence" for crystallographic groups. If Γ_1 and Γ_2 are two crystallographic subgroups of $\mathrm{Isom}(E)$, three successively finer equivalence relations can be studied.

(i) Γ_1 and Γ_2 are isomorphic as abstract groups.

(ii) Γ_1 and Γ_2 are conjugate subgroups of $\mathrm{Aff}(E)$.

(iii) Γ_1 and Γ_2 are subgroups of $\mathrm{Aff}(E)$ which are conjugate by an element in the oriented affine group, $\mathrm{Aff}^+(E)$.

The last possibility allows us to distinguish between right-handed and left-handed crystals. Here $\mathrm{Aff}^+(E) = \{x \in \mathrm{Aff}(E) \mid \det(\mathrm{ad}\, x) > 0\}$. It is clear that $|\mathrm{Aff}(E)\colon \mathrm{Aff}^+(E)| = 2$, so an equivalence class according to (ii) is split into at most two equivalence classes when orientation is taken into consideration. Given a crystallographic subgroup $\Gamma \subseteqq \mathrm{Isom}(E)$, by an *enantiomorph* of Γ we mean a crystallographic subgroup of $\mathrm{Isom}(E)$ conjugate to Γ in $\mathrm{Aff}(E)$ but not conjugate via an element in $\mathrm{Aff}^+(E)$.

Suprisingly, abstract isomorphism and conjugacy in the full affine group are the same relation on crystallographic groups.

THEOREM 19. *Every isomorphism $\beta\colon \Gamma_1 \to \Gamma_2$ of crystallographic subgroups of* $\mathrm{Isom}(E)$ *coincides with conjugation by some element of* $\mathrm{Aff}(E)$.

PROOF. Since the finite conjugate subgroup is characteristic, β sends $\Delta(\Gamma_1)$ to $\Delta(\Gamma_2)$. It is then easy to see that β can be extended to an isomorphism of the splitting groups. Thus we can assume that $\Gamma_1 = \Delta(\Gamma_1) \rtimes G_1$, $\Gamma_2 = \Delta(\Gamma_2) \rtimes G_2$ and $\beta(G_1) = G_2$. According to Theorem 8, there are points P and Q in E such that $G_1 \subseteqq \mathbb{O}(E_P)$ and $G_2 \subseteqq \mathbb{O}(E_Q)$.

The restriction $\mathrm{res}\,\beta\colon \Delta(\Gamma_1) \to \Delta(\Gamma_2)$ extends to an isomorphism of vector spaces, $1 \otimes \mathrm{res}\,\beta\colon \mathrm{Trans}(E) \to \mathrm{Trans}(E)$. Choose $b \in \mathrm{GL}(E_P)$ such that $\mathrm{ad}(b) = 1 \otimes \mathrm{res}\,\beta$.

Let g be an arbitrary element of G_1. Since Γ_1 is split, $\beta \circ \mathrm{ad}(g) =$

$\mathrm{ad}(\beta(g)) \circ \beta$ when restricted to $\Delta(\Gamma_1)$. Of course, the span of $\Delta(\Gamma_1)$ is Trans (E). Hence $\mathrm{ad}(b) \circ \mathrm{ad}(g) = \mathrm{ad}(\beta(g)) \circ \mathrm{ad}(b)$. Equivalently, $\mathrm{ad}(bgb^{-1}) = \mathrm{ad}(\beta(g))$. Apply the (AD) diagram and notice that bgb^{-1} already fixes P; $bgb^{-1} = \pi_P \beta(g)$. If we set $t = \mathbf{PQ}$, then $tbgb^{-1}t^{-1} = \pi_Q\, \beta(g) = \beta(g)$.

On the other hand, if $s \in \Delta(\Gamma_1)$, then $tbsb^{-1}t^{-1} = \mathrm{ad}(tb)(s) = \mathrm{ad}(b)(s) = \beta(s)$. We conclude that β and conjugation by tb are the same functions on Γ_1.

Among other things, this theorem implies that the property of having an enantiomorph is independent of embedding.

THEOREM 20. *Let Γ be a crystallographic subgroup of* Isom(E). *Then Γ has an enantiomorph if and only if the image of the representation for* $\mathrm{Aut}(\Gamma)$ *on $\Delta(\Gamma)$ lies in* $\mathrm{SL}(\Delta(\Gamma))$.

PROOF. By the previous theorem, every automorphism of Γ can be realized as conjugation by an element in N, the normalizer of Γ in $\mathrm{Aff}(E)$. The theorem follows if we prove that Γ has an enantiomorph if and only if $N \subseteqq \mathrm{Aff}^+(E)$.

N is the stabilizer of the element Γ for the transitive action of $\mathrm{Aff}(E)$ on the collection of conjugates of Γ. $\mathrm{Aff}^+(E)$ acts transitively if and only if $\mathrm{Aff}(E) = \mathrm{Aff}^+(E)N$. Since $\mathrm{Aff}^+(E)$ has index 2 in $\mathrm{Aff}(E)$, the collection of conjugates splits into two orbits precisely when $N \subseteqq \mathrm{Aff}^+(E)$.

Returning to Hilbert's problem, we see that it makes no difference which definition of equivalence we use if we only wish to show that there are finitely many crystals for a given dimension. (It is a much more difficult task to find the precise number of crystal classes. This problem has only recently been solved for four dimensional groups.) We shall try to bound the number of isomorphism classes of n-dimensional Bieberbach groups.

The strategy will be a familiar one. We first analyze isomorphism classes of split Bieberbach groups and then reduce the general case to this one. The data needed to describe a split group consists of the finite point group and a faithful action of the point group on $\Delta \cong \mathbf{Z}^n$. According to the discussion following Bieberbach's Characterization Theorem (Theorem 14), there are only finitely many isomorphism classes of point groups. It remains for us to show that given a finite group G there are only finitely many "essentially different" faithful actions of G on $\mathbf{Z}^n$. This is the content of a deep theorem of Jordan-Zassenhaus ([**24**], [**48**]). Rather than prove this result, we will suggest why such a theorem is plausible.

Suppose $\Gamma_1 = \Delta(\Gamma_1) \rtimes G_1$ and $\Gamma_2 = \Delta(\Gamma_2) \rtimes G_2$. We claim that if the action of G_1 on $\Delta(\Gamma_1)$ is "conjugate" to the action of G_2 on $\Delta(\Gamma_2)$, then Γ_1 and Γ_2 are isomorphic. More precisely, pick any isomorphism $f\colon \Delta(\Gamma_1) \to \Delta(\Gamma_2)$; this induces an isomorphism $\tilde{f}\colon \mathrm{GL}(\Delta(\Gamma_1)) \to \mathrm{GL}(\Delta(\Gamma_2))$. If

$x\tilde{f}(\text{ad } \Gamma_1)x^{-1} = \text{ad}(\Gamma_2)$ for some $x \in \text{GL}(\Delta(\Gamma_2))$, then Γ_1 and Γ_2 are isomorphic We shall write down the isomorphism and leave details to the reader. Assume $G_2 \subseteqq \mathbb{O}(E_Q)$ and e: $\text{GL}(\text{Trans}(E)) \to \text{GL}(E_Q)$ is the isomorphism from the (AD) diagram. The map from $\Delta(\Gamma_1) \rtimes G_1$ to $\Delta(\Gamma_2) \rtimes G_2$ is given by

$$t \cdot g \mapsto (x \circ f)(t) \cdot e[x\tilde{f}(\text{ad}g)x^{-1}].$$

The gist of this paragraph is that the number of isomorphism classes of split groups with point group G is bounded by the number of conjugacy classes in $\text{GL}(n, \mathbf{Z})$ of subgroups isomorphic to G.

We have a new question. Are there only finitely many conjugacy classes of finite subgroups of $\text{GL}(n, \mathbf{Z})$? If $\text{GL}(n, \mathbf{Z})$ were a crystallographic group, this would have been taken care of by Theorem 9. One interpretation of the theory of arithmetic groups is that it mimics the theory for crystallographic groups, once appropriate spaces for these other groups to act on are found. This viewpoint has been exploited, in a disguised form, by some crystallographers.

Suppose Γ is a crystallographic subgroup of $\text{Isom}(E)$ and L is the lattice $\Gamma \cap \text{Trans}(E)$. The point group of Γ is contained in the *holohedry* of L, the set $\{g \in \mathbb{O}(\text{Trans}(E)) | g(L) = L\}$. This subgroup describes which non-translational symmetries can be imposed on a lattice. Pick a basis $x_1, \ldots, x_n$ for L (which is a basis for $\text{Trans}(E)$) and let M_L be the matrix whose (i, j)-entry is $\langle x_i, x_j \rangle$. This matrix is frequently called the Gramian of the basis. It is always symmetric (${}^tM_L = M_L$) and positive definite ($\langle M_L(v), v \rangle > 0$ for all nonzero $v \in \mathbf{R}^n$). Write $a \in \mathbb{O}(\text{Trans}(E))$ as a matrix A with respect to the basis $x_1, \ldots, x_n$. The condition $\langle ax_i, ax_j \rangle = \langle x_i, x_j \rangle$ for all i and j becomes the equality ${}^tAM_LA = M_L$. $\{B \in \text{GL}(n, \mathbf{R}) | {}^tBM_LB = M_L\}$ is called the set of *automorphs* of M_L. Notice that if B is, in fact, an automorph, then $\det(B) = \pm 1$; the set of automorphs constitute a group. We have actually obtained an isomorphism between $\mathbb{O}(\text{Trans}(E))$ and the group of automorphs of M_L.

Under this correspondence, the holohedry of L maps to the group of *integral automorphs*, $\{B \in \text{GL}(n, \mathbf{Z}) | {}^tBM_LB = M_L\}$. An analogy is emerging. Let $\mathscr{H}$ be the set of all symmetric positive definite real matrices (or forms) and have $\text{GL}(n, \mathbf{R})$ act on $\mathscr{H}$ (this time on the right) by $M*B = {}^tBMB$. Intuitively, $\text{GL}(n, \mathbf{Z})$ ought to be a discrete subgroup of $\text{GL}(n, \mathbf{R})$. A difficult result of Hermite and Siegel states that there is a subset $\Omega \subseteqq \mathscr{H}$ such that $\Omega * \text{GL}(n, \mathbf{Z}) = \mathscr{H}$ and $\{x \in \text{GL}(n, \mathbf{Z}) | \Omega \cap \Omega * x \neq \emptyset\}$ is finite. This is just the information about covering domains that we need for Theorem 9. Only one piece of that argument is missing. Does every finite subgroup G of $\text{GL}(n, \mathbf{R})$ have a fixed point? The answer is a restatement of Theorem 11— $\sum_{A \in G} {}^tAA \in \mathscr{H}$ is the desired point. (It is an easier exercise to show that the group of integral automorphs is always finite.

Why is a discrete subgroup of Isom(E) which fixes a point finite?)

Our long-winded discourse is meant to convince the reader that there are only finitely many isomorphism classes of n-dimensional split Bieberbach groups. We turn to the non-split case.

LEMMA. *Fix a positive integer m. A finitely generated group has only finitely many subgroups of index m.*

PROOF. The action of left multiplication on the cosets of a subgroup of index m gives rise to a homomorphism from the group into the symmetric group, S_m, on m letters. Under this map, our subgroup is the complete inverse image of a subgroup of S_m. There are only finitely many such homomorphisms since each is determined by what it does to the generators of the group. There are only finitely many subgroups of the finite group S_m, and so only finitely many inverse images.

THEOREM 21. *There are finitely many isomorphism classes of n-dimensional Bieberbach groups.*

PROOF. We are assuming that there are only finitely many isomorphism classes of split groups. If a crystallographic group Γ is not already split, consider Γ^*. This larger group is certainly generated by n translations and its point group. Moreover $|\Gamma^*: \Gamma| = k^n$ where k is one of the finitely many possible orders of point groups for Γ. Apply the lemma.

REFERENCES. The Hilbert Problems are reviewed in [**28**]. Milnor's article is especially relevant.

Theorem 19 was proved by Bieberbach [**5**].

[**6**] is the standard reference on arithmetic groups. More expository descriptions of the subject matter are found in Borel's talks to various International Congresses.

Theorem 21 is again due to Bieberbach [**4**].

A modern treatment of the Jordan-Zassenhaus Theorem can be found in [**44**].

The best discussion, at an elementary level, of the classification of lattices is found in [**29**]. Also see [**9**].

VIII. Abstract Bieberbach groups. To begin with, we need some examples. We have already seen that if A is a finitely generated free abelian group and G is a finite group of automorphisms of A, then $A \rtimes G$ is a Bieberbach group. In this way, every finite group appears as the point group of some crystallographic group. More surprisingly, the same result holds when we restrict ourselves to torsion free crystallographic groups.

Suppose G is any finite group. To "present" G with generators and relations is to regard G as having the form F/R where F is a free group. We are primarily interested in the case when F is not cyclic, but has finitely many generators. Then $F/[R, R]$ is a torsion free Bieberbach group with point group G.

This observation follows from the most basic properties of free groups. If F is any free group, it is not difficult to see that $F/[F, F]$ is a free abelian group. Its rank, $d(F)$, coincides with the least number of elements required to generate F. A celebrated theorem of Schreier says that if R is a subgroup of finite index in F, then R is free and $d(R) = [F: R](d(F) - 1) + 1$. Consequently, if F is a finitely generated noncyclic free group, then $d(F) \neq d(R)$ for proper subgroups R of finite index in F.

We claim that, in this case, $\Delta(F/[R, R]) = R/[R, R]$. As we have noticed several times before, it suffices to prove that no nonidentity element of G acts trivially on $R/[R, R]$. Suppose $x \in F$ has the property that $[x, r] \in [R, R]$ for all $r \in R$. Then $\langle x, R\rangle$ is a free group with $d(\langle x, R\rangle) < \infty$. By dropping down to this subgroup we may assume G is cyclic and $R/[R, R]$ is in the center of $F/[R, R]$. Elementary group theory implies $F/[R, R]$ is abelian. That means $[F, F] = [R, R]$. Since $R/[R, R]$ has finite index in $F/[R, R]$, their ranks are the same. Hence $d(F) = d(R)$. It must be that $|G| = 1$, i.e., $x \in R$. The image of x in G is the identity element.

Applying Schreier's formula again, we see that $R/[R, R]$ is a finitely generated torsion free abelian group. Consequently $F/[R, R]$ is a Bieberbach group. Perhaps it should be christened a "free Bieberbach group on G".

We have claimed that $F/[R, R]$ is torsion free. This is tested one element at a time, so we can again reduce to the case that G is cyclic. In fact, if G is solvable, then $F/[R, R]$ has a normal series $[R, R] = S_0 \subseteqq S_1 \subseteqq \cdots \subseteqq S_\nu = F$ such that S_{i+1}/S_i is finitely generated, torsion free abelian. (This certainly proves that $F/[R, R]$ is torsion free, although in an inefficient way. We have chosen this line of reasoning for a later application.) By assumption there is a normal series

$$R = T_0 \subseteqq T_1 \subseteqq \cdots \subseteqq T_\nu = F$$

such that T_{i+1}/T_i is finite abelian. Set $S_i = [T_i, T_i]$. Since T_{i+1}/T_i is abelian, $T_i \supseteqq [T_{i+1}, T_{i+1}] \supseteqq [T_i, T_i]$. Therefore $[T_{i+1}, T_{i+1}]/[T_i, T_i]$ is a subgroup of the finitely generated torsion free abelian group $T_i/[T_i, T_i]$.

We turn now to a technique which has proved useful in induction arguments. Suppose the Bieberbach group Γ has an infinite cyclic homomorphic image. Γ then has the form $\Lambda \rtimes \mathbf{Z}$ where Λ is the kernel of the homomorphism. According to the lemma for Theorem 17, Λ is, itself, a Bieberbach group. We leave it to the reader to show that $\dim \Lambda = \dim \Gamma - 1$. Hence if enough is known (say, by induction) about Λ, then properties of Γ may be recovered. The idea works particularly well for the group ring of a Bieberbach group, since the group ring for Γ is essentially a twisted polynomial ring over the group ring for Λ.

Infinite cyclic homomorphic images will be produced by the *transfer map*. Suppose H is any group with an abelian subgroup A of index $m < \infty$. Pick right coset representatives $x_1, \ldots, x_m$ for A in H. If h is in H, we construct an $m \times m$ monomial matrix by placing $x_i h x_j^{-1}$ in the i-th row and j-th column if that element is in A and by placing 0 there otherwise. This induces an injective homomorphism from H to the group of matrices with precisely one nonzero entry, an element of A, in each row and column. The transfer map T: $H \to A$ is obtained by taking the product of the nonzero entries of the "monomial" representation. The reader may check that T is a homomorphism and is independent of coset representatives. Notice that if A is normal and $a \in A$, then $T(a)$ is no more than the product of the conjugates gag^{-1} as g runs over coset representatives for A in H. In particular, $T(a)$ is central.

LEMMA. *Suppose Γ is a Bieberbach group and T: $\Gamma \to \Delta(\Gamma)$ is the transfer map. Then* rank $(\Gamma/\mathrm{Ker}\ T)$ = rank $(\Gamma/[\Gamma, \Gamma])$.

PROOF. If $x \in \Gamma$, let $\bar{x}$ denote its image in $\Gamma/[\Gamma, \Gamma]$. Since $[\Gamma, \Gamma] \subseteq \mathrm{Ker}\ T$, it suffices to show that rank $(\bar{\Gamma}/\overline{\mathrm{Ker}\ T})$ = rank $(\bar{\Gamma})$. We do this by proving that $\overline{\mathrm{Ker}\ T}$ is periodic. Set $m = |\Gamma: \Delta|$. If $x \in \Gamma$, then $x^m \in \Delta$. As we have just observed, $T(x^m)$ is the product of m conjugates of x^m. Thus $\overline{T(x^m)} = (\bar{x})^{m^2}$. If $x \in \mathrm{Ker}\ T$, then $1 = T(x)^m = T(x^m)$.

THEOREM 22. *Let Γ be a Bieberbach group. Γ has an infinite cyclic homomorphic image if and only if Γ has a nontrivial center.*

PROOF. Suppose $z \neq 1$ is in the center of Γ. Then $T(z) = z^m$ where $m = |\Gamma: \Delta|$. Since $z^m \neq 1$, $T(\Gamma)$ is a finitely generated abelian group which is not finite.

Suppose the center of Γ is $\{1\}$. If $a \in \Delta$, then $T(a)$ is in the center of Γ. Therefore $T(x^m) = 1$ for all $x \in \Gamma$. Since T takes values in Δ, which is torsion free, T must be trival on all of Γ. In particular, rank$(\Gamma/\mathrm{Ker}\ T) = 0$. By the lemma, $|\Gamma: [\Gamma, \Gamma]| < \infty$; Γ has no infinite abelian images.

One might imagine using the theorem to "peel off" successive copies of **Z**. A group which is obtained by starting with the identity element and iteratively taking a semidirect product with **Z** a finite number of times

$$(\cdots((\mathbf{Z} \rtimes \mathbf{Z}) \rtimes \mathbf{Z})\cdots) \rtimes \mathbf{Z}$$

is called a *poly-**Z** group*. For instance, a free Bieberbach group with solvable point group is poly-**Z**. According to the theorem, if Γ is a Bieberbach group and every nontrivial subgroup of Γ has a nontrivial center, than Γ is a poly-**Z** group. We discuss the converse of this proposition.

Let us extract a crucial property of poly-**Z** groups. In general, a group H is said to be *right-ordered* if it can be totally ordered so that for any

a, b, and c in H, $a < b$ implies $ac < bc$. Certainly every subgroup of a right-ordered group is right-ordered under the inherited ordering. Moreover, if H is right-ordered, then so is $H \rtimes \mathbf{Z}$; make $(g, m) < (h, n)$ if either $m < n$ or $m = n$ with $g < h$. This certainly establishes that poly-$\mathbf{Z}$ groups can be right-ordered.

THEOREM 23. *Let Γ be a Bieberbach group. Then the following statements are equivalent.*

(i) *Every nontrivial subgroup of Γ has a nontrivial center.*

(ii) *Γ is a poly-$\mathbf{Z}$ group.*

(iii) *Γ can be right-ordered.*

PROOF. (i) $\Rightarrow$ (ii). This is the content of the previous theorem together with our general induction scheme.

(ii) $\Rightarrow$ (iii). See the previous paragraph.

(iii) $\Rightarrow$ (i). Let H be a nonidentity subgroup of Γ and set $A = H \cap \Delta(\Gamma)$. Then $|H\colon A| < \infty$; call this integer m. If $m = 1$, then H is abelian and so coincides with its center. Otherwise take coset representatives $x_1 < x_2 < \cdots < x_m$ for A in H. By multiplying each x_j on the right by x_1^{-1}, if necessary, we can assume that $x_1 = 1$. Let $a = (x_m)^m$, an element in A. Since $m > 1$, $a > x_m$. Thus $x_i > 1 \Rightarrow x_ia > a \Rightarrow x_ia > x_i \Rightarrow x_iax_i^{-1} > 1$. Consequently, if $T\colon H \to A$ is the transfer map, then $T(a) > 1$. But $T(a)$ lies in the center of H.

REFERENCES. The structure of $F/[R, R]$ is first described, under broader hypotheses, in **[2]**. $R/[R, R]$ is the so-called relation module for F/R; it is discussed in detail in K. Gruenberg's lecture notes **[20]**.

Elementary material about free groups can be found in **[26]**.

The normal series for $F/[R, R]$ when F/R is solvable appears in **[41]**.

Our application of the transfer technique is essentially found in *E.* Formanek's thesis **[18]**. Theorem 22 is explicitly recorded in **[14]**. The argument in Theorem 23 is extracted from a more general result of Rhemtulla **[34]**.

IX. The Group Ring. One of the most successful techniques for analyzing a group is the study of various representations of that group by matrices. The group operation becomes realized as matrix multiplication. It is a pleasant coincidence of nature that matrices can be added as well. How can one add group elements? The answer is, "as naively as possible".

Suppose k is any commutative ring with an identity element. (We are usually happy with k being a field or Z.) If H is a group, then the group ring $k[H]$consists of all finite formal sums $\sum a_hh$ where $a_h \in k$ and $h \in H$. One might equally well think of the elements of $k[H]$ as infinite linear combinations, $\sum a_hh$, of the member of H with the proviso that $a_h = 0$

for all but finitely many $h \in H$. Then it is clear that we can define addition by

$$(\sum_H a_h h) + (\sum_H b_h h) = \sum_H (a_h + b_h)h.$$

Define multiplication so that it distributively extends the group "multiplication" and the multiplication in k; we require that the coefficients in k commute with the group elements. (Technically speaking, $k[H]$ is an associative algebra over k with a basis made up of all of the elements in H.)

In this chapter we analyze the structure of the group ring of a Bieberbach group. It is crucial to understand the role played by the finite conjugate subgroup. If R is any ring, its center $Z(R)$, is the subring $\{a \in R \mid ar = ra \text{ for all } r \in R\}$. Suppose $a = \sum a_h h$ is in the center of the group ring $k[H]$. There is no harm in assuming that $a_h \neq 0$ for each h which actually appears in the finite sum. If $x \in H$, then $xax^{-1} = a$. Hence conjugation by x permutes the elements of H appearing in the finite sum. That is, each h with $a_h \neq 0$ has only finitely many conjugates! For all practical purposes we can conclude that $a \in k[\Delta(H)]$. Therefore $Z(k[H]) \subseteqq k[\Delta(H)]$.

Now suppose Γ is a Bieberbach group, $\Delta = \Delta(\Gamma)$, and $G = \Gamma/\Delta$. Although it is not strictly necessary, we shall assume that k is a field. We begin by studying $k[\Delta]$.

LEMMA. *If H is a right-ordered group and k is a field, then $k[H]$ has no zero divisors.*

(This means that if a and b are in $k[H]$ and $ab = 0$, then either $a = 0$ or $b = 0$).

PROOF. Write $a = \sum_{i=1}^n a_i x_i$ and $b = \sum_{j=1}^m b_j y_j$ where a_i and b_j are nonzero elements of k and $x_1 < x_2 < \cdots < x_n$. Choose y_d such that $x_n y_d$ is maximal among all of the $x_n y_j$. Then

$$x_i y_j \leqq x_n y_j \leqq x_n y_d,$$

with strict equality only when $i = n$ and $j = d$. Consequently, in multiplying a and b, the only contribution to the coefficient of $x_n y_d$ is $a_n b_d$ which is not zero.

We have already observed that Δ is right-ordered (in fact, poly-Z). Hence $k[\Delta]$ is an integral domain. We claim that if $0 \neq a \in k[\Delta]$ and $ab = 0$ for any b in $k[\Gamma]$, then $b = 0$. Take right coset representatives $x_1, \ldots, x_m$ for Δ in Γ and write $b = \sum_{i=1}^m b(i)x_i$ where $b(i) \in k[\Delta]$. Then the group elements which actually appear in $ab(i)x_i$ lie in a different coset than those in $ab(j)x_j$ for $i \neq j$. Consequently $ab = 0$ implies $ab(i) = 0$ for each i. By the lemma, $b(i) = 0$ for each i, as desired.

In particular, nonzero members of $Z(k[\Gamma])$ are not zero divisors in $k[\Gamma]$. This allows us to mimic the commutative construction and form the ring of fractions $Q(k[\Gamma])$ whose numerators are in $k[\Gamma]$ and whose denominators are nonzero elements in $Z(k[\Gamma])$. As in the commutative case, $k[\Gamma]$ is a subring of $Q(k[\Gamma])$.

If R is any subring of $k[\Gamma]$ containing $Z(k[\Gamma])$, we will let $Q(R)$ denote the ring of fractions with numerators in R and denominators in Z.

THEOREM 24. *$Q(k[\Gamma])$ is a simple ring of dimension $|G|^2$ over its center, $Q(Z)$.*

According to the Wedderburn Theorem, for some ν, $Q(k[\Gamma])$ is then isomorphic to the full ring of $\nu \times \nu$ matrices over a division ring D. Moreover, D is finite dimensional over $Q(Z)$.

PROOF. We study the interaction of three rings, $Q(Z) \subseteqq Q(k[\Delta]) \subseteqq Q(k[\Gamma])$. First notice that the action of G on Δ by $\overline{\text{ad}}$ extends to a faithful linear action of G on $k[\Delta]$ by

$$(\overline{\text{ad}}\, g)(\textstyle\sum a_u u) = \sum a_u(\overline{\text{ad}}\, g)(u) \text{ for } g \in G.$$

The subring $Z = Z(k[\Gamma])$ can then be characterized as the set of elements in the group ring fixed by the group G. Extend the action of G in the obvious way to an action on $Q(k[\Delta])$.

We claim that $Q(k[\Delta])$ is a field. Suppose $0 \neq a/z \in Q(k[\Delta])$. Then $w = \Pi_{g\in G}(\overline{\text{ad}}\, g)(a) \in Z$ and is nonzero by the lemma. Hence

$$a/z \cdot z \prod_{g\neq 1}(\overline{\text{ad}}\, g)(a)/w = 1.$$

Certainly $Q(Z)$ is in the fixed field for the action of G on $Q(k[\Delta])$. Indeed, if a/z is fixed by each $g \in G$, then so is a. Therefore the fixed field of $Q(k[\Delta])$ is precisely $Q(Z)$. (A similar argument explains why $Q(Z)$ is the center of $Q(k[\Gamma])$.) Elementary Galois Theory tells us that the dimension of $Q(k[\Delta])$ over $Q(Z)$ is finite, in fact $|G|$.

Let $1 = x_1, \ldots, x_m$ be right coset representatives for Δ in Γ. Then $Q(k[\Gamma]) = \sum Q(k[\Delta])x_i$. (Here we identify x_i with $x_i/1$.) By clearing denominators and keeping track of cosets, we find that each element of $Q(k[\Gamma])$ can be written in a unique way as $\sum b(i)x_i$, $b(i) \in Q(k[\Delta])$. Consequently

$$\dim_{Q(Z)} Q(k[\Gamma]) = \dim_{Q(Z)} Q(k[\Delta]) \cdot \dim_{Q(k[\Delta])} Q(k[\Gamma]) = |G| \cdot |G|.$$

The proof is completed once it is shown that every nonzero two-sided ideal of $Q(k[\Gamma]$ possesses a nonzero (necessarily invertible) element inside $Q(k[\Delta])$. Suppose $0 \neq I$ is an ideal of $Q(k[\Gamma])$ and $b = \sum b(i)x_i$ is a nonzero element of I with the smallest possible number of nonzero coefficients

$b(i)$. By multiplying b on the right by the inverse of one of the x_i's which "appears", we can ensure that $b(1) \neq 0$. Multiplying on the left by $b(1)^{-1}$, we can assume that $b(1)=1$. Since I is an ideal, $ubu^{-1} - b \in I$ for each $u \in \Delta$. But

$$ubu^{-1} - b = \sum(ub(i)x_iu^{-1}x_i^{-1} - b(i))x_i.$$

The coefficient of $x_1 = 1$ is 0. Thus $ubu^{-1} - b$ has fewer nonzero coefficients than b. We conclude that $ubu^{-1} - b = 0$ for all $u \in \Delta$. That is,

$$ub(i)\ x_iu^{-1}x_i^{-1} = b(i)$$

whenever $b(i) \neq 0$ in the original sum, $i > 1$. $Q(k[\Delta])$ is a (commutative) field, so the $b(i)$ may be cancelled. $x_iu\,x_i^{-1} = u$ for each $i \neq 1$ with $b(i) \neq 0$ and for each $u \in \Delta$. But Δ is self-centralizing.

One of the perplexing questions about $Q(k[\Gamma])$ is whether the size, $\nu \times \nu$, of the matrix ring has a group theoretic significance. Is ν an easily described invariant of Γ? The little that is known is due to K. A. Brown **[8]** and Linnell **[25]**; $\nu = 1$ if and only if Γ is torsion free. In other words, $Q(k[\Gamma])$ is a division ring precisely when Γ is torsion free. if $1 \neq x$ in Γ has finite order q, then $(1 - x)(1 + x + x^2 + \cdots + x^{q-1}) = 0$, so $Q(k[\Gamma])$ has zero divisors. The proof of the converse is well beyond the scope of these notes. But we can prove a special case of their theorem.

THEOREM 25. *If Γ is a free Bieberbach group for the point group G and k is a field, then $Q(k[\Gamma])$ is a division ring.*

PROOF. Write $G = F/R$ and $\Gamma = F/[R, R]$. If P_i is a Sylow p_i-subgroup of G, let F_i be the complete inverse image of P_i in F and set $\Gamma_i = F_i/[R, R]$. Notice that $Z = Z(k[\Gamma]) \subseteqq k[R/[R, R]]$, so $Z \subseteqq k[\Gamma_i]$. Γ_i is a free Bieberbach group for the solvable point group P_i. According to the lemma preceding Theorem 24, $k[\Gamma_i]$ has no zero divisors; it is then easy to see that $D_i = Q(k[\Gamma_i])$ has no zero divisors either. The previous theorem, when applied to Γ_i, implies that D_i is a division ring and

$$\dim_{Q(Z_i)} D_i = |P_i|^2.$$

Let I be any nonzero left ideal of $Q(k[\Gamma])$. I is certainly a left vector space over D_i.

$$\dim_{Q(Z)} I = (\dim_{Q(Z)} Q(Z_i) \cdot (\dim_{Q(Z_i)} D_i) \cdot (\dim_{D_i} I)$$

Since $Q(k[\Gamma])$ is finite dimensional over $Q(Z)$, the dimension of I over $Q(Z)$ is finite. Hence $|P_i|^2| \dim_{Q(Z)} I$. Note that the orders of the Sylow p_i-subgroups of G are relatively prime. Thus $|G|^2| \dim_{Q(Z)} I$. Since $|G|^2$ is the dimension of all of $Q(k[\Gamma]$, I is the entire ring. Now an elementary exercise implies that a ring with no nontrivial left ideals is, in fact, a division ring.

REFERENCES. The reader interested in group rings should consult [**33**]. Theorem 24 is due to M. Smith [**42**]. The proof that the group algebra of a free Bieberbach group has no zero divisors, appeared in [**14**] and used cohomological methods. These have been eliminated following a suggestion of K. A. Brown.

Appendix. Groups With Expansive Maps. We briefly discuss the elementary algebraic properties of groups which have an expansive map. Suppose $|\cdot|$ is a norm on the group H and f: $H \to H$ is expansive. There is a real number $s > 1$ with $|f(x)| \geqq s|x|$ for all $x \in H$.

First, f is injective. For if $f(x) = 1$, then $s\,|x| = 0$. But $|x| = 0$ implies $x = 1$.

Let us define some useful subsets of H. If r is a positive real number, then $B(r)$ will denote the ball $\{x \in H|\ |x| \leqq r\}$. We will say that H is *discrete* if $B(r)$ is a finite set for each $r > 0$. Clearly the geodesic norm on a torsion free crystallographic group makes that group discrete in this normed sense. Notice that a discrete group always has an element with smallest possible positive norm ε.

If H is a discrete group, then $\bigcap_{n\in\mathbf{N}} f^n(H) = 1$. Indeed, if $y \neq 1$ is in $f^n(H)$, then $y = f^n(x)$ implies $|y| \geqq s^n|x| \geqq s^n\varepsilon$. Since $\lim_{n\to\infty} s^n\varepsilon \to \infty$, y cannot be in the intersection.

Having an expansive map severely limits the type of presentation (by generators and relations) that a group can have. Suppose, for a moment, that G is any finitely generated group and $\{g_1, g_2 \ldots, g_t\}$ is a set of generators for G. Milnor [**30**] has introduced the *growth function* for G as follows: if $g \in G$, let its length $\ell(g)$ be the least non-negative integer m such that g can be written as a product $g = s_1\, s_2 \cdots s_m$ where either s_i or s_i^{-1} is a generator. The growth function γ: $\mathbf{N} \to \mathbf{N}$ is defined by setting $\gamma(m)$ to be the cardinality of $\{x \in G|\ell(x) \leqq m\}$. We shall say that G is *polynomially bounded* if $\gamma(m) \leqq Am^B$ for some constants A and B, B a positive integer. Although γ is dependent on the particular generating set used for G, it is not difficult to show that all growth functions on G are polynomially bounded when any one is. It has been conjectured [**30**] that a finitely generated group which is polynomially bounded is a finite extension of a nilpotent group.

THEOREM. (Franks, cf. [**19**]). *Suppose H is a finitely generated discrete normed group and f: $H \to H$ is an expansive map. If $|H{:}\,f(H)| < \infty$, then H is polynomially bounded.*

PROOF. By iterating f we may assume that $|f(x)| > 2\,|x|$ for all $x \neq 1$

in H. Mutiply the norm by a fixed scalar so that the smallest positive norm which can occur is 1. Then $|f^n(x)| > 2^n$ whenever $x \neq 1$.

Notice that if a and b are in $B(2^{n-1})$, then $|ab^{-1}| \leqq 2^n$. Hence $ab^{-1} \in f^n(H)$ implies $a = b$. In other words, distinct elements of $B(2^{n-1})$ are in distinct cosets of $f^n(H)$.

Now suppose each of the elements in a set of generators for H has norm most 2^J. If If $\ell(x) \leqq 2^k$, then $x \in B(2^{k+J})$. Therefore

$$\begin{aligned}\gamma(2^k) &\leqq |H: f^{k+J+1}(H)| \\ &= |H: f(H)|^{k+J+1} \\ &= |H: f(H)|^{J+1} \cdot (2^k)^{\log_2 |H: f(H)|}.\end{aligned}$$

A similar inequality for integers which are not necessarily powers of 2 follows from the monotonicity of γ.

Bibliography

1. E. Artin, *Geometric Algebra*, Wiley, N.Y., 1957.

2. M. Auslander and R.C. Lyndon, *Commutator subgroups of free groups*, Amer, J. Math. **77** (1955), 929–931.

3. L. Auslander, *An account of the theory of crystallographic groups*, PAMS **16** (1965), 1230–1236.

4. L. Bieberbach, *Über die Bewegungsgruppen des n-dimensionalen euklidischen Raumes mit einem endlichen Fundamentalbereich*, Gött. Nachr. (1910), 75–84.

5. ———, *Über die Bewegungsgruppen der Euklidschen Räume* II, Math. Ann. **72** (1912), 400–412.

6. A. Borel, *Introduction aux Groups Arithmétiques*, Hermann, Paris, 1969

7. C.J. Bradley and A.P. Cracknell, *The Mathematical Theory of Symmetry in Solids*, Clarendon Press, Oxford, 1972.

8. K.A. Brown, *On zero divisors in group rings*, Bull. London Math. Soc **8** (1976), 251–256.

9. J.J. Burckhardt, *Die Bewegungsgruppe der Kristallographie*, Birkhauser, Basel, 1957.

10. L.S. Charlap, *Compact flat Riemannian manifolds* I, Annals of Math. **81** (1965), 15–30.

11. P.E. Conner and F. Raymond, *Deforming homotopy equivalences to homeomorphisms in aspherical manifolds*, BAMS **83** (1977), 36–85.

12. J.D. Dixon, *The Structure of Linear Groups*, Van Nostrand Reinhold Co., N.Y., 1971.

13. D. Epstein and M. Shub, *Expanding endomorphisms of flat manifolds*, Topology **7** (1968), 139–141.

14. D. R. Farkas, *Miscellany on Bieberbach group algebras*, Pac. J. **59** (1975), 427–435.

15. F.T. Farrell and W.C. Hsiang, *The topological-Euclidean space form problem*, Invent. Math. **45** (1978), 181–192.

16. E.S. Fedorov, *Symmetry of Crystals* (transl.), Amer. Cryst. Assoc., N.Y., 1971.

17. L. Fejes (Toth), *Regular Figures*, Pergamon Pr., 1964.

18. E. Formanek, *Matrix Techniques in Polycyclic Groups*, Ph.D. Dissertation, Rice Univ., 1970.

19. J. Franks, *Anosov diffeomorphisms*, Global Analysis, Symp. in Pure Math., AMS, Providence, R.I., 1970.

20. K.W. Gruenberg, *Relation Modules of Finite Groups*, CBMS #25, AMS, Providence, R.I., 1976.

21. P. Halmos, *Finite Dimensional Vector Spaces*, Van Nostrand Co., Princeton, N.J., 1958.

22. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, 1938.

23. C. Jordan, *Mémoires sur les équations differentielles linéaires à intégrale algébrique*, J. für Math. **84** (1878), 89–215.

24. C. Jordan, *Mémorires sur léquivalence des formes*, J.de L'École Poly. **29** (1880), 111–150.

25. P.A. Linnell, *Zero divisors and idempotents in group rings*, Math. Proc. Camb. Phil. Soc. **81** (1977), 365–368.

26. R. C. Lyndon and P.E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, Berlin, 1977.

27. C.H. MacGillavry, *Fantasy and Symmetry: The Periodic Drawings of M.C. Escher*, Harry N. Abrams, Inc., N.Y., 1976.

28. *Mathematical Developments Arising From Hilbert Problems*, AMS, Providence, R.I., 1976.

29. W. Miller, Jr. *Symmetry Groups and Their Applications*, Academic Press, N.Y., 1972.

30. J. Milnor, *A note on curvature and the fundamental group*, J. Diff. Geom. **2** (1968), 1–7.

31. H. Minkowski, *Diskontinuitätsbereich für arithmetische Äquivalenz*, J. Reine Angew. **129** (1905), 220–274.

32. J. Niman and J. Norman, *Mathematics and Islamic art*, Amer. Math. Monthly **85** (1978), 489–490.

33. D. S. Passman, *The Algebraic Structure of Group Rings*, Wiley Interscience, N.Y., 1977.

34. A. H. Rhemtulla, *Right-ordered groups*, Can. J. XXIV, 891–895.

35. J. J. Rotman, *Notes on Homological Algebra*, Van Nostrand Reinhold Co., N.Y., 1970.

36. D. Schattschneider, *The plane symmetry groups: their recognition and notation*, Amer. Math. Monthly **85** (1978), 439–450.

37. A. Schönfliess, *Kristallsysteme und Kristallstruktur*, Leipzig, 1891.

38. I. Schur, *Über Gruppen periodischer Substitutionen*, Sitzber. Preuss. Adak. Wiss. (1911), 619–627.

39. M. Shub, *Expanding maps*, Global Analysis, Symp. in Pure Math., Providence, R.I., 1970.

40. A. V. Shubnikov and V.A. Koptsik, *Symmetry in Science and Art*, Plenum Press, N.Y., 1974.

41. D. M. Smirnov, *On a generalization of solvable group and their group rings*, Mat. Sb. **67** (1965), 366–383.

42. M. K. Smith, *Group algebras*, J. Alg. **18** (1971), 477–499.

43. D.A. Suprunenko, *Matrix Groups* (transl.), AMS, Providence, R.I., 1976.

44. R. G. Swan, *K-Theory of Finite Groups and Orders*, Lecture Notes #149, Springer-Verlag, Berlin, 1970.

45. H. Weyl, *Symmetry*, Princeton Univ. Press, Princeton, N.J., 1952.

46. J. A. Wolf, *Spaces of Constant Curvature*, McGraw-Hill, N.Y., 1967.

47. P. B. Yale, *Geometry and Symmetry*, Holden-Day, San Francisco, 1968.

48. H. Zassenhaus, *Neuer Beweis der Endlichkeit der Klassenzahl bei unimodularer Äquivalenz endlicher ganzzahliger Substitutionsgrouppen*, Hamb. Abh. **12** (1938), 276–288.

49. ———, *Über einen Algorithmus zur Bestimmung der Raumgruppen*, Commentarii **21** (1948), 117–141.

Virginia Polytechnic Institute and State University, Blacksburg, VA 24060