ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 16, Number 2, Spring 1986

COMPACTNESS IN SPACES OF GROUP-VALUED CONTENTS, THE VITALI-HAHN-SAKS THEOREM AND NIKODYM'S BOUNDEDNESS THEOREM

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0. Introduction. The starting point of this paper is the following theorem of W.G. Graves and W. Ruess [**11**, Theorem 7].

THEOREM. *For a locally convex space E and a subset K of the space of E-valued measures on a σ-algebra Σ, K is relatively compact in the topology of pointwise convergence (on each $A \in \Sigma$) if and only if $K(A)$ is relatively compact for each $A \in \Sigma$ and K is uniformly σ-additive.*

Graves and Ruess proved this theorem in the setting of Graves theory [**10**] of s-bounded measures with values in a locally convex space, the main idea of which is a topological linearization of the study of such measures, using as central device the "universal measure space" and its topology.

In this paper the theorem mentioned above is proved completely elementarily and generalized for group-valued measures. The essential part of this theorem, namely that the compactness of K implies the uniform s-boundedness, may be considered as a generalization of the Vitali-Hahn-Saks theorem (in the σ-additive case), for which there are elementary, transparent proofs (see, e.g., [**16, 17**]). This part is here proved by a refinement of the methods in the proofs for the Vitali-Hahn-Saks theorem. The proof is carried through in such a way that it yields, without extra work, the Vitali-Hahn-Saks theorem for s-bounded (finitely additive) contents, Nikodym's boundedeness theorem (for contents with values in a quasi-normed group), Rosenthal's lemma, and a criterion for uniform s-boundedness of A.B. d'Andrea de Lucia and P. de Lucia.

The paper is structured as follows. In §2, certain $[0, \infty]$-valued functions on a Boolean ring R are studied. As the main result of this section we get, in Theorem 2.4, a criterion for s-boundedness, from which the compactness criterion mentioned above and the Vitali-Hahn-Saks theorem can be easily deduced. It is of interest that no further assumption for R (like σ-completeness) is needed in Theorem 2.4. In §3.1 we obtain, as a

Received by the editors on February 6, 1984.

corollary of Theorem 2.4, a characterization of relatively compact subsets of $ca(R, G)$ in the topology of pointwise convergence, with G being a topological group and R a σ-complete Boolean ring. §3.2 contains an analogical result for the space $sa(R, G)$ of all G-valued s-bounded contents on an arbitrary Boolean ring R. The role, which the uniform s-boundedness plays for a compactness criterion in $ca(R, G)$, is taken over for a compactness criterion in $sa(R, G)$ by the notion of quasi-uniform boundedness, which is here introduced according to the notion of quasi-uniform convergence. §4 contains three criteria for s-boundedness, Theorem 4.2 as the group-valued version of Theorem 2.4, the result (Corollary 4.3) of A. B. d'Andrea de Lucia and P. de Lucia as corollary of Theorem 4.2 and the Vitali-Hahn-Saks theorem (Theorem 4.1) as immediate consequence of Theorem 3.1.2 as well as of Theorem 4.2. §5 deals with several versions of Nikodym's boundedness theorem. Since functions are admitted here which are not necessarily s-bounded, we do not only obtain a sufficient, but an equivalent condition to uniform boundedness. A version of Nikodym's boundedness theorem, with G being quasi-normed, easily follows from a result of §2. In §6, boundedness in the sense of **[17]** and in the sense of **[2**, p.210], **[4]** are examined and both characterized by quasi-norms. This examination together with the version of Nikodym's boundedness theorem for quasi-normed groups yields, in §5, a generalization of versions of Landers and Rogge **[17]** and Constantinescu **[4]**; in both papers the σ-additive case is considered. Further, they do not reduce their boundedness theorems to the quasi-normed case. §7 briefly discusses generalizations of results of §2 to §5. In §7.1 we study, how the condition of σ-completeness of R, in the results in which this is assumed, can be weakened. In §7.2 we briefly go into possibilities of generalization for contents with values, e.g., in uniform semigroups. Such possibilities can easily be seen by taking into account that many of our considerations for group-valued contents are based on results for certain subadditive functionals (cf. §2, Lemma 5.1 to Theorem 5.3), which can also be used in the semigroup-valued case.

1. Basic assumptions and notations. Throughout the paper let R be a Boolean ring (not necessarily with unit) and $(G, +)$ a commutative Hausdorff topological group. $\mathbf{N}$ ($\mathbf{Z}$, $\mathbf{R}$) denotes the set of all positive integers (integers, real numbers), Γ the set of all subsequences of 1, 2, 3, . . . , i.e., of all strictly increasing functions from $\mathbf{N}$ into $\mathbf{N}$ and $\mathscr{P}(\mathbf{N})$ the power set of $\mathbf{N}$.

A set K of G-valued or $[0, \infty]$-valued functions on R is called uniformly s-bounded if, for every disjoint sequence $(a_k)_{k\in\mathbf{N}}$ in R, the sequences $(\mu(a_k))_{k\in\mathbf{N}}$ converge to 0 uniformly in $\mu \in K$; K is said to be uniformly σ-additive if, for every disjoint sequence (a_k) in R such that the supremum

$\bigvee_{k=1}^{\infty} a_k$ exists in R, $(\sum_{i=1}^{k} \mu(a_i))_{k\in\mathbf{N}}$ converges to $\mu(\bigvee_{k=1}^{\infty} a_k)$ uniformly in $\mu \in K$. Each $\mu \in K$ is then s-bounded or σ-additive, respectively.

A function μ: $R \to G$ is called a content if μ is additive, i.e., if $\mu(a \vee b) = \mu(a) + \mu(b)$, for disjoint $a, b \in R$. A σ-additive content is called a measure. And $a(R, G)$, $ca(R, G)$, $sa(R, G)$ denote the group of all contents, of all measures, of all s-bounded contents on R with values in G.

A function $|\ |$ on a semigroup $(S, +)$ with zero element 0 is called a quasi-norm if $|x| \in [0, \infty]$, $|0| = 0$, $|x + y| \leqq |x| + |y|$ and $|x| \leqq |x + y| + |y|$ for all $x, y \in S$; the inequalities $|x + y| \leqq |x| + |y|$ and $|x| \leqq |x + y| + |x|$ together are equivalent to $||x + y| - |x|| \leqq |y|$, when you put $\infty - \infty := 0$. A $[0, \infty]$-valued function $|\ \ |$ on the group G is obviously a quasi-norm if and only if $|0| = 0$, $|-x| = |x|$ and $|x + y| \leqq |x| + |y|$, for all $x, y \in G$. It is well-known that the topology of G is generated by a family of quasi-norms; G can be embedded in a product of Hausdorff quasi-normed groups.

Compactness and functional analytic notions are used as in [**15**]. For $U \subset G$ we put $U^{(1)} := U$ and $U^{(n+1)} := U + U^{(n)} (n \in \mathbf{N})$.

2. $[0, \infty]$-valued set functions. This section is the foundation for §3.1, 4 and Theorem 5.4.

The lemmata 2.1, 2.2, 2.3 can be shortly described in the following way if you consider the numbers $\eta_n(\{k\})$ or $\varphi_n(\{k\})$ appearing in them as coefficients of a matrix$(p_{nk})_{n,k\in\mathbf{N}}$. By successively crossing out certain rows and columns, one obtains matrices of the form$(p_{\gamma(n),\gamma(k)})$with $\gamma \in \Gamma$, whose diagonal elements are also diagonal elements of the original matrix (p_{nk}); in the first step one achieves that the part of the new matrix above the diagonal is "small", in the second step that, additionally, the part of the matrix below the diagonal is "small" and in the third step that the diagonal elements are small. This yields, in Theorem 2.4 and Corollary 2.5, criteria for uniform s-boundedness of certain functions ψ_n: $R \to [0, \infty]$.

In view of Theorem 5.4, we here consider set functions, which satisfy a condition slightly weaker than s-boundedness.

LEMMA 2.1. *Let $\varepsilon > 0$, and for each $n \in \mathbf{N}$, let η_n: $\mathscr{P}(\mathbf{N}) \to [0, \infty]$ be a monotone function such that, for every disjoint sequence (A_k) in $\mathscr{P}(\mathbf{N})$, there is an $l \in \mathbf{N}$ with $\eta_n(A_l) \leqq \varepsilon$. Then there exists a $\gamma \in \Gamma$ such that $\eta_{\gamma(n)}(\{\gamma(k)$: $k \in \mathbf{N}, k > n\}) \leqq \varepsilon$, for all $n \in \mathbf{N}$.*

PROOF. (i). For $n \in \mathbf{N}$, every infinite subset M of $\mathbf{N}$ contains an infinite subset A with $\eta_n(A) \leqq \varepsilon$. There is, namely, a disjoint sequence (A_k) of infinite subsets of M and so $\eta_n(A_l) \leqq \varepsilon$, for some one $1 \in \mathbf{N}$, by assumption.

(ii). Put $A_0 := \mathbf{N}$ and $l_0 := 0$. For all $n \in \mathbf{N}$, you can choose, by induction, $l_n \in \mathbf{N}$ and, because of (i), infinite sets A_n with $\{l_n\} \cup A_n \subset \{l \in A_{n-1}$:

$l > l_{n-1}\}$ and $\eta_{l_n}(A_n) \leqq \varepsilon$. Then $\gamma := (l_n)_{n\in\mathbf{N}} \in \Gamma$, and, for $n \in \mathbf{N}$, we have $\{l_k : k > n\} \subset A_n$; hence $\eta_{\gamma(n)}(\{\gamma(k) : k > n\}) \leqq \eta_{\gamma(n)}(A_n) \leqq \varepsilon$.

LEMMA 2.2. *Assume that*

(i) $(\eta_n)_{n\in\mathbf{N}}$ *and* ε *satisfy the assumption of* (2.1);

(ii) $\eta_n(A) \leqq \eta_n(A\backslash\{k\}) + \eta_n(\{k\})$, *for any* $A \subset \mathbf{N}$ *and* $n, k \in \mathbf{N}$; and

(iii) $\lim_k \lim_n \eta_n(\{k\}) = 0$ (i.e., the double limit $\lim_{k\to\infty}\lim_{n\to\infty}\eta_n(\{k\})$ exists and is equal to 0).

Then, for every $\delta > \varepsilon$, *there exists a* $\gamma \in \Gamma$ *such that* $\eta_{\gamma(n)}(\{\gamma(k) : k \in \mathbf{N}, k \neq n\}) \leqq \delta$, *for all* $n \in \mathbf{N}$.

PROOF. Put $p_k := \lim_n \eta_n(\{k\})$. Since $\lim_k p_k = 0$, there is an $\alpha \in \Gamma$ with $\sum_{k=1}^{\infty} p_{\alpha(k)} < \delta - \varepsilon$. By (2.1), α has a subsequence β with $\eta_{\beta(n)}(\{\beta(k) : k > n\}) \leqq \varepsilon$. Since $\lim_n \sum_{k\in F} \eta_{\beta(n)}(\{k\}) = \sum_{k\in F} p_k < \delta - \varepsilon$, for every finite subset F of $\beta(\mathbf{N})$, you can choose for $n \in \mathbf{N}$, by induction, $l_n \in \beta(\mathbf{N})$ with $l_n > l_{n-1}$, $l_0 := 0$ and $\sum_{k=1}^{n-1} \eta_{l_n}(\{l_k\}) < \delta - \varepsilon$. Then $\gamma := (l_n)_{n\in\mathbf{N}}$ is a subsequence of β. If $n \in \mathbf{N}$, choose $m \in \mathbf{N}$ with $\beta(m) = \gamma(n)$; then $\eta_{\gamma(n)}(\{\gamma(k) : k \neq n\}) \leqq \eta_{\gamma(n)}(\{\gamma(k) : k > n\}) + \sum_{k=1}^{n-1} \eta_{\gamma(n)}(\{\gamma(k)\}) \leqq \eta_{\beta(m)}(\{\beta(l) : l > m\}) + (\delta - \varepsilon) \leqq \delta$.

Lemma 2.2 may be considered as a generalization of Rosenthal's lemma (see [**5**, p.18]). For, if (μ_n) is a uniformly bounded sequence of real-valued contents on a ring $\mathscr{R}$ of subsets of $\mathbf{N}$ containing all finite subsets of $\mathbf{N}$, define s-bounded functions $\tilde{\mu}_n$ on $\mathscr{P}(\mathbf{N})$ by $\tilde{\mu}_n(M) := \sup\{|\mu_n(A)| : A \in \mathscr{R}, A \subset M\}$, then the assumptions of Lemma 2.2 are satisfied for any $\varepsilon > 0$ and a suitable subsequence (η_n) of $(\tilde{\mu}_n)$. In connection with Rosenthal's lemma, cf. also Corollaries 3.2.6, 3.2.7 and Proposition 3.2.1 (1) $\Rightarrow$ (5).

LEMMA 2.3. *Assume that*

(i) $\mathscr{R}$ *is a ring of subsets of* $\mathbf{N}$ *with* $\{k\} \in \mathscr{R}$, *for all* $k \in \mathbf{N}$, $\varepsilon > 0$;

(ii) *For each* $n \in \mathbf{N}$, $\varphi_n : \mathscr{R} \to [0, \infty]$ *is a function such that* $|\varphi_n(A) - \varphi_n(\{k\})| \leqq \varphi_n(A\backslash\{k\})$, *for* $k \in A \in \mathscr{R}$; *and, for every disjoint sequence* (A_k) *in* $\mathscr{R}$, *there is an* $l \in \mathbf{N}$ *with* $\varphi_n(A_l) \leqq \varepsilon$;

(iii) $\lim_k \lim_n \varphi_n(\{k\}) = 0$; *and*

(iv) $\inf\{\varphi_m(A) : m \in A \subset M, A \in \mathscr{R}\} \leqq \varepsilon$, *for any infinite set* $M \subset \mathbf{N}$.

Then $\limsup \varphi_n(\{n\}) \leqq 2\varepsilon$.

PROOF. If $\limsup \varphi_n(\{n\}) > 2\varepsilon$, then you can choose a real number $\delta > \varepsilon$ and $\alpha \in \Gamma$ such that $\varphi_{\alpha(n)}(\{\alpha(n)\}) > 2\delta$, for all $n \in \mathbf{N}$. Define $\eta_n(A) := \sup\{\varphi_n(B) : A \supset B \in \mathscr{R}\}$ for $A \subset \mathbf{N}$, $n \in \mathbf{N}$. By (2.2), α has a subsequence γ with $\eta_{\gamma(n)}(\{\gamma(k) : k \neq n\}) \leqq \delta$ $(n \in \mathbf{N})$. $\gamma(\mathbf{N})$ contains, by (iv), a subset $A \in \mathscr{R}$ such that $\varphi_m(A) \leqq \delta$, for some $m \in A$. Now we get $\varphi_m(\{m\}) \leqq \varphi_m(A) + \varphi_m(A\backslash\{m\}) \leqq \delta + \eta_m(\gamma(\mathbf{N})\backslash\{m\}) \leqq 2\delta$, a contradiction to $\varphi_m(\{m\}) > 2\delta$.

THEOREM 2.4. *Let $\psi_n: R \to [0, \infty]$ be s-bounded functions with $|\psi_n(b) - \psi_n(a)| \leqq \psi_n(b\backslash a)$ for $a, b \in R$, $a \leqq b$ and $n \in \mathbf{N}$. Then the following two statements are equivalent:*

(1) *For every disjoint sequence (a_k) in R and every $\alpha \in \Gamma$, $\inf \{\psi_{\alpha(n)}(\bigvee_{k \in A} a_{\alpha(k)}): n \in A \subset \mathbf{N}, \bigvee_{k \in A} a_{\alpha(k)}$ exists in $R\} = 0$ and α has a subsequence β such that $\lim_k \lim_n \psi_{\beta(n)}(a_{\beta(k)}) = 0$; and*

(2) *$\{\psi_n: n \in \mathbf{N}\}$ is uniformly s-bounded.*

PROOF. (1) $\Rightarrow$ (2). If $\{\psi_n: n \in \mathbf{N}\}$ is not uniformly s-bounded, then there is a disjoint sequence (a_n) in R, $\varepsilon > 0$ and $\alpha \in \Gamma$ with $\psi_{\alpha(n)}(a_{\alpha(n)}) \geqq 3\,\varepsilon$. Choose a subsequence β of α with $\lim_k \lim_n \psi_{\beta(n)}(a_{\beta(k)}) = 0$ and apply (2.3) to the functions $\varphi_n: \mathscr{R} \to [0, \infty]$, where $\mathscr{R} := \{A \subset \mathbf{N}: \bigvee_{k \in A} a_{\beta(k)}$ exists in $R\}$ and $\varphi_n(A) := \psi_{\beta(n)}(\bigvee_{k \in A} a_{\beta(k)})$ $(A \in \mathscr{R}, n \in \mathbf{N})$. Lemma 2.3 yields $\limsup \psi_{\beta(n)}(a_{\beta(n)}) = \limsup \varphi_n(\{n\}) \leqq 2\,\varepsilon$, a contradiction.

(2) $\Rightarrow$ (1). Put $\psi(a) := \sup_n \psi_n(a)$, for $a \in R$. ψ is s-bounded by (2). If (a_n) is a disjoint sequence in R and $\alpha \in \Gamma$, then $\inf \{\psi_{\alpha(n)}(\bigvee_{k \in A} a_{\alpha(k)}):$ $n \in A \subset \mathbf{N}, \bigvee_{k \in A} a_{\alpha(k)}$ exists in $R\} \leqq \inf_n \psi(a_{\alpha(n)}) = 0$. Since $[0, \infty]$ is sequentially compact, α has a subsequence β such that the following double limit exists and $\lim_k \lim_n \psi_{\beta(n)}(a_{\beta(k)}) \leqq \lim_k \psi(a_{\beta(k)}) = 0$.

A function $\psi: R \to [0, \infty]$ is called σ-subadditive if $\psi(\bigvee_{k \in \mathbf{N}} a_k) \leqq \sum_{k \in \mathbf{N}} \psi(a_k)$, whenever (a_k) is a disjoint sequence such that the supremum $\bigvee_{k \in \mathbf{N}} a_k$ exists in R.

COROLLARY 2.5. *Let R be σ-complete and $\psi_n: R \to [0, \infty]$ be s-bounded functions with $|\psi_n(a) - \psi_n(b)| \leqq \psi_n(b\backslash a)$, for $a, b \in R$, $a \leqq b$ and $n \in \mathbf{N}$.*

Then we have:

(a) *If every subsequence of (ψ_n) has a σ-subadditive, s-bounded cluster point with respect to the topology of pointwise convergence, then $\{\psi_n: n \in \mathbf{N}\}$ is uniformly s-bounded;*

(b) *If $\psi: R \to [0, \infty]$ is s-bounded and $(\psi_n(a))$ converges to $\psi(a)$, for all $a \in R$, then $\{\psi_n: n \in \mathbf{N}\}$ is uniformly s-bounded.*

PROOF. We check that, in both cases (a) and (b), the condition (1) of 2.4 is fulfilled. Let (a_k) be a disjoint sequence in R and $\alpha \in \Gamma$.

(a). Choose an s-bounded cluster point ψ of $(\psi_{\alpha(n)})$ and a subsequence β of α such that $(\psi_{\beta(n)}(a_k))_{n \in \mathbf{N}}$ converges to $\psi(a_k)$, for all $k \in \mathbf{N}$. Then $\lim_k \lim_n \psi_{\beta(n)}(a_{\beta(k)}) = \lim_k \psi(a_{\beta(k)}) = 0$. Now let be $\varepsilon > 0$, $\bar{\beta}$ a subsequence of β with $\sum_{k=1}^{\infty} \psi(a_{\bar{\beta}(k)}) \leqq \varepsilon$, $a := \bigvee_{k=1}^{\infty} a_{\bar{\beta}(k)}$, γ a subsequence of $\bar{\beta}$ such that $(\psi_{\gamma(n)}(a))$ is convergent, and ψ_0 a σ-subadditive cluster point of $(\psi_{\gamma(n)})$. Then $\psi_0(a_k) = \psi(a_k)$ for $k \in \mathbf{N}$, so $\inf \{\psi_{\alpha(n)}(\bigvee_{k \in A} a_{\alpha(k)}): n \in A \subset \mathbf{N}\} \leqq \lim_n \psi_{\gamma(n)}(a) = \psi_0(a) \leqq \sum_{k=1}^{\infty} \psi_0(a_{\bar{\beta}(k)}) \leqq \varepsilon$.

(b). $\lim_k \lim_n \psi_{\alpha(n)}(a_{\alpha(n)}) = \lim_k \psi(a_{\alpha(k)}) = 0$, since ψ is s-bounded. Let $\varepsilon > 0$ and B be an infinite subset of $\alpha(\mathbf{N})$ with $\psi(\bigvee_{k \in B} a_k) \leqq \varepsilon$ (cf. (i) in the proof of (2.1)). Then $\inf\{\psi_{\alpha(n)}(\bigvee_{k\in A} a_{\alpha(k)}): n \in A \subset \mathbf{N}\} \leqq \inf_{m\in B} \psi_m(\bigvee_{k\in B} a_k) \leqq \psi(\bigvee_{k\in B} a_k) \leqq \varepsilon$.

For an application of Theorem 2.4 and Corollary 2.5 to group-valued contents we will put $\psi_n := |\mu_n(\cdot)|$ when $|\ \ |$ is a quasi-norm on G and $\mu_n \in a(R, G)$.

3. Compactness in spaces of contents. In this section, τ_p denotes the product topology on G^R, i.e., the topology of pointwise convergence. The topology on subsets of G^R induced by τ_p is also denoted by τ_p. For $K \subset G^R$, let $\bar{K}$ denote the topological closure of K in (G^R, τ_p).

3.1. *Compactness in* $(ca(R, G), \tau_p)$.

STATEMENT 3.1.1. *For $K \subset M \subset G^R$, K is relatively compact in (M, τ_p) if and only if $\bar{K} \subset M$ and $K(a) := \{\mu(a): \mu \in K\}$ is relatively compact in G, for every $a \in R$.*

This immediately follows from Tychonoff's theorem.

In Theorem 3.1.4 we will show that the inclusion $\bar{K} \subset M$ can here be replaced by the uniform s-boundedness of K, if $M = ca(R, G)$ and R is σ-complete.

THEOREM 3.1.2. *Let R be σ-complete and $K \subset sa(R, G)$ such that every infinite subset of K has a σ-additive cluster point with respect to τ_p. Then K is uniformly s-bounded.*

PROOF. We have to show that, for every sequence (μ_n) of pairwise different contents from K and every continuous quasi-norm $|\ \ |$ on G, the functions $\psi_n := |\mu_n(\cdot)|$ $(n \in \mathbf{N})$ are uniformly s-bounded. But this is a direct consequence of Corollary 2.5 (a). (Observe that $ca(R, G) \subset sa(R, G)$ if R is σ-complete.)

COROLLARY 3.1.3. *If R is σ-complete and K a relatively countably compact subset of $(ca(R, G), \tau_p)$, then K is uniformly s-bounded.*

THEOREM 3.1.4. *Let R be σ-complete and $K \subset ca(R, G)$. Then the following statements are equivalent:*

(1) *K is relatively compact in $(ca(R, G), \tau_p)$;*

(2) *K is uniformly s-bounded, and $K(a)$ is relatively compact in G, for all $a \in R$;*

(3) *K is uniformly σ-additive, and $K(a)$ is relatively compact in G, for all $a \in R$; and*

(4) *K is relatively countably compact, and $K(a)$ is relatively compact in G, for all $a \in R$.*

If every relatively countably compact subset of G is relatively compact, then a further equivalent statement is

(5) *K is relatively countably compact in $(ca(R, G), \tau_p)$.*

Note that relative countable compactness and relative compactness in G are equivalent if G is metrizable or complete or if G is a locally convex linear space quasi-complete with respect to its Mackey topology, see [**15**, p. 39 (3) and p. 316, (1)].

PROOF OF THEOREM 3.1.4. (1) $\Rightarrow$ (4) is obvious and (4) $\Rightarrow$ (2) holds by Corollary 3.1.3.

(2) $\Rightarrow$ (3). It is well-known that a uniformly s-bounded subset K of $ca(R, G)$ is uniformly σ-additive; if (a_k) is a sequence in R decreasing to 0, then $(\mu(a_k))$ is Cauchy convergent uniformly in $\mu \in K$ (cf. [**5**, p. 9]). Since $(\mu(a_k))$ converges to 0 for every $\mu \in K$, it follows that $(\mu(a_k))$ converges to 0 uniformly in $\mu \in K$.

(3) $\Rightarrow$ (1). Since K is uniformly σ-additive, we have $\bar{K} \subset ca(R, G)$, so (1) follows from statement 3.1.1.

(4) $\Leftrightarrow$ (5) obviously holds if G satisfies the given additional assumption.

3.2. *Compactness in* $(sa(R, G), \tau_p)$. Let δ_n denote the Dirac measure on $\mathscr{P}(\mathbf{N})$ located at $\{n\}$; then $\{\delta_n \colon n \in \mathbf{N}\}$ is a relatively compact subset of $sa(\mathscr{P}(\mathbf{N}), \mathbf{R})$ with respect to the topology of pointwise convergence, but not uniformly s-bounded. In connection with compactness in $(sa(R, G), \tau_p)$, the notion of s-boundedness is not of interest, but rather the notion of "quasi-uniform s-boundedness", which is based on the following definition.

DEFINITION. Let S be a uniform space, K a set of S-valued sequences and $l(f) \in S$, for $f \in K$. We say that $(f(k))_{k \in \mathbf{N}}$ converges to $l(f)$ quasi-uniformly in $f \in K$, if $\lim_k f(k) = l(f)$, for all $f \in K$, and if, for every entourage N of S and every $k_0 \in \mathbf{N}$, there exists a finite number of indices $k_1, \ldots, k_n \geqq k_0$ such that, for each $f \in K$, at least one of the pairs $(f(k_1), l(f)), \ldots, (f(k_n), l(f))$ belongs to N [**8**, p. 268–269 and **22**, p. 72–74]).

Here, above all, we are interested in the case that, for all $\alpha \in \Gamma$, the sequences $(h(k))_{k \in \mathbf{N}}$ converge quasi-uniformly in $h \in K \circ \alpha := \{f \circ \alpha \colon f \in K\}$, too. That corresponds to the notion of "almost uniform convergence" in the sense of Fichtenholz, Kantorovitch and Sirvint [**22**, p.73] for which we shall give some other equivalent formulations.

With the same notion as above, put $l(f \circ \alpha) := l(f)$, for $f \in K$, and $\alpha \in \Gamma$. Then the following statements are equivalent:

(1) For every $\alpha \in \Gamma$, $(h(k))$ converges to $l(h)$ quasi-uniformly in $h \in K \circ \alpha$;

(2) For any infinite subset M of $\mathbf{N}$ and every entourage N of S, there exists a finite subset F of M such that, for each $f \in K$, at least one of the pairs $(f(k), i(f))$, $k \in F$, belongs to N;

(3) For every $\alpha \in \Gamma$, every sequence (h_n) in $K \circ \alpha$ and every entourage

N of S, there exists a $\gamma \in \Gamma$ with $\{(h_{\gamma(n)}(\gamma(k)), l(h_{\gamma(n)})) : k, n \in N; k \neq n\} \subset N$; and

(4) For every $\alpha \in \Gamma$, every sequence (h_n) in $K \circ \alpha$, and every uniformly continuous quasi-metric, ρ for S (i.e., a map $\rho: S \times S \to [0, \infty]$ with $\rho(x, x) = 0$, $\rho(y, x) = \rho(x, y) \leqq \rho(x, z) + \rho(z, y)$), there exists a $\gamma \in \Gamma$ with $\lim_k \lim_n \rho(h_{\gamma(n)}(\gamma(k)), l(h_{\gamma(n)})) = 0$.

If $\{l(f) : f \in K\}$ and $K(k)$ for $k \in \mathbf{N}$ are relatively sequentially compact in S, a further equivalent statement is

(5) For every $\alpha \in \Gamma$ and every sequence (h_n) in $K \circ \alpha$, there exists a $\gamma \in \Gamma$ with $\lim_k \lim_n h_{\gamma(n)}(\gamma(k)) = \lim_n \lim_k h_{\gamma(n)}(\gamma(k))$.

We omit the proof in this generality. The statements (1) to (4) above correspond to the statements (1) to (4) of Proposition 3.2.1, the equivalence of which will be proved.

DEFINITION. We call a subset K of $a(R, G)$ quasi-uniformly s-bounded, if, for every disjoint sequence $(a_k)_{k \in \mathbf{N}}$ in R, the sequences $(\mu(a_k))_{k \in \mathbf{N}}$ converge to 0 quasi-uniformly in $\mu \in K$.

Of course, uniform s-boundedness implies quasi-uniform s-boundedness, and every quasi-uniformly s-bounded subset of $a(R, G)$ is contained in $sa(R, G)$.

PROPOSITION 3.2.1. *For $K \subset a(R, G)$, the following statements are equivalent*:

(1) *K is quasi-uniformly s-bounded*;

(2) *For every disjoint sequence (a_k) in R and every 0-neighbourhood U in G, there is an $n \in \mathbf{N}$ such that, for each $\mu \in K$, at least one of the elements $\mu(a_1), \ldots, \mu(a_n)$ belongs to U*;

(3) *For every disjoint sequence (a_k) in R, every continuous quasi-norm $|\ |$ on G and every sequence (μ_n) in K, there is a $\gamma \in \Gamma$ such that* $\lim_k \lim_n |\mu_{\gamma(n)}(a_{\gamma(k)})| = 0$;

(4) *For every disjoint sequence (a_k) in R, every 0-neighbourhood U in G and every sequence (μ_n) in K, there is a $\gamma \in \Gamma$ such that $\{\mu_{\gamma(n)}(a_{\gamma(k)}) : k, n \in \mathbf{N}; k \neq n\} \subset U$; and*

(5) *For every disjoint sequence (a_k) in R, every 0-neighbourhood U in G and every sequence (μ_n) in K, there is a $\gamma \in \Gamma$ such that $\{\mu_{\gamma(n)}(\bigvee_{k \in A} a_{\gamma(k)}) : n \in \mathbf{N},\ A \subset \mathbf{N} \setminus \{n\},\ \bigvee_{k \in A} a_{\gamma(k)}$ exists in $R\} \subset U$.*

PROOF. (1) $\Leftrightarrow$ (2) and (5) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (2). If (2) does not hold, then there exists a disjoint sequence (a_k) in R, a 0-neighbourhood U in G and, for each $n \in \mathbf{N}$, a content $\mu_n \in K$ such that $\mu_n(a_k) \notin U$ $(k \leqq n)$. This contradicts (4).

(1) $\Rightarrow$ (3). Let (a_k), $|\ |$, (μ_n) be given as stated in (3). Then there is a $\gamma \in \Gamma$ such that the double limit $l := \lim_k \lim_n |\mu_{\gamma(n)}(a_{\gamma(k)})|$ exists, since $[0, \infty]$ is sequentially compact. Assume that $l > \varepsilon > 0$. Then you can

choose a $k_0 \in \mathbf{N}$ with $\lim_n |\mu_{\gamma(n)}(a_{\gamma(k)})| > \varepsilon$, for all $k \geqq k_0$. By (1), there is an integer $k_1 > k_0$ such that for each $\mu \in K$, $|\mu(a_{\gamma(k)})| < \varepsilon$ for at least one k with $k_0 \leqq k \leqq k_1$. On the other hand, since $\lim_n |\mu_{\gamma(n)}(a_{\gamma(k)})| > \varepsilon$, for $k \geqq k_0$, there is an $m \in \mathbf{N}$ such that $|\mu_{\gamma(m)}(a_{\gamma(k)})| > \varepsilon$, for $k_0 \leqq k \leqq k_1$, a contradiction.

(3) $\Rightarrow$ (5). Observe first that the condition (3) for $K = \{\mu\}$ means exactly that μ is s-bounded; hence every $\mu \in K$ is s-bounded. Let (a_k), U, (μ_n) be given as stated in (5) and $|\ |$ a continuous quasi-norm on G with $\{x \in G\colon |x| \leqq 1\} \subset U$. Define $\eta_n\colon \mathscr{P}(\mathbf{N}) \to [0, \infty]$ by $\eta_n(A) := \sup \{|\mu_n(\bigvee_{k \in B} a_k)|\colon B \subset A, \bigvee_{k \in B} a_k$ exists in $R\}$ $(A \subset \mathbf{N}, n \in \mathbf{N})$. By assumption, $\lim_k \lim_n \eta_{\alpha(n)}(a_{\alpha(k)}) = 0$, for some $\alpha \in \Gamma$. The application of Lemma 2.2 to the s-bounded functions $\eta_{\alpha(n)}$, $n \in \mathbf{N}$, yields a $\gamma \in \Gamma$ such that $\eta_{\gamma(n)}(\{\gamma(k)\colon k \neq n\}) \leqq 1$, for all $n \in \mathbf{N}$; hence $\mu_{\gamma(n)}(\bigvee_{k \in A} a_{\gamma(k)}) \in U$, for all $n \in \mathbf{N}$ and $A \subset \mathbf{N}\backslash\{n\}$ for which $\bigvee_{k \in A} a_{\gamma(k)}$ exists in R.

The equivalence (1) $\Leftrightarrow$ (5) of Proposition 3.2.1 may be interpreted as follows. K is quasi-uniformly s-bounded if and only if K satisfies the assertion of a group-valued version of Rosenthal's lemma. So Proposition 3.2.2 (a) and Corollaries 3.2.6 and 3.2.7 are generalization of Rosenthal's lemma [**5**, p. 18].

PROPOSITION 3.2.2. *Let $K \subset sa(R, G)$.*

(a) *If K is relatively countably compact in $(sa(R, G), \tau_p)$, then K is quasi-uniformly s-bounded.*

(b) *If K is quasi-uniformly s-bounded, then also $\bar{K}$ and therefore $\bar{K} \subset sa(R, G)$.*

PROOF. (a) If K is not quasi-uniformly s-bounded, then, by Proposition 3.2.1 (1) $\Leftrightarrow$ (2), there is a disjoint sequence (a_k) in R, a sequence (μ_n) in K and an open 0-neighbourhood U with $\mu_n(a_k) \notin U$ $(k \leqq n)$. For every cluster point μ of (μ_n) in $(a(R, G), \tau_p)$, we have therefore $\mu(a_k) \notin U$ $(k \in \mathbf{N})$ and so μ is not s-bounded. Hence K is not relatively countably compact in $(sa(R, G), \tau_p)$.

(b) Obviously, $\bar{K}$ satisfies the condition (2) of Proposition 3.2.1, if K does so.

THEOREM 3.2.3. *For $K \subset sa(R, G)$, the following statements are equivalent:*

(1) *K is relatively compact in $(sa(R, G), \tau_p)$;*

(2) *K is quasi-uniformly s-bounded, and $K(a)$ is relatively compact in G, for all $a \in R$; and*

(3) *K is relatively countably compact, and $K(a)$ is relatively compact in G, for all $a \in R$.*

If every relatively countably compact subset of G is relatively compact, then a further equivalent statement is

(4) *K is relatively countably compact.*

PROOF. (1) $\Rightarrow$ (3) is obvious; (3) $\Leftrightarrow$ (4), too, if G satisfies the given additional assumption. (3) $\Rightarrow$ (2) holds by Proposition 3.2.2 (a). (2) $\Rightarrow$ (1) follows from Proposition 3.2.2 (b) and Statement 3.1.1.

In the following we examine the connection between quasi-uniform s-boundedness of K and (total) boundedness of $K(R) := \{\mu(a) \colon \mu \in K, a \in R\}$.

EXAMPLES 3.2.4. (a) $\{n\,\delta_n \colon n \in \mathbf{N}\}$ is a quasi-uniformly s-bounded subset of $ca(\mathscr{P}(\mathbf{N}), \mathbf{R})$, which is not uniformly bounded.

(b) Define $\mu_n \colon \mathscr{P}(\mathbf{N}) \to l_\infty$ by $\mu_n(A) := \chi_{A \cap \{1,\ldots,n\}}$ where χ_M denotes the characteristic function of a set $M \subset \mathbf{N}$ with domain $\mathbf{N}$. Then $\{\mu_n \colon n \in \mathbf{N}\}$ is a closed subset of $ca(\mathscr{P}(\mathbf{N}), l_\infty)$ with respect to the topology of pointwise convergence; $\{\mu_n \colon n \in \mathbf{N}\}$ is uniformly bounded, but not quasi-uniformly s-bounded.

PROPOSITION 3.2.5. (cf. Postscript). *If $\mu \in a(R, G)$ has a relatively countably compact range in G and $\mathbf{N}x := \{nx \colon n \in \mathbf{N}\} \not\subset \overline{\mu(R)}$, for every $x \in G\backslash\{0\}$, then μ is s-bounded.*

PROOF. Let (a_k) be a disjoint sequence in R. We show that 0 is a cluster point of $(\mu(a_k))$. By assumption, $(\mu(a_k))$ has a cluster point x. The next lemma yields $\mathbf{N}x \subset \overline{\mu(R)}$, hence, by assumption $x = 0$.

LEMMA. *Let $\mu \in a(R, G)$, (a_k) be a disjoint sequence in R and x a cluster point of $(\mu(a_k))$. Then $\mathbf{N}x \subset \overline{\mu(R)}$.*

PROOF. Let be $n \in \mathbf{N}$, U and V 0-neighbourhoods in G with $V^{(n)} \subset U$ and (b_k) a subsequence of (a_k) with $x \in \mu(b_k) + V$ $(k \in \mathbf{N})$. Then $nx \in \mu(\bigvee_{k=1}^{n} b_k) + V^{(n)} \subset \mu(R) + U$. Since U is an arbitrary 0-neighbourhood, we get $nx \in \overline{\mu(R)}$.

COROLLARY 3.2.6. *Let $K \subset a(R, G)$, $K(R)$ be relatively compact in G and $\mathbf{N}x \not\subset \overline{K(R)}$, for every $x \in G\backslash\{0\}$. Then K is a relatively compact subset of $(sa(R, G), \tau_p)$ and therefore quasi-uniformly s-bounded.*

PROOF. Every $\mu \in \bar{K}$ is s-bounded by Proposition 3.2.5, since $\mu(R)$ is contained in the compact set $\overline{K(R)}$. Hence $\bar{K} \subset sa(R, G)$. Since further, $K(a)$ is relatively compact for every $a \in R$, we get, by Statement 3.1.1, that K is relatively compact in $(sa(R, G), \tau_p)$ and therefore quasi-uniformly s-bounded by Proposition 3.2.2 (a).

COROLLARY 3.2.7. *Let E be a Hausdorff topological linear space over $\mathbf{R}$, $K \subset a(R, E)$ and $K(R)$ totally bounded. Then K is quasi-uniformly s-bounded.*

PROOF. Apply Corollary 3.2.6 to the completion G of E.

EXAMPLE 3.2.8. Let p, q be prime numbers with $q < p$ and $| \ | := | \ |_p + | \ |_q$, where $| \ |_p$ ($| \ |_q$) denotes the p-adic (q-adic) valuation on $\mathbf{Z}$. Let $\mathscr{R}$ be the algebra of all finite and cofinite subsets of $\mathbf{N}$ and $\mu \in a(\mathscr{R}, \mathbf{Z})$ defined by $\mu(\mathbf{N}) = 0$, $\mu(\{n\}) = p^n$ ($n \in \mathbf{N}$). Then $(\mathbf{Z}, | \ |)$ is a totally bounded, Hausdorff, quasi-normed group; $\mathbf{N}x \not\subset \mu(\mathscr{R})$ for every $x \in \mathbf{Z}\backslash\{0\}$ and $\mu(\mathscr{R})$ is closed, but μ is not s-bounded.

We only prove that $\mu(\mathscr{R})$ is closed in $(\mathbf{Z}, | \ |)$. The image $\mu(\mathscr{R})$ is even closed in $(\mathbf{Z}, | \ |_p)$. If x is a cluster point of $\mu(\mathscr{R})$ in the completion $\mathbf{Z}_p$ of $(\mathbf{Z}, | \ |_p)$, the ring of p-adic integers, then x can be written as a in the $\mathbf{Z}_p$ convergent series $x = a \cdot \sum_{n=1}^{\infty} a_n p^n$, where $a \in \{1, -1\}$ and $a_n \in \{0, 1\}$ for all $n \in \mathbf{N}$ and $a_n = 1$ for an infinite number of indices. Since $p > 0$, it follows that $x \notin \mathbf{Z}$, i.e., $\mu(\mathscr{R})$ has no cluster point is $(\mathbf{Z}, | \ |_p)$.

Example 3.2.8 shows that in the assumptions of Proposition 3.2.5 the relative countable compactness cannot be weakened to total boundedness. Especially, Corollary 3.2.7 doesn't hold in the group-valued case. If you consider, in Example 3.2.8, the completion of $(\mathbf{Z}, | \ |)$ as range space of μ, then $\mu(\mathscr{R})$ is no longer closed, but is relatively compact, and you see that in the assumptions of Proposition 3.2.5, $\mathbf{N}x \not\subset \overline{\mu(\mathscr{R})}$ cannot be replaced by $\mathbf{N}x \not\subset \mu(\mathscr{R})$.

4. The Vitali-Hahn-Saks theorem and further criteria for uniform s-boundedness. Theorem 3.1.2 contains the Vitali-Hahn-Saks Theorem 4.1 in the special case that (μ_n) converges pointwise to 0. This special version of the Vitali-Hahn-Saks theorem yields Theorem 4.1 with the help of a standard argument.

THEOREM 4.1 (VITALI-HAHN-SAKS). *Assume that R is σ-complete, $\mu_n \in sa(R, G)$ for $n \in \mathbf{N}$, and $(\mu_n(a))$ is a Cauchy sequence for all $a \in R$. Then $\{\mu_n \colon n \in \mathbf{N}\}$ is uniformly s-bounded.*

PROOF. Assume that $\{\mu_n \colon n \in \mathbf{N}\}$ is not uniformly s-bounded. Then there is a continuous quasi-norm on G, a disjoint sequence (a_k) in R, and a subsequence (ν_n) of (μ_n) with $|\nu_n(a_n)| \geqq 2$. Application of Lemma 2.1 to the function η_n defined by $\eta_n(A) := \sup_{k \in A} |\nu_n(a_k)|$ ($A \subset \mathbf{N}$, $n \in \mathbf{N}$) yields a $\gamma \in \Gamma$ such that $|\nu_{\gamma(n)}(a_{\gamma(k)})| \leqq 1$, for $k > n$. Since $\lambda_n := \nu_{\gamma(n+1)} - \nu_{\gamma(n)} \in sa(R, G)$ and $\lim_n \lambda_n(a) = 0$ for all $a \in R$, $\{\lambda_n \colon n \in \mathbf{N}\}$ is uniformly s-bounded, by Theorem 3.1.2, a contradiction to $|\lambda_n(a_{\gamma(n+1)})| \geqq |\nu_{\gamma(n+1)}(a_{\gamma(n+1)})| - |\nu_{\gamma(n)}(a_{\gamma(n+1)})| \geqq 1$ ($n \in \mathbf{N}$).

Instead of Theorem 3.1.2, which is based on Theorem 2.4, it is also possible to use the following group-valued version of Theorem 2.4 in the proof of Theorem 4.1.

THEOREM 4.2. *A subset K of $a(R, G)$ is uniformly s-bounded if and only if*

(i) *K is quasi-uniformly s-bounded; and*

(ii) *For every 0-neighbourhood U in G, every disjoint sequence (a_k) in R, and every sequence (μ_n) of different members of K, there is a set $A \subset \mathbf{N}$ and $n \in A$ such that the supremum $\bigvee_{k\in A} a_k$ exists in R and $\mu_n(\bigvee_{k\in A} a_k) \in U$.*

PROOF. If K is uniformly s-bounded, then (i) holds, and (ii) holds even for $A = \{n\}$ if n is large enough. For the proof of the other (non-trivial) implication we may assume that $K = \{\mu_n: n \in \mathbf{N}\}$ and $\mu_n \neq \mu_m$ $(n \neq m)$. We have to prove that, for every continuous quasi-norm $|\ \ |$ on G, the functions $\phi_n := |\mu_n(\,\cdot\,)|$ $(n \in \mathbf{N})$ are uniformly s-bounded. This follows directly from Theorem 2.4, since the condition (1) of Theorem 2.4 is fulfilled because of Proposition 3.2.1 (1) $\Rightarrow$ (3).

From Theorem 4.2 we deduce the following result of A.B. d'Andrea de Lucia and P. de Lucia [3, (1.4)].

COROLLARY 4.3. *Let R be σ-complete. Then a subset K of $sa(R, G)$ is uniformly s-bounded if and only if, for every 0-neighbourhood U in G and every disjoint sequence (a_k) in R, there exists an $l \in \mathbf{N}$ such that the set $\{\mu \in K: \mu(a_l) \notin U\}$ is finite.*

PROOF OF THE NON-TRIVIAL IMPLICATION. We check that the conditions (i) and (ii) of Theorem 4.2 are fulfilled. Let U be a 0-neighbourhood in G and (a_k) a disjoint sequence in R.

(i) Choose $l \in \mathbf{N}$ and a finite set $K_0 \subset K$ with $\mu(a_l) \in U$, for $\mu \in K\backslash K_0$. Since K_0 is a finite subset of $sa(R, G)$, there is an integer $n \geqq l$ with $\mu(a_n) \in U$ $(\mu \in K_0)$. Hence K is quasi-uniformly s-bounded by Proposition 3.2.1 (2) $\Rightarrow$ (1).

(ii). Let (μ_n) be a sequence of different members from K and (A_n) a sequence of disjoint infinite subsets of $\mathbf{N}$. Put $b_n := \sup\{a_k: k \in A_n\}$. By assumption, there are integers l, $m \in \mathbf{N}$ with $\mu_i(b_l) \in U$ $(i \geqq m)$. Choose $n \in A := A_l$ with $n \geqq m$. Then $\mu_n(\bigvee_{k\in A} a_k) \in U$.

5. The boundedness theorem of Nikodym. We here use Lemma 2.3 to get various versions of Nikodym's boundedness theorem. Since we admit functions which are not necessarily s-bounded, we get in Theorems 5.3, 5.4, and 5.7 not only a sufficient but an equivalent condition to the uniform boundedness. We first consider certain $[0, \infty[$-valued functions.

LEMMA 5.1. [1, Lemma 3.1]. *Let $\phi: R \to [0, \infty[$ be a function such that $|\phi(b) - \phi(a)| \leqq \phi(b\backslash a)$, for all $a, b \in R$ with $a \leqq b$. Then* $\sup \phi(R) < \infty$ *if and only if* $\sup_{k\in\mathbf{N}} \phi(a_k) < \infty$ *for every disjoint sequence (a_k) in R.*

COROLLARY 5.2. *Let Ψ be a set of* $[0, \infty[$*-valued functions on R such* *that* $|\psi(b) - \psi(a)| \leqq \psi(b \backslash a)$, *for all* $\psi \in \Psi$ *and all* $a, b \in R$ *with* $a \leqq b$. *Then* $\sup \Psi(R) < \infty$ *if and only if* $\sup \Psi(a) < \infty$ *for all* $a \in R$ *and* $\sup_{n \in \mathbf{N}} \psi_n(a_n) < \infty$ *for every sequence* (ψ_n) *in* Ψ *and every disjoint sequence* (a_n) *in* R.

PROOF. Apply Lemma 5.1 to ψ_0, where $\psi_0(a) := \sup \Psi(a)$, for $a \in R$.

THEOREM 5.3. *Let* R *be* σ*-complete and* Ψ *a set of* $[0, \infty[$*-valued functions on* R *such that* $|\psi(b) - \psi(a)| \leqq \psi(b \backslash a)$, *for all* $\psi \in \Psi$ *and all* $a, b \in R$ *with* $a \leqq b$. *Then the following two statements are equivalent*:

(1) (i) $\sup \Psi(a) < \infty$, *for all* $a \in R$, and

(ii) *there exists a* $p \in \mathbf{R}$ *such that, for every* $\psi \in \Psi$ *and every disjoint sequence* (a_k) *in* R, *there is an* $l \in \mathbf{N}$ *with* $\psi(a_l) \leqq p$; *and*

(2) $\sup \Psi(R) < \infty$.

PROOF. (2) $\Rightarrow$ (1) is obvious. (1) $\Rightarrow$ (2). Assume that $\sup \Psi(R) = \infty$. Then there is, by Corollary 5.2, a sequence (ψ_n) in Ψ and a disjoint sequence (a_n) in R with $\psi_n(a_n) \geqq n^2$, for all $n \in \mathbf{N}$. Choose p for (a_n) according to (ii), define φ_n by $\varphi_n(A) := (1/n)\, \psi_n(\bigvee_{k \in A} a_k)$, for $A \subset \mathbf{N}$ and $n \in \mathbf{N}$, and apply Lemma 2.3. Observe that assumptions (iii) and (iv) of Lemma 2.3 are fulfilled, since $\lim_n \varphi_n(M) = 0$, for all $M \subset \mathbf{N}$, because of Theorem 5.3 (1) (i). Lemma 2.3 yields $\limsup \varphi_n(\{n\}) \leqq 2p$, a contradiction to $\varphi_n(\{n\}) = (1/n)\psi_n(a_n) \geqq n$ $(n \in \mathbf{N})$.

If every $\psi \in \Psi$ is s-bounded, then the condition (ii) of Theorem 5.3 (1) is fulfilled, and this condition implies, by Lemma 5.1, that every $\psi \in \Psi$ is bounded. On the other hand, in Theorem 5.3 (1) the condition (ii) cannot be weakened to the boundedness of ψ, for all $\psi \in \Psi$, cf. Example 5.10. Analogous statements hold for the condition (ii) of Theorem 5.4 (1).

THEOREM 5.4 (NIKODYM'S boundedness theorem for quasi-normed groups.) *Let* R *be* σ*-complete,* $|\ \ |$ *a quasi-norm on* G *and* $K \subset a(R, G)$. *Then the following two statements are equivalent*:

(1) (i) $K(a)$ *is* $|\ \ |$*-bounded* (i.e., $\sup \{|\mu(a)|: \mu \in\} K < \infty$), *for every* $a \in R$, *and*

(ii) *there exists a* $p \in \mathbf{R}$ *such that, for every* $\mu \in K$ *and every disjoint sequence* (a_k) *in* R, *there is an* $l \in \mathbf{N}$ *with* $|\mu(a_l)| \leqq p$; *and*

(2) $K(R)$ *is* $|\ \ |$*-bounded.*

PROOF. Apply Theorem 5.3 to $\Psi := \{|\mu(\cdot)| : \mu \in K\}$.

In the case that every $\mu \in K$ is σ-additive or s-bounded, Theorem 5.4 was proved, e.g., by Mikusinski **[18]** or Drewnowski **[6]**, respectively. Theorem 5.4 also contains the version **[17**, Theorem 5] of Nikodym's theorem of Landers and Rogge, which uses the notion of "bounding systems". For, if $\mathscr{B} = \{B_n : n \in \mathbf{N}\}$ is a bounding system in the sense of

[17], then $|x| := \min\{n \in \mathbf{N} \cup \{0\} \colon x \in B_n\}$ ($x \in G$, $B_0 := \{0\}$) defines a quasi-norm on G, $\{x \in G \colon |x| \leqq 1\}$ is a 0-neighbourhood in G, and a subset M of G is $\mathscr{B}$-bounded in the sense of [17] if and only if M is $|\ \ |$-bounded; note that any set $K \subset sa(R, G)$ satisfies (ii) of Theorem 5.4 (1) (with $p = 1$), but a content $\mu \in K$ is not necessarily s-bounded with respect to the $|\ \ |$-topology, which is discrete.

In contrast to $|\ \ |$-boundedness, the boundedness notion of Definition 5.5 only depends on the group-topology.

DEFINITION 5.5. *A* subset M of G is called bounded in G if, for every 0-neighbourhood U in G, there is a finite subset F of G and an integer $n \in \mathbf{N}$ with $M \subset F + U^{(n)}$ [2; II, 4, Exercise 7; p. 210].

REMARK 5.6. In the definition above, the set F can be chosen as a subset of M.

The following theorem is an immediate consequence of Theorem 5.4 if you use the characterization (Theorem 6.8 (a)) of boundedness by quasi-norms. In the case that each $\mu \in K$ is σ-additive (hence s-bounded), (1) (i) $\Leftrightarrow$ (3) of Theorem 5.7 was proved by Constantinescu [4, Theorem 1.7] without using quasi-norms.

THEOREM 5.7. (NIKODYM'S boundedness theorem for topological groups.) *Let R be σ-complete and $K \subset a(R, G)$. Then the following two statements are equivalent:*

(1) (i) *$K(a)$ is bounded in G for every $a \in R$ and*

(ii) *for every 0-neighbourhood U in G, there exists a finite set $F \subset G$ and an $n \in \mathbf{N}$ such that for every $\mu \in K$ and every disjoint sequence (a_k) in R, there is an $l \in \mathbf{N}$ with $\mu(a_l) \in F + U^{(n)}$.*

(2) *$K(R)$ is bounded in G.*

PROOF. (2) $\Rightarrow$ (1) is obvious. (1) $\Rightarrow$ (2). By Theorem 6.8 (a), we have to show that, for any real-valued, continuous quasi-norm $|\ \ |$ on G, $K(R)$ is $|\ \ |$-bounded. But this follows immediately from Theorem 5.4 (1) $\Rightarrow$ (2). Observe that the condition (ii) of Theorem 5.4 (1) is fulfilled with $p := \max\{|x| \colon x \in F\} + n$, if you choose F and n according to Theorem 5.7 (1) (ii) for $U := \{x \in G \colon |x| \leqq 1\}$.

For a discussion of the condition (ii) of Theorem 5.7 (1) we use (b) of the following proposition, which obviously follows from Corollary 5.2 and Theorem 6.8 (a).

PROPOSITION 5.8. (a) *For $K \subset a(R, G)$, $K(R)$ is bounded in G if and only if $\{\mu_n(a_n) \colon n \in \mathbf{N}\}$ and $K(a)$ are bounded in G, for every sequence (μ_n) in K, every disjoint sequence (a_n) in R and every $a \in R$.*

(b) *For $\mu \in a(R, G)$, $\mu(R)$ is bounded in G if and only if $\{\mu(a_n) \colon n \in \mathbf{N}\}$ is bounded in G for every disjoint sequence in R.*

PROOF. (a). Apply (5.2) to $\Psi := \{|\mu(\cdot)| : \mu \in K\}$ for an arbitrary real-valued, continuous quasi-norm $|\ \ |$ on G and observe Theorem 6.8 (a).

(b). This follows from (a).

Proposition 5.8 was proved by Constantinescu [**4**, Proposition 1.2] without using quasi-norms. For further references see [**4**, Remark on p. 57].

REMARK 5.9. (a) If $K \subset sa(R, G)$, then the condition (ii) of Theorem 5.7 (1) is fulfilled.

(b) For $K \subset a(R, G)$, the condition (ii) of Theorem 5.7 (1) implies that the range of each $\mu \in K$ is bounded in G.

PROOF OF (b). Let $\mu \in K$. For a 0-neighbourhood U in G, choose F and n according to Theorem 5.7 (1) (ii). Then, for any disjoint sequence (a_k) in R, $\mu(a_l) \notin F + U^{(n)}$ holds at most for finitely many integers l. Hence $\{\mu(a_n): n \in \mathbf{N}\}$ is bounded in G. So $\mu(R)$ is bounded in G by Proposition 5.8 (b).

In contrast to the locally convex case (see Theorem 5.11 (a)), (ii) of Theorem 5.7 (1) cannot be weakened in general to the condition that the range of each $\mu \in K$ is bounded in G, as can be seen by the following example.

EXAMPLE 5.10. Let E be the real linear space of $\mathbf{R}$-valued functions on $[0, \infty[$, which is generated by the functions $\chi_{[a,b[}(0 \leqq a < b < +\infty)$, endowed with the topology of convergence in measure with respect to the Lebesgue measure λ, which is generated by the F-norm $f \mapsto \|f\| := \inf\{p > 0: \lambda(\{x \in [0, \infty[: |f(x)| \geqq p\}) \leqq p\}$. Let $\mu: \mathscr{P}(\mathbf{N}) \to \mathbf{R}$ be an unbounded content and μ_n be defined by $\mu_n(A) := \mu(A) \cdot \chi_{[0,n[}(n \in \mathbf{N}, A \subset \mathbf{N})$.

(a) Then $\{\mu_n: n \in \mathbf{N}\} \subset a(\mathscr{P}(\mathbf{N}), E)$, $\sup_n \|\mu_n(A)\| < \infty$, for all $A \subset \mathbf{N}$, $\sup\{\|\mu_n(A)\|: A \subset \mathbf{N}\} < \infty$, for all $n \in \mathbf{N}$, but $\sup\{\|\mu_n A)\|: n \in \mathbf{N}, A \subset \mathbf{N}\} = \infty$.

(b) A subset of E is bounded in E (in the sense of Definition 5.5) if and only if it is $\|\ \ \|$-bounded.

PROOF. (a). Observe that $\|\mu_n(A)\| = \min\{|\mu(A)|, n\}$ and μ is unbounded.

(b). One implication ($\Rightarrow$) is obvious. Let M be a $\|\ \ \|$-bounded subset of E, $\varepsilon > 0$ and $U := \{f \in E: \|f\| \leqq \varepsilon\}$. Choose $m \in \mathbf{N}$ with $1/m < \varepsilon$ and $\|f\| < m$, for all $f \in M$; $n := m^2$. If $f \in M$, then $\lambda(A) \leqq m$, for $A := \{x \in \mathbf{R}: |f(x)| \geqq m\}$. Hence there are disjoint subsets A_i of A with $\chi_{A_i} \in E$, $\lambda(A_i) \leqq \varepsilon$ and $\bigcup_{i=1}^n A_i = A$. f can be written as $f = \sum_{i=1}^n (f \cdot \chi_{A_i} + (1/n) f \cdot \chi_{\mathbf{R}\setminus A})$. Since $\|f \chi_{A_i}\| \leqq \varepsilon$ and $\|(1/n) f \chi_{\mathbf{R}\setminus A}\| \leqq \varepsilon$, we get $M \subset U^{(2n)}$. So M is bounded in E.

THEOREM 5.11. *Let R be σ-complete, $E\,(= G)$ a Hausdorff locally convex real linear space and $K \subset ba(R, E) := \{\mu \in a(R, E)\colon \mu(R)$ is bounded$\}$.*

(a) *Then $K(R)$ is bounded if and only if $K(a)$ is bounded for every $a \in R$,* (cf. [5, p. 14]).

(b) *K is relatively compact in $(ba(R, E), \tau_p)$ if and only if $K(a)$ is relatively compact in E, for every $a \in R$.*

(c) *If relative countable compactness and relative compactness coincide in E, then they coincide in $(ba(R, E), \tau_p)$, too.*

PROOF. (a). Observe that boundedness with respect to the original topology τ of E and with respect to the weak topology $\sigma = \sigma(E, E')$ are equivalent, and that $ba(R,(E, \tau)) = sa(R, (E, \sigma))$. Take K as subset of $sa(R, (E, \sigma))$, apply Theorem 5.7 and use statement 6.3.

(b). If $K(a)$ is relatively compact for all $a \in K$, then $K(R)$ is bounded, by (a). So $\overline{K(R)}$ is also bounded, and, because of $\bar{K}(R) \subset \overline{K(R)}$, we get $\bar{K} \subset ba(R, E)$. Now the assertion follows from Statement 3.1.1.

(c). This follows from (b).

In connection with Nikodym's boundedness theorem, Landers and Rogge [17] call a subset M of G bounded if, for every 0-neighbourhood U, there is an $n \in \mathbf{N}$ with $M \subset U^{(n)}$. Theorem 5.7, Proposition 5.8, and Remark 5.9 remain valid if you replace boundedness in the sense of Definition 5.5 by boundedness in the sense of [17]; at the same time you can change condition (ii) of Theorem 5.7 (1) by putting $F := \{0\}$. For a proof, use Theorem 5.7, Proposition 5.8, Remark 5.9, and statement 6.1. Instead of Theorem 5.7, Proposition 5.8, and Remark 5.9 you can also use Theorem 5.3, Corollary 5.2, and Theorem 6.8 (b). Note that, in Example 5.10, a subset of E is bounded in the sense of [17] if and only if it is $\|\ \ \|$-bounded.

Theorem (5.7), Proposition (5.8), and Remark (5.9) don't remain valid if G is a topological linear space and you replace boundedness in the sense of Definition 5.5 by boundedness in the sense of [15, p.156]. For Turpin [23] has given an example of a measure on a σ-algebra, whose range is an unbounded subset of a topological linear space (in the sense of [15, p.156]).

6. Boundedness in topological groups. In this section boundedness in G (in the sense of Definition 5.5) and boundedness in the sense of [17] are compared and characterized by quasi-norms.

If M_1, M_2 are subsets of G and bounded in G, then $M_1 \cup M_2$, $M_1 + M_2$, $-M_1$, $\overline{M_1}$ and subsets of M_1 are bounded in G. Let H be a subgroup of G and $M \subset H$. If M is bounded in H, then M is bounded in G, but the opposite is not true in general (see Example 6.9). If $\bar{H} = G$, then M is bounded in H if and only if M is bounded in G. M is bounded in G if and only if every countable subset of M is bounded in G. (The last two statements can easily be proved with Remark 5.6 or follow directly from Theorem 6.8.) Analogous statements hold true for the boundedness notion of **[17]**.

If M is a totally bounded subset of G, then M is bounded in G, but not necessarily bounded in the sense of **[17]**. For, if the topology of G is discrete, then $\{0\}$ is the only bounded subset of G in the sense of **[17]**, whereas the finite subsets of G are exactly the sets which are bounded in G. As an immediate consequence of Remark 5.6, we get

STATEMENT 6.1. For $M \subset G$, the following statements are equivalent:

(1) M is bounded in the sense of **[17]**;

(2) M is bounded in G and $\{x\}$ is bounded in the sense of **[17]**, for all $x \in M$; and

(3) M is bounded in G and $M \subset \bigcup_{n=1}^{\infty} U^{(n)}$, for every 0-neighbourhood U in G.

COROLLARY 6.2. *Boundedness in G* (in the sense of Definition 5.5) *and boundedness in the sense of* **[17]** *are equivalent if and only if* $G = \bigcup_{n=1}^{\infty} U^{(n)}$, *for every* 0-*neighbourhood U in G* (i.e., *G is chained*, **[12**, Definition 1.1]).

It follows that the boundedness notions of Definition 5.5 and **[17]** are equivalent if G is connected, especially, if G is a topological linear space over **R**.

STATEMENT 6.3. For subsets of a topological linear space G over **R**, boundedness in the sense of [15, p.156] implies boundedness in the sense of Definition 5.5, but the opposite is not true in general (see Example 6.9). Both notions are equivalent if G is pseudoconvex (i.e., there exists a 0-neighbourhood base $(U_\alpha)_{\alpha \in A}$ in G and a family $(p_\alpha)_{\alpha \in A}$ in $]0, 1]$ such that U_α is p_α-convex for all $\alpha \in A$).

The following is an auxiliary result for linking boundedness in G with $|\ \ |$-boundedness (for certain quasi-norms $|\ \ |$).

PROPOSITION 6.4. *Every real-valued quasi-norm* $|\ \ |$ *defined on a subgroup H of G has an extension to a real-valued quasi-norm on G.*

PROOF. In view of Zorn's lemma we may assume that $G = \mathbf{Z}x_0 + H$, for some $x_0 \in G \backslash H$. Choose $m \in \mathbf{Z}$ with $\{n \in \mathbf{Z}\colon nx_0 \in H\} = m\mathbf{Z}$ and $p \in \mathbf{R}$ with $|mx_0| \leqq |m| \cdot p$. Obviously, $\|y\| := \inf\{|np| + |x| : n \in \mathbf{Z}, x \in H, y = nx_0 + x\}$ $(y \in G)$ defines a real-valued quasi-norm on G with $\|y\| \leqq |y|$, for $y \in H$. We have to show that $|y| \leqq \|y\|$, for $y \in H$, and for this that $x, y \in H$, $n \in \mathbf{Z}$, $y = nx_0 + x$ imply $|y| \leqq |np| + |x|$. Because of $nx_0 = y - x \in H$ we have $n = m \cdot k$, for some $k \in \mathbf{Z}$, hence, $|y| \leqq |k| \cdot |mx_0| + |x| \leqq |k| \cdot |m| \cdot p + |x| = |np| + |x|$.

Even if the $|\ \ |$-topology is the topology induced on H by the topology of G, $|\ \ |$ does not necessarily have an extension to a continuous quasi-norm on G (see Example 6.9 (c)).

PROPOSITION 6.5. *For every infinite subset M of G, there exists a real-valued quasi-norm $|\ \ |$ on G such that M is not $|\ \ |$-bounded.*

PROOF. By Proposition 6.4, we may assume that G is generated by M and $M = \{x_n: n \in \mathbf{N}\}$ is countable. G_n denotes the subgroup of G generated by $\{x_1, \ldots, x_n\}$ ($n \in \mathbf{N}$), where $G_0 := \{0\}$. Define a quasi-norm $|\ \ |: G \to [0, \infty[$ by $|x| := \min\{n \in \mathbf{N} \cup \{0\} : x \in G_n\}$. If $\sup\{|x|: x \in M\} = \infty$, the proof is finished. Otherwise G is finitely generated, and on grounds of the structure theorem for finitely generated abelian groups, we may assume that $G = \mathbf{Z}^n \times F$, for some $n \in \mathbf{N}$ and a finite group F. In this case, $\|(k_1, \ldots, k_n, x)\| := \max\{|k_1|, \ldots, |k_n|\}$ ($k_i \in \mathbf{Z}$, $x \in F$) defines a suitable quasi-norm on G.

LEMMA 6.6. *For $n \in \mathbf{Z}$, let U_n be subsets of G with $U_{n+1} + U_{n+1} \subset U_n = -U_n \neq \emptyset$.* (Notation: $f(A) := \sum_{n \in A} 2^{-n}$, $U_A := \sum_{n \in A} U_n$, for any finite subset A of $\mathbf{Z}$; $|x| := \inf\{f(A): A \subset \mathbf{Z}, A$ is finite, $x \in U_A\}$ for $x \in G$, where $\inf \emptyset = +\infty$.)
Then $|\ \ |: G \to [0, \infty]$ is a quasi-norm with $\{x \in G: |x| < 2^{-n}\} \subset U_n \subset \{x \in G: |x| \leqq 2^{-n}\}$, for $n \in \mathbf{Z}$.

The proof is easy and well-known. If U_0 is a symmetric 0-neighbourhood in G and you choose symmetric 0-neighbourhoods U_n in G with $U_n + U_n \subset U_{n-1}$ and $U_{-n} := U_0^{(2^n)} (n \in \mathbf{N})$, then Lemma 6.6 yields.

COROLLARY 6.7. *For every symmetric 0-neighbourhood U in G there exists a continuous quasi-norm $|\ \ |$ on G such that $\{x \in G: |x| < 1\} \subset U \subset \{x \in G: |x| \leqq 1\}$, $\{x \in G: |x| < \infty\} = \bigcup_{n=1}^{\infty} U^{(n)}$ and a subset of G is $|\ \ |$-bounded if and only if it is contained in $U^{(n)}$, for some $n \in \mathbf{N}$.*

THEOREM 6.8. (a) *A subset of G is bounded in G* (in the sense of Definition 5.5) *if and only if it is $|\ \ |$-bounded for every continuous, real-valued quasi-norm on G.* (see Postscript)
(b) *A subset of G is bounded in the sense of* [17] *if and only if it is $|\ \ |$-bounded for every continuous quasi-norm on G.*

PROOF. (a). The implication $\Rightarrow$ is obvious. ($\Leftarrow$). Let M be a subset of G, which is $|\ \ |$-bounded for every continuous, real-valued quasi-norm on G, and U a symmetric 0-neighbourhood in G. Then $H := \bigcup_{n=1}^{\infty} U^{(n)}$ is an open subgroup of G, so G/H is discrete. We first show that $M \subset F + H$, for some finite set $F \subset G$. Otherwise, for the canonical map $\pi: G \to G/H$, the range $\pi(M)$ is infinite, and so, by Proposition 6.5, there is a real-valued quasi-norm $|\ \ |$ on G/H such that $\sup\{|\pi(x)|: x \in M\} =$

∞. Hence, $\|x\| := |\pi(x)|$ defines a quasi-norm $\| \quad \|: G \to [0, \infty]$ with $\sup\{\|x\| : x \in M\} = \infty$, a contradiction. So we have proved that there are finitely many $x_1, \ldots, x_k \in G$ and $M_1, \ldots, M_k \subset H$ with $M = \bigcup_{i=1}^k (x_i + M_i)$. We show that $M_0 := \bigcup_{i=1}^k M_i \subset U^{(n)}$, for some $n \in \mathbf{N}$. Assume that $M_0 \not\subset U^{(n)}$, for all $n \in \mathbf{N}$. Then, by Corollary 6.7, there is a continuous real-valued quasi-norm $| \quad |$ on H such that $\sup\{|x| : x \in M_0\} = \infty$. By Proposition 6.4, $| \quad |$ has an extension to a real-valued quasi-norm $\| \quad \|$ on G. Then $\| \quad \|$ is continuous, since $| \quad |$ is continuous and H open, and $\sup\{\|x\| : x \in M_0\} = \infty$, a contradiction.

(b). The implication $\Rightarrow$ is obvious. ($\Leftarrow$). This follows directly from Corollary 6.7. You can also deduce $\Leftarrow$ from (a) and statement 6.1, using the fact that any quasi-norm $| \quad |$ defined on a subgroup H of G is extended to a quasi-norm on G by $\|x\| := \infty$, for $x \in G \backslash H$.

EXAMPLE 6.9. (a) Let $\mathscr{F}(G)$ be the group of all functions f: $[0, 1[\to G$, for which there is a decomposition $0 = a_0 < a_1 \cdots < a_n = 1$ of $[0, 1]$ such that f is constant on $[a_i, a_{i+1}]$ $(i = 0, 1, \ldots, n-1)$. Let λ denote the Lebesgue measure on $[0, 1]$. Then the sets $\{f \in \mathscr{F}(G) : \lambda(\{t \in [0, 1] : f(t) \notin U\}) \leqq \varepsilon\}$, where $\varepsilon > 0$ and U is an arbitrary 0-neighbourhood in G, form a 0-neighbourhood base of an arcwise connected, locally arcwise connected group topology on $\mathscr{F}(G)$. G is a closed subgroup of $\mathscr{F}(G)$, if you identify the constant functions from $\mathscr{F}(G)$ with elements of G.

(b) $\mathscr{F}(G)$, hence G, is bounded in $\mathscr{F}(G)$.

(c) For $G = \mathbf{R}$ (with the usual topology), $\mathscr{F}(\mathbf{R})$ is bounded in $\mathscr{F}(\mathbf{R})$ (in the sense of Definition 5.5), but not bounded in the sense of [**15**, p. 156]. $\mathbf{R}$ is bounded in $\mathscr{F}(\mathbf{R})$, but not bounded in $\mathbf{R}$. The usual absolute value on $\mathbf{R}$ has no extension to a continuous quasi-norm on $\mathscr{F}(\mathbf{R})$.

PROOF. (a). This is exactly the statement [**14**, (7.20)] of Hartman and Mycielski. The proof of (b) is similar to the proof of Example 5.10 (b). The last statement of (c) follows from the fact that, for every continuous quasi-norm $\| \quad \|$ on $\mathscr{F}(\mathbf{R})$, $\sup\{\|f\| : f \in \mathscr{F}(\mathbf{R})\} < \infty$ because of (b).

7. Generalizations.

7.1. *Weakening of the σ-completeness of R.* Consider the following two properties for R, which are weaker than σ-completeness:

(P1) For any disjoint sequence $(a_k)_{k \in \mathbf{N}}$ in R, there is an infinite subset A of $\mathbf{N}$ such that $\bigvee_{k \in A} a_k$ exists in R;

(P2) For any disjoint sequence $(a_k)_{k \in \mathbf{N}}$ in R and any infinite subset M of $\mathbf{N}$ there is an infinite subset A of M and an element $b \in R$ such that $a_k \leqq b$, for all $k \in A$ and $a_k \wedge b = 0$, for all $k \in \mathbf{N} \backslash A$.

If R is a ring of sets and if, in (P1) the supremum $\bigvee_{k \in A} a_k$ is replaced by the union $\bigcup_{k \in A} a_k$, you get exactly the property which definies a quasi-σ-ring in the sense of Constantinescu [**4**, p. 52]. (P1) introduced by Haydon

[**26**] is weaker than the property (E) considered by Schachermayer [**20**, p. 20]. The property (P2) is chosen in such a way that it is weaker than (P1) and weaker than the interpolation property considered in [**1**], [**21**], and [**9**]. Independently, the property (P2) has recently been introduced by I. Fleischer and T. Traynor in a paper which is in preparation.

All numbered statements of this paper, in which R is assumed to be σ-complete, remain valid, if you assume (P1) instead of the σ-completeness; only the proofs of (2.5), (4.3) and (5.3) require an obvious modification.

The statement Corollary 2.5 (b), Theorem 4.1 of Vatali-Hahn-Saks, Corollary 4.3 of A.B. d' Andrea de Lucia and P. de Lucia and Nikodym's boundedness theorem(5.11) in the locally convex case remain valid if you only assume (P2) instead of the σ-completeness. Observing the next proposition (7.1.1), the proofs can easily be reduced to the case of R satisfying (P1) (cf. the proof of Theorem 7.1.2).

PROPOSITION 7.1.1 (a) *If R satisfies the countable chain condition CCC* (*i.e., every set of disjoint elements of R is at most countable*) *and* (P2), *then R satisfies* (P1).

(b) *If N is an ideal in R and R satisfies* (P2), *then the quotient ring R/N satisfies* (P2).

PROOF. (a) Let $(a_k)_{k\in\mathbf{N}}$ be a disjoint sequence in R. By Zorn's lemma there is a maximal set D of disjoint elements of R, which are disjoint to all a_k $k\in\mathbf{N}$. By CCC, D is at most countable and therefore, by (P2), there is an infinites et $A \subset \mathbf{N}$ and $b \in R$ such that $a_k \leqq b$ for $k \in A$ and $a_k \wedge b = d \wedge b = 0$ for $k \in \mathbf{N} \setminus A$ and $d \in D$. So b is an upper bound of $\{a_k\colon k \in A\}$. If $c \in R$ is another upper bound of $\{a_k\colon k \in A\}$, then $(b \setminus c) \wedge a_k = (b \setminus c) \wedge d = 0$, for all $k \in \mathbf{N}$ and $d \in D$. Hence $b \setminus c = 0$, because of the maximality of D and $b \leqq c$. Hence $b = \vee_{k\in A}\, a_k$.

(b) is obvious, cf. the proof of [**21**, Theorem 2.1].

If you consider only s-bounded functions in the versions (5.3), (5.4), (5.7) of Nikodym's bondedness theorem, then the σ-completeness can be replaced by (P2) in the Assumptions. Here we only formulate the result, which corresponds to (5.4).

THEOREM 7.1.2. *Assume that R satisfies* (P2), $G = (G, +, |\ \ |)$ *is a quasi-normed group, $K \subset sa\ (R, G)$ and $K\ (a)$ is $|\ \ |$-bounded for every $a \in R$. Then $K(R)$ is $|\ \ |$-bounded.*

We have to reduce the proof to the case of R satisfying (P1). We may assume that K is countable. Then $N := \{a \in R\colon \mu(b) = 0$, for all $\mu \in K$ and $b \in R$ with $b \leqq a\}$ is an ideal and $\hat{R} := R/N$ satisfies CCC, hence (P1), by Theorem 7.1.2. Now observe that $K(R) = \hat{K}(\hat{R})$, where $\hat{K} := \{\hat{\mu}\colon \mu \in K\}$ and $\hat{\mu} \in sa(\hat{R}, G)$ is defined by $\hat{\mu}(a \triangle N) := \mu(a)$.

7.2. *Generalization of the range space of the contents.*

STATEMENT 7.2.1. Several results of this paper are also true for contents with values in a commutative Hausdorff uniform semigroup $(S, +)$ with zero element 0. By [**24**, (1.1)], the uniformity of S is generated by a family $(\rho_L)_{L \in I}$ of $[0, 1]$-valued quasi-metrics satisfying $\rho_L(x + y, x' + y') \leqq \rho_L(x, x') + \rho_L(y, y')$ ($L \in I$; $x, x', y, y' \in S$). Then $|x|_L := \rho_L(x, 0)$ define quasi-norms on S. (E.g., $|x|_L \leqq |x + y|_L + |y|_L$ follows from $\rho_L(x, 0) \leqq \rho_L(x + 0, x + y) + \rho_L(x + y, 0) \leqq \rho_L(x, x) + \rho_L(0, y) + \rho_L(x + y, 0)$.) Note that, for a quasi-norm $|\ \ |: S \to [0, \infty]$ and a content $\mu: R \to S$, we have $||\mu(b)| - |\mu(a)|| \leqq |\mu(b \backslash a)|$ ($a, b \in R$; $a \leqq b$) as in the group-valued case.

All numbered results for G-valued contents of §3 and Theorem 4.2, Corollary 4.3 remain valid for S-valued contents with the following modification. In §3.1 you always have to replace σ-additivity by σ-smoothness ("$a_k \downarrow 0$ implies $\mu(a_k) \to 0$"), which is equivalent to σ-additivity together with s-boundedness if R satisfies (P1); so replace $ca(R, G)$, by $sca(R, S) := ca(R, S) \cap sa(R, S)$ and, in Theorem 3.1.4 (3), "uniformly σ-additive" by "uniformly σ-smooth".

Theorem 4.1 of Vitali-Hahn-Saks doesn't remain valid in the semigroup-valued case without any additional assumption. For, $\mu_n := \sum_{i=1}^{n} \delta_i$ are s-bounded measures on $\mathscr{P}(\mathbf{N})$, which converge to the counting measure, but $\{\mu_n: n \in \mathbf{N}\}$ is not uniformly s-bounded. Orlicz and Urbański [**19**] proved the Vitali-Hahn-Saks theorem for contents with values in a topological semigroup which satisfies a certain "topological cancellation law"; as Drewnowski [**7**] later showed, the topological semigroup considered can be embedded in a topological group. In the following theorem, the additional assumption is the s-boundedness of the limit function.

THEOREM. *Assume that R satisfies* (P2), $\mu_n \in sa(R, S)$ *for* $n \in \mathbf{N} \cup \{0\}$, *and* $\lim_n \mu_n(a) = \mu_0(a)$ *for all* $a \in R$. *Then* $\{\mu_n: n \geqq 0\}$ *is uniformly s-bounded.*

For the proof in the case where R is σ-complete, apply Corollary 2.5 (b) to $\psi_n := |\mu_n(\cdot)|$ for an arbitrary continuous quasi-norm on S. For the weakening of the σ-completeness, see §7.1.

Evidently, Nikodym's boundedness theorem (5.4), which uses the notion of $|\ \ |$-boundedness, is also true in the semigroup-valued case. On the other hand, the boundedness notion of [**2**, p. 210] as in Theorem 5.7 cannot be used in Nikodym's boundedness theorem in the semigroup-valued case.

EXAMPLE. Choose $S := \mathbf{N} \cup \{0, \infty\}$ with the usual addition and the discrete uniformity. Put $\mu_1 := 0$, $\mu_2 := \delta_1$, $\mu_{n+1} := \mu_n + \infty \cdot \delta_{n-1} + n \cdot \delta_n$

$(n \geqq 2)$ and $K := \{\mu_n : n \in \mathbf{N}\}$. Then $K \subset sca(\mathscr{P}(\mathbf{N}), S)$. $K(A) = \{0, \min A, \infty\}$ for every non-empty subset A of $\mathbf{N}$, but $K(R) = S$.

STATEMENT 7.2.2. Let S be a Hausdorff uniform space, $+ : S \times S \to S$ a uniform continuous map, $0 \in S$ and $1 < q < \infty$. Then the uniformity of S is generated by a family $(\rho_L)_{L \in I}$ of quasi-metrics satisfying $\rho_L(x + y, x' + y') \leqq q(\rho_L(x, x') + \rho_L(y, y'))$. (In general, that is not true for $q = 1$ [**24**, p. 422].) The functions $|x|_L := \rho_L(x, 0)$ $(x \in S)$ are not necessarily quasi-norms, but for $\mu \in sa(R, S)$ we still have the inequalities $|\mu(b)|_L \leqq q(|\mu(a)|_L + |\mu(b\backslash a)|_L)$, $|\mu(a)|_L \leqq q(|\mu(b)|_L + |\mu(b\backslash a)|_L$ $(a, b \in R, a \leqq b)$ and $|\mu(0)|_L = 0$. All other statements of Statement 7.2.1 also hold in the more general situation of (7.2.2). The constant q appearing here requires modifications of some proofs, which are near at hand.

Postscript. After Z. Lipeoki had read the first version of this paper, he called my attention to [**25**], [**26**] and the papers [**12**, **13**] of Hejcman and informed me that Theorem 6.8 (a) was already proved in [**12**, Theorem 2.5]. He further informed me that results related to Proposition 3.2.5 are contained in the thesis of W. Herer (Warszawa) and the paper of M.P. Kats, *On extension of vector measures* (in Russian), Sib. Mat. Zhurn. **13** (1972), 1158–1168.

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