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ASYMPTOTIC BEHAVIOR OF SINGULAR VALUES OF CONVOLUTION OPERATORS

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1. Introduction. In [1] a study was made of the singular values and singular functions of the convolution operator

$$\tilde{K} \cdot = \int_0^x K(x-y) \cdot dy, \quad 0 \leqq x \leqq 1, \tag{1.1}$$

under the condition that $K(u)$ is reasonably smooth and $K(0) \neq 0$. Asymptotic estimates of the singular functions and values were obtained. A somewhat heuristic argument was made to suggest that quite different behaviors are to be expected in the event that $K(0) = 0$.

In this paper we treat the case

$$K(u) = u^n k(u), \quad 0 \leqq u \leqq 1, \tag{1.2}$$

where n is a positive integer, $k(u) \in C^n [0, 1]$, and $k(0) \neq 0$. We are unable to obtain asymptotic estimates for the singular functions, but we do obtain such results for the singular values. This is done by showing that the singular values of $K(u)$ and those of $k(0)\, u^n$ differ little for large indices.

2. Some preliminaries. It is shown in [1] that instead of studying the nonsymmetric operator $\tilde{K}$ we may confine our attention to the symmetric operator

$$K \cdot = \int_{1-x}^1 K(x+y-1) \cdot dy, \; 0 \leqq x \leqq 1. \tag{2.1}$$

The singular values of $\tilde{K}$ are just the absolute values of the eigenvalues of K. It is also convenient to assume

$$k(0) = 1. \tag{2.2}$$

The "comparison operator" now becomes

$$K_n \cdot = \int_{1-x}^1 K_n(x+y-1) \cdot dy, \tag{2.3}$$

with

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$$K_n(u) = u^n, \quad 0 \leqq u \leqq 1. \tag{2.4}$$

We denote the eigenvalues of K_n by μ_j. (There is no need to exhibit the index n).

THEOREM 1. *There exists a constant A, dependent on n, such that*

$$|\mu_j| = \frac{A}{j^{n+1}}\left(1 + 0\left(\frac{1}{j^2}\right)\right). \tag{2.5}$$

PROOF. In order to avoid interrupting the basic chain of reasoning, we postpone this proof until §4.

3. The principal results. Let λ_j be the eigenvalues of K (see (2.1)). Write

$$K(x + y - 1) = (x + y - 1)^n + (x + y - 1)^n\{k(x + y - 1) - 1\}. \tag{3.1}$$

The last term in (3.1) is the kernel of a symmetric operator whose eigenvalues we denote by σ_j. Then (see [2])

$$|\lambda_j| \leqq |\mu_q| + |\sigma_p|, \quad j = p + q - 1. \tag{3.2}$$

Here we follow the convention that all eigenvalues are indexed according to decreasing absolute value.

From (2.5) we get

$$|\lambda_j| \leqq \frac{A}{q^{n+1}}\left(1 + O\left(\frac{1}{q^2}\right)\right) + |\sigma_p|. \tag{3.3}$$

Recall that σ_p is associated with the kernel

$$\hat{K}(u) = u^n(k(u) - 1), \quad 0 \leqq u \leqq 1. \tag{3.4}$$

Then, because $k(u) \in C^n\ [0, 1]$,

$$\frac{d^n}{du^n}\hat{K}(u) = n!(k(u) - 1) + B_1\, u\, k'(u) + \cdots + B_n u^n k^{(n)}(u). \tag{3.5}$$

Here the B_j's are easily calculated constants. Because $k(0) = 1$ (see (2.2)), we have

$$\frac{d^n}{du^n}\hat{K}(0) = 0. \tag{3.6}$$

Clearly, for $j < n$.

$$\frac{d^j \hat{K}(0)}{du^j} = 0. \tag{3.7}$$

Now we extend $\hat{K}(u)$ so that $\hat{K}(u) \equiv 0$, $u < 0$. Thus

$$\int_{1-x}^{1} \hat{K}(x + y - 1) \cdot dy = \int_0^1 \hat{K}(x + y - 1) \cdot dy. \tag{3.8}$$

The extended $\hat{K}(x + y - 1)$ is n times continuously differentiable with respect to x on the square, $0 \leqq x, y \leqq 1$. By a known result (see [3]) the eigenvalues of $\hat{K}$ satisfy

$$|\sigma_p| < \frac{\varepsilon_p}{p^{n+3/2}}, \tag{3.9}$$

where $\varepsilon_p \to 0$. We rewrite (3.3) as

$$|\lambda_j| \leqq \frac{A}{q^{n+1}}\left(1 + D\left(\frac{1}{q^2}\right)\right) + \frac{\varepsilon_p}{p^{n+3/2}}, \quad j = p + q - 1. \tag{3.10}$$

We now select

$$p = [j^s], \quad q = j + 1 - [j^s], \tag{3.11}$$

where the square bracket means "largest integer in." As yet, s is unspecified, although we require $0 < s < 1$. Because interest lies in large j and

$$[j^s] = \alpha_j j^s, \alpha_j \to 1 \text{ as } j \to \infty, \tag{3.12}$$

we shall simplify notation by simply writing j^s for $[j^s]$.

Now write

$$|\lambda_j| - \frac{A}{j^{n+1}} \leqq A\left(\frac{1}{q^{n+1}} - \frac{1}{j^{n+1}}\right) + \frac{\varepsilon_p}{p^{n+3/2}} + O\left(\frac{1}{q^{n+3}}\right) = R. \tag{3.13}$$

We attempt to find the largest t such that $j^t R \to 0$ as $j \to \infty$. This implies (see (3.11) and (3.12))

$$j^t A\left(\frac{1}{(j+1-j^s)^{n+1}} - \frac{1}{j^{n+1}}\right) + \frac{j^t \varepsilon_p}{j^{s(n+3/2)}} + 0\left(\frac{j^t}{(j+1-j^s)^{n+3}}\right) \to 0. \tag{3.14}$$

To guarantee proper behavior of the second term in (3.14), we require

$$t \leqq s\left(n + \frac{3}{2}\right), \tag{3.15}$$

and the third term implies that we need

$$t < n + 3. \tag{3.16}$$

(Recall that we require $0 < s < 1$.) Observe that if (3.15) holds, then (3.16) does also.

To examine the first term of (3.14) we write

$$\frac{1}{(j + 1 - j^s)^{n+1}} - \frac{1}{j^{n+1}} = \frac{1}{j^{n+1}}\left\{\frac{1}{\left(1 + \dfrac{1 - j^s}{j}\right)^{n+1}} - 1\right\}$$

$$(3.17)\qquad = -\frac{1}{j^{n+1}}\left\{(n+1)\left(\frac{1-j^s}{j}\right)+O\left(\frac{1}{j^{2(1-s)}}\right)\right\}$$
$$= -\frac{(n+1)}{j^{n+2-s}}+O\left(\frac{1}{j^{3-2s+n}}\right).$$

Thus

$$(3.18)\qquad j^t\left\{\frac{1}{(j+1-j^s)^{n+1}}-\frac{1}{j^{n+1}}\right\}=\frac{-(n+1)}{j^{n+2-s-t}}+O\left(\frac{1}{j^{3-2s+n-t}}\right).$$

If we require

$$(3.19)\qquad t+s<n+2,$$

then the expression in (3.18) approaches zero. Conditions (3.15) and (3.19) are both satisfied if and only if

$$(3.20)\qquad t<n+1+\frac{1}{2n+5}.$$

(Note that this choice gives $0<s<1$.) From (3.13), (3.14), and (3.20) we conclude that

$$(3.21)\qquad \overline{\lim_{j\to\infty}}\left\{|\lambda_j|-\frac{A}{j^{n+1}}\right\}j^{n+1+1/2(n+3)}\leqq 0.$$

We now wish to reverse the inequality in (3.21). Write

$$(3.22)\qquad (x+y-1)^n=(x+y-1)^n k(x+y-1)+(x+y-1)^n\{1-k(x+y-1)\}$$

and obtain

$$(3.23)\qquad \begin{aligned}\frac{A}{j^{n+1}}\left(1+O\left(\frac{1}{j^2}\right)\right)&=|\mu_j|\leqq|\lambda_q|+|\sigma_p|\\ &\leqq|\lambda_q|+\frac{\varepsilon_p}{p^{n+3/2}},\quad j=p+q-1,\end{aligned}$$

or

$$(3.24)\qquad \frac{A}{q^{n+1}}-|\lambda_q|\leqq A\left(\frac{1}{q^{n+1}}-\frac{1}{j^{n+1}}\right)+\frac{\varepsilon_p}{p^{n+3/2}}+O\left(\frac{1}{j^{n+3}}\right)=\tilde{R}.$$

Now, precisely the arguments employed in obtaining (3.21) show that, for t as in (3.20),

$$(3.25)\qquad \lim_{j\to\infty} j^t\tilde{R}=0.$$

From (3.11) we note that

$$(3.26)\qquad \lim_{j\to\infty} q/j=1.$$

Therefore

$$\overline{\lim_{q\to\infty}}\, q^t\left\{\frac{A}{q^{n+1}} - |\lambda_q|\right\} = \overline{\lim_{q\to\infty}}\, q^t\,\tilde{R} = \overline{\lim_{q\to\infty}}\left(\frac{q}{j}\right)^t\left\{j^t\,\tilde{R}\right\} = 0. \tag{3.27}$$

Upon combining (3.21) and (3.27) we obtain our desired estimate.

THEOREM 2. *Let* $K(u) = u^n k(u)$, $k(0) = 1$, *and* $k(u) \in C^n\,[0, 1]$. *Then the singular values* $|\lambda_j|$ *of the operator* $\tilde{K}$ *defined by* (1.1) *satisfy*

$$\left|\frac{A}{j^{n+1}} - |\lambda_j|\right| \leqq \varepsilon_j j^{-(n+1+1/2(n+3))}, \tag{3.28}$$

where $\varepsilon_j \to 0$ and A is a known constant dependent upon n.

It is interesting to compare this result with that in [1] where n was zero and $k(u)$ was slightly more restricted. There it was found that

$$|\lambda_j| = \frac{1}{\left(j + \frac{1}{2}\right)\pi} + O\left(\frac{1}{j^3}\right) = \frac{\pi^{-1}}{j} + O\left(\frac{1}{j^2}\right). \tag{3.29}$$

Clearly the result obtained from (3.28) with $n = 0$ is much less satisfactory. This suggests that (3.28) can be improved, especially if additional hypotheses are imposed on $k(u)$. The approach employed in [1] was completely different (and considerably more subtle) than the methods of this paper.

4. The Proof of Theorem 1. We propose to calculate the eigenvalues of K_n. Write

$$\mu_j\phi_j(x) = \int_{1-x}^{1} (x + y - 1)^n\, \phi_j(y)dy. \tag{4.1}$$

Differentiation gives

$$\mu_j\phi_j^{(k)}(x) = n(n-1)\cdots(n-k+1)\int_{1-x}^{1}(x+y-1)^{n-k}\,\phi_j(y)dy, \quad k = 1, 2, \cdots, n. \tag{4.2}$$

Note that

$$\phi_j^{(k)}(0) = 0, \quad k = 0, 1, 2, \cdots, n, \tag{4.3}$$

and that

$$\mu_j\phi_j^{(n)}(x) = n!\int_{1-x}^{1}\phi_j(y)\,dy. \tag{4.4}$$

Thus

$$\mu_j\phi_j^{(n+1)}(x) = n!\,\phi_j(1-x) \tag{4.5}$$

and

$$\mu_j\phi_j^{(n+k)}(x) = n!(-1)^{k-1}\,\phi_j^{(k-1)}(1-x), \quad k = 1, 2, \ldots, n. \tag{4.6}$$

From (4.3) we find

$$\mu_j\phi_j^{(n+k)}(1) = n!(-1)^{k-1}\phi_j^{(k-1)}(0) = 0, \quad k = 1, 2, \ldots, n. \tag{4.7}$$

Differentiating (4.6) once more gives

$$\mu_j^2\phi_j^{(2n+1)}(x) = n!(-1)^n \mu_j\phi_j^{(n)}(1-x) \tag{4.8}$$

so that

$$\mu_j^2\phi_j^{(2n+1)}(1) = 0. \tag{4.9}$$

A final differentiation of (4.8) followed by use of (4.5) yields

$$\mu_j^2\phi_j^{(2n+2)}(x) = n!(-1)^{n+1}\mu_j\phi_j^{(n+1)}(1-x) = (n!)^2(-1)^{n+1}\phi_j(x). \tag{4.10}$$

Summarizing, if $\mu_j \neq 0$ and $\phi_j(x)$ are eigenvalues and eigenfunctions of K_n, then

$$\phi_j^{(2n+2)}(x) + (-1)^{n+2}(n!)^2\tilde{\mu}_j\phi_j(x) = 0, \tag{4.11a}$$

$$\tilde{\mu}_j = \frac{1}{\mu_j^2}, \tag{4.11b}$$

$$\phi_j^{(k)}(0) = 0, \quad k = 0, 1, 2, \ldots, n, \tag{4.11c}$$

$$\phi_j^{(k)}(1) = 0, \quad k = n+1, n+2, \ldots, 2n+1. \tag{4.11d}$$

It must be noted that the system (4.11) may have eigenvalues that do not belong to K_n, a matter that we shall address shortly. It is shown in [**4**] that the eigenvalues $\tilde{\mu}_j$ of (4.11) satisfy

$$|\tilde{\mu}_j|^{1/2(n+1)} = M_n j + O\left(\frac{1}{j}\right), \tag{4.12}$$

where the M_n are constants given explicitly in [**4**]. We obtain at once

$$|\mu_j| = \frac{A}{j^{n+1}}\left(1 + O\left(\frac{1}{j^2}\right)\right). \tag{4.13}$$

We must now show that every eigenvalue of (4.11) corresponds to one of K_n. From (4.11a, b)

$$\begin{aligned} &\mu_j^2 \int_{1-x}^{1} (x+z-1)^n \phi_j^{(2n+2)}(x)\,dx \\ &= (n!)^2(-1)^{n+1}\int_{1-x}^{1}(x+z-1)^n\phi_j(x)\,dx \\ &= (n!)^2(-1)^{n+1}K_n\phi_j. \end{aligned} \tag{4.14}$$

If we integrate the left side of (4.14) by parts $(n + 1)$ times and use (4.11d), we find

$$(-1)^n \mu_j^2 n! \phi_j^{(n+1)}(1 - z) = (n!)^2(-1)^{n+1} K_n \phi_j. \tag{4.15}$$

Next apply K_n to both sides of (4.15):

$$\begin{aligned} -\frac{\mu_j^2}{n!}\int_{1-x}^{1} (x + z - 1)^n \phi^{(n+1)}(1 - z)dz \\ = K_n^2 \phi_j = \frac{\mu_j^2}{n!}\int_0^x (x - t)^n \phi^{(n+1)}(t)\, dt. \end{aligned} \tag{4.16}$$

Integrating the last integral by parts $(n + 1)$ times and using (4.11c) produces

$$\mu_j^2 \phi_j(x) = K_n^2 \phi_j(x). \tag{4.17}$$

Thus μ_j^2 is an eigenvalue of K_n^2. Because K_n is symmetric, it follows (see [5]) that either μ_j or $(-\mu_j)$ is an eigenvalue of K_n, and so $|\mu_j|$ is a singular value of $\tilde{K}$.

Finally, we must show that no eigenvalue of K_n can be zero, an assumption made in deriving (4.11). Suppose for some $\psi \not\equiv 0$,

$$\int_{1-x}^{1} (x + y - 1)^n \psi(y)dy \equiv 0. \tag{4.18}$$

Differentiating (4.18) $(n + 1)$ times yields

$$\psi(1 - x) = 0, \quad 0 \leqq x \leqq 1, \tag{4.19}$$

a contradiction.

This completes the proof of Theorem 1.

5. Summary and remarks. We have shown that when $K(u) = u^n k(u)$, $k(0) \neq 0$, $k(u) \in C^n [0, 1]$, the singular values $|\lambda_j|$ of the operator

$$\tilde{K} \cdot = \int_0^x K(x - y) \cdot dy \tag{5.1}$$

behave asymptotically like A/j^{n+1}. Roughly speaking, the behavior of $K(u)$ near $u = 0$ is all important in determining the behavior of the singular values of $\tilde{K}$. This has very important implications when one is interested in the approximate solution of convolution type integral equations of the first kind.

All efforts to obtain analogous results for the singular functions of such operators have failed. Numerical studies suggest strongly that these functions are basically sinusoidal except near the interval end points. A proof would be most welcome.

Extension of the present results to non-integer values of n seems possible, but depends on material in a forth coming paper [6].

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