ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 1, Spring 2002

THE HAUSDORFF DIMENSION OF THE NONDIFFERENTIABILITY SET OF A NONSYMMETRIC CANTOR FUNCTION

JERRY MORRIS

ABSTRACT. Each choice of numbers a and c in the segment $(0,(1/2))$ produces a Cantor set C_{ac} by recursively removing segments from the interior of the interval $[0,1]$ so that intervals of relative length a and c remain on the left and right sides of the removed segment, respectively. A Cantor function Φ_{ac} is obtained from C_{ac} in much the same way that the standard Cantor function, Φ, is obtained from the Cantor ternary set. When $a = c = (1/3)$, C_{ac} is the Cantor ternary set, C, and Φ_{ac} is the standard Cantor function, Φ. The derivative of Φ is zero off C, and the upper derivative is infinite on C; the set $N = \{x \in C \mid \text{ the lower derivative of } \Phi \text{ is finite}\}$ has Hausdorff dimension $[\ln 2/\ln 3]^2$. In this paper similar results are established for N_{ac}, the nondifferentiability set of Φ_{ac}. The Hausdorff dimension of N_{ac} is the maximum of the real numbers satisfying the following equations: $x(\ln(1/c))^2 = \ln((a+c)/c)\ln((a/c)^x+1)$, and $x(\ln(1/a))^2 = \ln((a+c)/a)\ln((c/a)^x+1)$.

1. Introduction. For any numbers a and c satisfying $0 < a$, $c < 1$, we generate a Cantor set in $[0,1]$ by recursively removing open intervals of relative length $b = 1-a-c$ so that closed intervals of relative length a and c remain to the left and right, respectively, of the removed interval:

$$C_{ac}^0 = [0,1],$$
$$C_{ac}^1 = [0,a] \cup [1-c,1],$$
$$C_{ac}^2 = [0,a^2] \cup [a-ac,a] \cup [1-c,1-c+ac] \cup [1-c^2,1],$$

etc., and $C_{ac} = \cap_{n\geq 1} C_{ac}^n$. We will refer to the set C_{ac}^n as the nth *stage in the construction of* C_{ac} and the 2^n closed intervals comprising C_{ac}^n will be called *stage* n *black intervals* or nth *stage black intervals*. The closures of the open intervals removed at various stages in the construction of C_{ac} will be called *complementary intervals* of the appropriate stage.

Received by the editors on December 13, 2000.

For the set C_{ac}, we describe a corresponding Cantor function $\Phi = \Phi_{ac}$ as a pointwise limit of the functions Φ_{ac}^n, where Φ_{ac}^n is the distribution function of the uniform probability measure on C_{ac}^n. This Cantor function is continuous and non-decreasing and therefore differentiable almost everywhere on the interval $[0, 1]$. We will denote upper left and right derivatives by Φ^- and Φ^+, respectively, and lower left and right derivatives by Φ_- and Φ_+. Another important property of Φ is the following: If x and y are both endpoints of complementary intervals at stage n or less with $x < y$, then the difference $\Phi(y) - \Phi(x)$ is equal to the sum of the lengths of all stage n black intervals between x and y divided by $(a + c)^n$. (An interval I is between x and y if no point of I is strictly less than x or strictly greater than y.)

Since the Lebesgue measure of the set C_{ac}^n is $(a+c)^n$, we see that C_{ac} has Lebesgue measure zero. Chapter 1 in [**4**] introduces the topic of Hausdorff measure and dimension, while a more detailed account can be found in [**3**], [**4**], [**5**] and [**7**]. It can be shown that the Hausdorff dimension of the underlying Cantor set C_{ac} is the unique real number d such that $a^d + c^d = 1$.

In [**2**], Darst treats the case where $a = c$, generating a Cantor function, Φ, based on a symmetric Cantor set, C. Darst showed that the upper derivatives of Φ are infinite on C, so that the set of points, N at which Φ is not differentiable, is the set of points in C at which the lower derivative of Φ is finite. Membership in the set N was characterized in terms of lengths of constant strings of zeroes or twos in the locator sequences of points in C, and this characterization was used to show that the Hausdorff dimension of N is $[\ln(2)/\ln(1/a)]^2$. An analogue of this characterization is given below as Theorem 1.3 and is then used to compute the Hausdorff dimension of N_{ac}, the nondifferentiability set of Φ_{ac}.

Ternary locator sequences for points and intervals. Recall the standard ternary representation for a point in the standard Cantor set using zeroes and twos. For any point t in C_{ac}, we will associate a similar ternary representation $\{t\} = (t(1), t(2), t(3), \dots)$, where $t(i)$ equals 0 or 2, which locates the position of t in C_{ac}. (0 means 'left side' and 2 means 'right side.') We will refer to the above expansion as the *locator sequence* for t.

We will also assign a finite locator sequence of length n to each stage n black interval as follows: The expansion for an interval I will be the first n digits in the locator sequence of any point x contained in the intersection of C_{ac} and I.

The composition of N_{ac}. Our first claim is that for non-endpoints in C_{ac}, the upper derivatives are infinite. The specific result is summarized in the following theorem.

Theorem 1.1. *At any point x in C_{ac} that is a non-right endpoint of C_{ac}, the upper right derivative of Φ_{ac} is infinite.*

Proof. Consider any non-right endpoint x in C_{ac} with locator sequence

$$x = (x(1), x(2), x(3), \dots).$$

For each positive integer n, let $z(n)$ denote the position of the nth zero in the locator sequence of x, and define $\{x_n\} = (x(1), \dots, x(z(n) - 1), 0, 2, 2, \dots)$, so that the locator sequences of x and x_n agree up to the $[z(n+1)-1]$st position. Since there is one stage $z(n+1)$ black interval of length $a^n c^{z(n+1)-n}$ between x_{n+1} and x_n, we have $\Phi(x_n) - \Phi(x) \geq \Phi(x_n) - \Phi(x_{n+1}) = a^n c^{rz(n+1)-n}$, where $r = \ln((a + c)/c)/\ln(1/c)$. Also, since x and x_n share the same stage $z(n + 1) - 1$ black interval, $x_n - x \leq a^n c^{z(n+1)-(n+1)}$. Thus,

$$(\Phi(x_n) - \Phi(x))/(x_n - x) \geq c^{1-(1-r)z(n+1)},$$

Since $r < 1$ and $c < 1$, this proves the theorem. □

The upper left derivatives are handled using the "complementary" Cantor set C_{ca} (in which the intervals of relative length c are on the left and those of length a are on the right). If Φ_{ca} denotes the Cantor function associated with C_{ca}, then $\Phi_{ac}(x) = 1 - \Phi_{ca}(1 - x)$. Thus, the upper left derivative of Φ_{ac} at a non-left endpoint x of C_{ac} is equal to the upper right derivative at the non-right endpoint $1-x$ of C_{ca}, which by the above theorem is infinite.

Fix a and c in the segment $(0, (1/2))$. Let $N^+(N^-)$ denote the set of non-endpoints of C_{ac} at which Φ_{ac} is not differentiable from the right (left). Using a nearly identical argument to that found in [**2**], we arrive at the decomposition

$$N_{ac} = N^+ \cup N^- \cup \{\text{endpoints of } C_{ac}\}. \tag{1.2}$$

The set of endpoints of C_{ac} is countable and therefore has Hausdorff dimension zero, so we may focus our attention on N^+ and then use the complementary Cantor function Φ_{ca} to treat N^-.

Characterization of the nondifferentiability set N_{ac}.

Theorem 1.3. *Let Φ denote the Cantor function associated with the Cantor set C_{ac}, and let t be a non-endpoint of C_{ac}. Let N^+ denote the set of non-endpoints in C_{ac} at which Φ is not differentiable from the right, and set $r = \ln((a+c)/c)/\ln(1/c)$.*

1. *If $t \in N^+$, then $\limsup_{n\to\infty}[z(n+1)/z(n)] \geq r^{-1}$.*
2. *If $\limsup_{n\to\infty}[z(n+1)/z(n)] > r^{-1}$, then $t \in N^+$,*

where $z(n)$ denotes the position of the nth zero in the locator sequence of t.

Proof. 1. Let t be any non-endpoint in C_{ac} with $\limsup\{z(n+1)/z(n)\} < r^{-1}$. Then we can choose a positive real number q and a positive integer m_0 such that

$$r^{-1} - \frac{z(n+1)}{z(n)} \geq q > 0 \quad \text{for all } n \geq m_0. \tag{1.4}$$

Consider any positive integer $n \geq m_0$, let u_n be a positive number satisfying

$$u_n < \text{distance}\,(t, [0,1] - C_{ac}^{z(n)}),$$

and choose any point x in the interval $(t, t+u_n)$. Then t and x share the same stage $z(n)$ black interval, and there exists a positive integer $n_0 > n$ such that $z(n_0)$ is the first stage in the construction of C_{ac} at which t and x are not contained in the same black interval. Thus, $x - t \leq a^{n_0-1}c^{z(n_0)-n_0}$ because x and t share the same stage $z(n_0)-1$ black interval.

Now, let t' and x' be the left and right endpoints, respectively, of the stage $z(n_0+1)$ black interval of length $a^{n_0}c^{z(n_0+1)-n_0}$ which lies immediately to the right of the stage $z(n_0+1)$ black interval containing t. Clearly, $t < t' < x' < x$, so we have

$$\Phi(x) - \Phi(t) \geq \Phi(x') - \Phi(t') = a^{n_0}c^{rz(n_0+1)-n_0}.$$

Combining the above observations and using (1.4), we obtain

$$\begin{aligned}(\Phi(x) - \Phi(t))/(x-t) &\geq ac^{rz(n_0+1)-z(n_0)} \\ &\geq ac^{-rqz(n)}\end{aligned}$$

for all x satisfying $t < x < t + u_n$. But the above holds for all $n \geq m_0$, so we see that $\Phi_+(t) = \infty$, which completes the proof of part one.

2. Take any non-endpoint $t \in C_{ac}$ with $\limsup\{z(n+1)/z(n)\} > r^{-1}$. Then there exist a subsequence $\{n_k\}$ of positive integers, a real number $q > 0$ and a positive integer m_0 such that

$$r^{-1} - \frac{z(n_k+1)}{z(n_k)} \leq -q < 0 \quad \text{for all } k \geq m_0. \tag{1.5}$$

We define two sequences of points, $u(n)$ and $w(n)$, such that $u(n)$ decreases to t and $w(n)$ increases to t. The points $u(n)$ and $w(n)$ are defined below in terms of locator sequences, where $(t(1), t(2), \ldots)$ is the locator sequence of t.

$$\begin{aligned}\{u(n)\} &= \{(t(1), t(2), \ldots, t(z(n)-1), 2, 0, 0, \ldots)\} \\ \{w(n)\} &= \{(t(1), t(2), \ldots, \ldots, t(z(n+1)-1), 0, 0, 0, \ldots)\}.\end{aligned}$$

Now, $w(n) < u(n)$, and there is one stage $z(n)$ complementary interval of length $ba^{n-1}c^{z(n)-n}$ between t and $u(n)$, so that $u(n) - t \geq ba^{n-1}c^{z(n)-n}$. Similarly, there is one stage $z(n+1)-1$ black interval of length $a^n c^{z(n+1)-n-1}$ between $w(n)$ and $u(n)$, so we have

$$\Phi(u(n)) - \Phi(t) \leq \Phi(u(n)) - \Phi(w(n)) = a^n c^{rz(n+1)-r-n}.$$

Thus, for all $k \geq m_0$, we have

$$\begin{aligned}(\Phi(u(n_k)) - \Phi(t))/(u(n_k) - t) &\leq ab^{-1}c^{-r}c^{rz(n_k+1)-z(n_k)} \\ &\leq ab^{-1}c^{-r}c^{qrz(n_k)},\end{aligned}$$

the last inequality following from equation (1.5). It follows that $\Phi_+(t) < \infty$, so t is contained in N^+. $\square$

2. The Hausdorff dimension of N_{ac}. Let $\mathcal{H}^t$ denote t-dimensional Hausdorff outer measure. We will show that the Hausdorff dimension of N_{ac} is the maximum of the unique real numbers d and $\tilde{d}$ defined below:

$$d(\ln(1/c))^2 = \ln((a+c)/c)\ln((a/c)^d + 1)$$
$$\tilde{d}(\ln(1/a))^2 = \ln((a+c)/a)\ln((c/a)^{\tilde{d}} + 1).$$

To accomplish this, we will first show that the Hausdorff dimension of N^+ is d by verifying the following two facts:

(A) If $d < t < t'$, then $\mathcal{H}^t(N^+) = 0$, where t' will be specified later.

(B) If $0 < t < d$, then $\mathcal{H}^t(N^+) > 0$.

As in [**2**], we will verify (A) by constructing a set E (dependent on t) that contains N^+ and satisfies $\mathcal{H}^t(E) = 0$. To verify (B), we will construct a subset E of N^+ satisfying $\mathcal{H}^t(E) > 0$; it will then follow that $\mathcal{H}^{t'}(N^+) \geq H^{t'}(E) = \infty$ for $0 < t' < t$. Combining these observations shows that the Hausdorff dimension of N^+ is d.

Next we will treat N^- by applying the above argument to the complementary Cantor function Φ_{ca}. With this function, we obtain a decomposition

$$N_{ca} = \tilde{N}^+ \cup \tilde{N}^- \cup \{t : t \text{ is an endpoint of } C_{ca}\}$$

analogous to that of (1.2), and we note that

$$\dim_H(N^-) = \dim_H(1 - \tilde{N}^+) = \dim_H(\tilde{N}^+).$$

Since applying Theorems 1.3, 2.2 and 2.5 to Φ_{ca} simply reverses the roles of a and c, we see that the Hausdorff dimension of N^- is $\tilde{d}$. It then follows that the Hausdorff dimension of N_{ac} is the maximum of d and $\tilde{d}$.

Lemma 2.1. *Let $0 < a$, $c < (1/2)$, let $r = \ln((a+c)/c)/\ln(1/c)$, let d be defined as it was above, and define*

$$f(x) = \frac{\ln(a^x + c^x)}{x\ln(1/c)} + 1.$$

1. *The function f is strictly decreasing on the segment $(0, \dim_H(C_{ac}))$.*

2. *The number $x = d$ is the unique solution of the equation $f(x) = r^{-1}$ on the segment $(0, \dim_H(C_{ac}))$.*

Proof. 1. Let x be any number in the segment $(0, \dim_H(C_{ac}))$. Then

$$\begin{aligned} f'(x) = x^2[\ln(1/c)]^{-1}&[a^x + c^x]^{-1} \\ &\cdot [x(a^x \ln a + c^x \ln c) - (a^x + c^x)\ln(a^x + c^x)]. \end{aligned}$$

Since $\dim_H(C_{ac})$ is the unique real number t satisfying $a^t + c^t = 1$, we see that the expression $a^x + c^x$ is strictly greater than one. Therefore, since $\ln a$ and $\ln c$ are both negative numbers, we obtain $f'(x) < 0$.

2. We note that $f(x)$ approaches positive infinity as x approaches zero through positive values and that $f(\dim_H(C_{ac})) = 1$. Therefore, since $r^{-1} > 1$, it follows from the intermediate value theorem that $f(x) = r^{-1}$ has a solution on the desired interval, and uniqueness follows from part one above. Finally, rearranging and using properties of logs reveals that the defining relation of d is equivalent to the equation $f(d) = r^{-1}$. □

The Hausdorff dimension of N^+ is bounded above by d.

Theorem 2.2. *Fix a and c in $(0, (1/2))$, and set $r = \ln((a+c)/c)/\ln(1/c)$. Then the Hausdorff dimension of N^+ is less than or equal to d.*

Proof. By Lemma 2.1 there exists a real number $t' > d$ such that $f(t') > 1$. Choose any real number t satisfying $d < t < t'$. Then it also follows from Lemma 2.1 that

$$1 < f(t) < r^{-1}, \tag{2.3}$$

and we note that the theorem will follow if it can be shown that $\mathcal{H}^t(N^+) = 0$.

Now, letting a point x in C_{ac} be represented once again by its locator sequence $(x(1), x(2), \dots)$, we define a sequence of sets

$$E_k = \{x \in C_{ac} \mid x(k) = 0 \text{ and } x(j) = 2 \text{ for } k < j \leq u_k\}, \quad k = 1, 2, \dots,$$

where u_k is the integer defined by $u_k = \lceil kf(t) + \sigma \ln k \rceil$ and σ is a positive real number (independent of k) to be specified later. Also define

$$E^\infty = \bigcap_{m=1}^{\infty} \bigcup_{k \geq m} E_k = \limsup_{k \to \infty} E_k.$$

Since for each value of k, we have

$$\frac{u_k}{k} \leq \frac{kf(t) + \sigma \ln k + 1}{k} = f(t) + \frac{\sigma \ln k + 1}{k},$$

it follows from (2.3) that there exists a positive real number ε such that $u_k/k \leq r^{-1} - \varepsilon$ holds for k sufficiently large. Therefore, by the first part of Theorem 1.3, $N^+ \subset E^\infty$.

Next, note that the $(k-1)$st stage in the construction of C_{ac} consists of 2^{k-1} black intervals, with $C(k-1, i)$ black intervals of length $a^{k-1-i}c^i$ for $i = 0, 1, \ldots, k-1$. ($C(k-1, i)$ denotes the binomial coefficient $k-1$ choose i). Thus, E_k can be covered by the following collection of intervals: one interval of length $a^k c^{u_k - k}$, $C(k-1, 1)$ distinct intervals of length $a^{k-1}c^{u_k-(k-1)}, \ldots,$ and $C(k-1, k-1) = 1$ interval of length $ac^{u_k - 1}$. We therefore see that $\mathcal{H}^t(E^\infty) = 0$ will hold if

(2.4)
$$c^{(u_k - k)t}(a^t + c^t)^k = a^{-t}(a^t + c^t) \sum_{i=0}^{k-1} C(k-1, i)[a^{k-i}c^{i+(u_k-k)}]^t \leq k^{-p}$$

is true for some fixed $p \geq 2$. Define $p = t\sigma \ln(1/c)$, where σ is a positive real number large enough to make $p \geq 2$. Taking logs and rearranging, we see that equation (2.4) is equivalent to

$$u_k \geq kf(t) + \sigma \ln k,$$

which is clearly true by our definition of u_k. This concludes the proof. $\square$

The Hausdorff dimension of N^+ is bounded below by d.

Theorem 2.5. *Fix a and c in $(0, (1/2))$, and set $r = \ln((a + c)/c)/\ln(1/c)$. Then the Hausdorff dimension of N^+ is greater than or equal to d.*

Proof. Choose any positive real number t satisfying $t < d$. For convenience of notation, we set $q = \ln(a/c)/ln(1/c)$, $\alpha = f(t)$ and $\beta = \alpha - 1$, where f is the function defined in Lemma 2.1. We note that this same lemma guarantees that α and β are both positive.

The set E to be used will correspond to a sequence $0 < k_1 < u_1 < k_2 < u_2 < \cdots$ of positive integers as follows:

$$E = \{x = (x(1), x(2), x(3), \dots) \mid x(k_i) = 0 \text{ and } x(j) = 2 \text{ for } k_i < j \le u_i, \quad i \ge 1\},$$

where $x = (x(1), x(2), x(3), \dots)$ is the locator sequence of $x \in C_{ac}$. It is easy to show that C_{ac} is closed and contains no endpoints of complementary intervals in the construction of C_{ac}. When $k_i \le j \le u_i$, we call j a *fixed choice* for E; otherwise, j is a *free choice*. Let $F(p, q)$ denote the number of free choices j with $p < j \le q$.

To verify (B) and prove the theorem, we will make use of an outer measure μ^t identical to Hausdorff outer measure, except that covers are restricted to only black intervals in the construction of C_{ac} (see Lemma 2.18 following the proof). It is enough to show that $\mathcal{H}^t(E) \ge RPQ$, where P and Q are positive constants to be chosen later, and $R > 0$ is chosen so that $\mathcal{H}^t \ge R\mu^t$ (see Lemma 2.18). We can therefore verify (B) by showing that the following two equations hold:

$$E \subset N^+ \tag{2.6}$$

$$\mu^t(E) \ge PQ. \tag{2.7}$$

Our goal will be to choose k_i and u_i so that the strings of fixed choices are long enough to guarantee that (2.6) holds and the strings of free choices are long enough to guarantee that (2.7) holds.

We are now ready to specify our definitions of u_i and k_i, which will depend on t. We begin by choosing $k_1 = 1$ and $u_1 = 2$. For each $i \ge 2$, we will choose integers k_i and u_i and a real number ν_i such that (2.8), (2.9) and (2.10) hold, where our choices will be made so that $0 \le r_i$, $s_i < 1$ is always satisfied, and ε is a positive real number independent of i.

$$\alpha > \nu_i \ge \max\{\alpha - 1, r^{-1} + \varepsilon\}, \quad i \ge 2 \tag{2.8}$$

$$\alpha F(0, k_i) - \nu_i k_i + qi = s_i, \quad i \ge 2 \tag{2.9}$$

$$u_i = \nu_i k_i + r_i, \quad i \ge 2. \tag{2.10}$$

We note that, by Lemma 2.1, there exists $\varepsilon > 0$ such that $\alpha > r^{-1}+\varepsilon$ so we can immediately choose a sequence of real numbers $\{\nu_i\}_{i\geq 2}$, which satisfies equation (2.8). To see that such choices for k_i and u_i are always possible, fix $i \geq 2$ and suppose that $k_1 < u_1 < k_2 < u_2 < \cdots < k_{i-1} < u_{i-1}$ have been chosen using the method described above. Once k_i has been chosen, we can set $u_i = \lceil \nu_i k_i \rceil$, which satisfies (2.10) and the condition $0 \leq r_i < 1$, so it remains to show that k_i can be chosen so that equation (2.9) is satisfied.

To see this, first rewrite (2.9) in the equivalent form

$$\alpha(F(0,k_i) - k_i) + qi + (\alpha - \nu_i)k_i = s_i, \tag{2.11}$$

and temporarily set $k_i = u_{i-1} + 1$. Clearly $F(0, k_i) - k_i$ is bounded above by $-2(i-1)$, so since $\alpha > 1$, $q < 1$ and $i \geq 2$, the sum of the first two terms on the left-hand side of (2.11) is negative. Thus, there exists a choice for ν_i (possibly larger than our initial choice) which still satisfies (2.8) and makes the entire left-hand side of (2.11) negative. Finally, since $F(0, k_i) - k_i$ remains fixed as the value of k_i is increased, there exists a choice for k_i, greater than u_{i-1}, which yields $s_i \geq 0$; in fact, since $\alpha - \nu_i \leq 1$, this choice for k_i can be made so that $0 \leq s_i < 1$ also holds.

Now since (2.8) holds for all integers $i \geq 2$, it follows from the second part of Theorem (1.3) that (2.6) holds, so the proof will be complete if we can verify (2.7). Let $\{[a_j, b_j]\}_{j\geq 1}$ be any countable cover of E using only black intervals in the construction of C_{ac}. Since E contains no endpoints of C_{ac}, $\{(a_j, b_j)\}_{j\geq 1}$ is an open cover of E, and since E is closed and bounded (and thus compact), we can extract (by re-indexing if necessary) a finite subcover $\{(a_j, b_j)\}_{j=1}^{n}$ of E such that $[a_j, b_j] \cap E \neq \varnothing$ for $1 \leq j \leq n$. Let the term *m-interval* describe any black interval among the 2^m black intervals at the mth stage in the construction of C_{ac}, and let w be the largest value of m for which one of the covering intervals $\{(a_j, b_j)\}_{j=1}^{n}$ is an m-interval. We wish to bound from below the Hausdorff sum associated with our covering intervals; we will accomplish this by estimating the Hausdorff sum using smaller u_i-intervals.

Choose any positive integer $i \geq 2$ large enough so that $u_i, k_i > w$. Let $\mathcal{A}_i$ denote the collection of all u_i-intervals that intersect E, and let $\mathcal{A}_{ij}$ denote the collection of all u_i-intervals contained in $[a_j, b_j]$ that intersect E. Then the proof will be complete if we can show that

$$\sum_{j=1}^{n}(b_j-a_j)^t \geq P\sum_{j=1}^{n}\sum_{I\in\mathcal{A}_{ij}}|I|^t \geq P\sum_{I\in\mathcal{A}_i}|I|^t \geq PQ, \tag{2.12}$$

where P and Q are positive constants, depending on t, to be specified later, and where $|I|$ denotes the diameter of the set I. The second inequality in (2.12) is trivial, so it remains to verify the first and the third, which we will do by rewriting the involved sums in terms of u_i and k_i.

Note first that any u_i-interval is determined by choosing a sequence of zeros and twos of length u_i. To narrow our collection to only those u_i-intervals that intersect E, note that there are $F(0,k_i)$ free choices to be made and i fixed zero choices, meaning that the combined length of all such intervals in $\mathcal{A}_i$ is given by $(a+c)^{F(0,k_i)}a^ic^{u_i-i-F(0,k_i)}$. Similarly, the combined length of the intervals in the collection $\mathcal{A}_{ij}$ is given by the product of the length (b_j-a_j) and $(a+c)^{F(s,k_i)}a^{i-p(s)}c^{[(u_i-s)-(i-p(s))-F(s,k_i)]}$, where s is chosen so that $[a_j,b_j]$ is an s-interval, and $p(s)$ is defined to be the largest positive integer such that $k_{p(s)}\leq s$. Using the expressions for combined length indicated above, it is easy to write down associated Hausdorff sums for the collections $\mathcal{A}_i$ and $\mathcal{A}_{ij}$. We conclude that the third inequality in (2.12) is equivalent to (2.14) below, while the first inequality in (2.12) will follow from (2.13) if P can be chosen independent of s.

$$(a^t+c^t)^{F(s,k_i)}a^{t[i-p(s)]}c^{t[(u_i-s)-(i-p(s))-F(s,k_i)]} \leq P^{-1} \tag{2.13}$$

$$(a^t+c^t)^{F(0,k_i)}a^{ti}c^{t[u_i-i-F(0,k_i)]} \geq Q. \tag{2.14}$$

Rewriting in terms of a common base, we find that the left-hand side of (2.14) becomes

$$(a^t+c^t)^{\beta^{-1}(\alpha F(0,k_i)-\nu_ik_i+qi-r_i)}.$$

Using (2.9) and performing a similar computation for (2.13), we find that equations (2.13) and (2.14) are equivalent to equations (2.15) and (2.16) below, where $\gamma(s,i) := \alpha F(s,k_i)+q(i-p(s))-(u_i-s)$.

$$(a^t+c^t)^{\beta^{-1}\gamma(s,i)} \leq P^{-1} \tag{2.15}$$

$$(a^t+c^t)^{\beta^{-1}(s_i-r_i)} \geq Q. \tag{2.16}$$

It immediately follows that we can choose $Q = (a^t + c^t)^{-\beta^{-1}}$, and the proof will be complete if we can bound the quantity $\gamma(s, i)$ from above. Our claim is that $\gamma(s, i) \leq \gamma(u_p, i)$ where $p = p(s)$. To see that this is the case, it is enough to show that the expression $\gamma(u_p, i) - \gamma(s, i)$ is nonnegative. First note that

$$\gamma(u_p, i) - \gamma(s, i) = (u_p - s) + \alpha[F(u_p, k_i) - F(s, k_i)]. \tag{2.17}$$

If $k_p \leq s \leq u_p$, then $F(u_p, k_i) = F(s, k_i)$, so the right-hand side of (2.17) simplifies to $u_p - s$, which is clearly nonnegative. On the other hand, if $u_p < s < k_{p+1}$, then $F(u_p, k_i) - F(s, k_i) = (s - u_p)$, and since $\alpha > 1$, the desired expression is still nonnegative, which verifies our claim. It therefore suffices to show that $\gamma(u_m, i)$ is bounded for all m such that $u_m < k_i$.

Choose any positive integer m such that $u_m < k_i$. Then

$$\begin{aligned}\gamma(u_m, i) &= \alpha F(u_m, k_i) + q(i - m) - (u_i - u_m)\\ &= (\alpha F(0, k_i) + qi - \nu_i k_i - r_i) - (\alpha F(0, k_m) + qm - \nu_m k_m - r_m)\\ &= (s_i - r_i) - (s_m - r_m) \leq 2,\end{aligned}$$

so we can choose $P = (a^t + c^t)^{-2\beta^{-1}}$. This completes the verification of (2.7), and the proof is complete. □

Lemma 2.18. *Let C_{ac} denote the relevant Cantor set, and let $\mathcal{T}$ be the collection of all black intervals in the construction of C_{ac}. Define, for each subset A of C_{ac} and each $z > 0$,*

$$\mu^z_\delta(A) = \inf \sum_{i \geq 1} |U_i|^z,$$

where the infimum is taken over all at most countable δ-covers $\{U_i\}_{i\geq 1}$ of A, where $U_i \in \mathcal{T}$ for all i. Similarly, define

$$\mu^z(A) = \lim_{\delta \to 0^+} \mu^z_\delta(A).$$

Then μ^z is an outer measure on the collection of subsets of C_{ac}. Furthermore, there exists a constant $R > 0$, depending on z, such that

$$\mathcal{H}^z \geq R\mu^z. \tag{2.19}$$

Proof. Routine checks show that μ^z is an outer measure on the collection of subsets of C_{ac}. Now, choose a positive integer N large enough so that

$$L := \max\{a^N, c^N\} < b,$$

where $b = 1 - (a + c)$, and set $R = 2^{-N}$. Next, take any $A \subset C_{ac}$, choose $\delta > 0$, and consider any δ-cover $\{U_i\}$ of A. Since A is contained in C_{ac}, we may assume that the sets U_i in our δ-cover are subsets of C_{ac}. We may also assume that U_i contains at least two points for each i, since otherwise we would have $|U_i| = 0$, so the set would have no effect on the associated sums.

As a result of our assumptions, we see that for each i, there exists a largest nonnegative integer $k(i)$ such that U_i is a subset of one of the $2^{k(i)}$ black intervals at the $k(i)$th stage in the construction of C_{ac}; suppose that this particular black interval has length L_i. Since $k(i)$ is maximal, U_i cannot be contained in either of the two stage $k(i)+1$ black intervals which are subsets of the interval of length L_i, so it follows that $|U_i| \geq bL_i$, and we have

$$LL_i < bL_i \leq |U_i|.$$

Now, consider the collection of stage $k(i) + N$ black intervals which are subsets of the original interval of length L_i. Clearly, there are 2^N intervals in this collection, each having length less than or equal to LL_i. Let $\{S_{ij}\}_{j=1}^{2^N}$ denote this covering collection, and note that for each i the associated collection forms a δ-cover of U_i. It follows that $\{S_{ij}\}_{i,j}$ is a δ-cover of A, and we have

$$\mu_\delta^z(A) \leq \sum_{i,j} |S_{ij}|^z = \sum_{i\geq 1} \sum_{j=1}^{2^N} |S_{ij}|^z \leq 2^N \sum_{i \geq 1} |U_i|^z.$$

Finally, if we take the infimum over all δ-covers $\{U_i\}$, and let $\delta \to 0^+$ in the above equation, we obtain (2.19). □

REFERENCES

1. R.B. Darst, *The Hausdorff dimension of the nondifferentiability set of the Cantor function is* $[\ln 2/\ln 3]^2$, Proc. Amer. Math. Soc. **119** (1993), 105–108.

2. ———, *Hausdorff dimension of sets of non-differentiability points of Cantor functions*, Math. Proc. Cambridge Philos. Soc. **117** (1995), 185–191.

3. G.A. Edgar, *Measure, Topology, and Fractal Geometry*, Springer-Verlag, New York, 1990.

4. K.J. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, London, 1985.

5. ———, *Fractal Geometry*, John Wiley, New York, 1990.

6. Frank Jones, *Lebesgue Measure on Euclidean Space*, Jones and Bartlett Publ., Inc., Boston, 1993.

7. C.A. Rogers, *Hausdorff Measures*, Cambridge University Press, Cambridge, 1970.

Department of Mathematics, Colorado State University, Ft. Collins, CO 80523-1874

and

Department of Mathematics, Sonoma State University, Rohnert Park, CA 94928-3609
E-mail address: `jmorris@math.arizona.edu`