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ON THE ŁOŚ-MARCZEWSKI EXTENSION OF FINITELY ADDITIVE MEASURES

Abstract

In this short note, we give a relatively simple proof of the Łoś-Marczewski Extension of finitely additive measures. In particular, we extend the Łoś-Marczewski Extension to Dedekind complete Riesz-space-valued functions.

1 Introduction.

Recall that a ring $\mathcal{R}$ on a nonempty set X is a subset of the power set 2^X such that $A \cup B$ and $A \setminus B$ belong to $\mathcal{R}$ whenever A and B do. A map $\mu : \mathcal{R} \to [-\infty, \infty]$ is called a *charge* if $\mu(\emptyset) = 0$, and $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{R}$ are disjoint. If for each $A \in \mathcal{R}$ the value $\mu(A)$ is positive, then μ is said to be a *finitely additive measure*.

In [2] Łoś and Marczewski has proved the following well-known extension theorem: if $\mathcal{R}$ is a ring on X and $\mu : \mathcal{R} \to [0, \infty)$ is a finitely additive measure, then μ has a finitely additive extension $\overline{\mu} : 2^X \to [0, \infty)$. One can consult on [3] for a proof (see also [4]). In case where one takes $\mathbb{R}$ instead of $[0, \infty)$, a proof of this extension theorem can be given as follows: let

$$E := \left\{ \sum_{i=1}^{n} \alpha_i \chi_{E_i} \mid n \in \mathbb{N},\ \alpha_i \in \mathbb{R},\ E_i \in \mathcal{R} \right\};$$

then E is a vector subspace of

$$F := \left\{ \sum_{i=1}^{n} \alpha_i \chi_{E_i} \mid n \in \mathbb{N},\ \alpha_i \in \mathbb{R},\ E_i \subset X \right\}.$$

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Define the linear map $t : E \to \mathbb{R}$ by

$$t\left(\sum_{i=1}^{n} \alpha_i \chi_{E_i}\right) := \sum_{i=1}^{n} \alpha_i \mu(E_i),$$

which is readily seen to be well-defined. Choose a vector subspace G of F such that $F = E \oplus G$. Let P be the projection from F into E. Define now the map $T : F \to F$ by $T := t \circ P$. It then follows that the map $\overline{\mu} : 2^X \to \mathbb{R}$ given by $\overline{\mu}(A) := T(\chi_A)$ is a charge and an extension of μ. But there is no guarantee that μ is a finitely additive measure, whence the Łoś-Marczewski extension is partially lost at this point. But, as will be seen below, we can balance this. First let us recall a version of the Hahn-Banach Theorem (see [1, Theorem 2.1]). For all unexplained terminology for Riesz spaces, we refer to [1].

Theorem 1 (Hahn-Banach). *Let G be a vector space, F be a Dedekind complete Riesz space, and $p : G \to F$ be a sublinear function. If H is a vector subspace of G and $S : H \to F$ is an operator satisfying $s(x) \leq p(x)$ for all $x \in H$, then there exists an operator $T : G \to F$ such that*

(i) *$T = S$ on H;*

(ii) *$T(x) \leq p(x)$ holds for all $x \in G$.*

2 The Extension Theorem.

Now using the version of the Hahn-Banach Theorem given in Theorem 1, we can extend and give an elementary proof of the well-known Łoś-Marczewski Extension Theorem.

Theorem 2. *Let F be a Dedekind complete Riesz space, $\mathcal{R}$ be a ring on a set X, and $\mu : \mathcal{R} \to E^+$ be a finitely additive measure. Then μ has a finitely additive extension $\overline{\mu} : 2^X \to F^+$, where F^+ denotes the set of positive elements of E.*

Proof. Set

$$H := \left\{\sum_{i=1}^{n} \alpha_i \chi_{E_i} \mid n \in \mathbb{N},\ \alpha_i \in \mathbb{R},\ E_i \in \mathcal{R}\right\}.$$

Clearly, H is a Riesz subspace of the Riesz space

$$E := \left\{\sum_{i=1}^{n} \alpha_i \chi_{E_i} \mid n \in \mathbb{N},\ \alpha_i \in \mathbb{R},\ E_i \subset X\right\}.$$

Define $S : H \to F$ by

$$S\left(\sum_{i=1}^{n} \alpha_i \chi_{E_i}\right) := \sum_{i=1}^{n} \alpha_i \mu(E_i),$$

and observe that S is a positive operator. Also, define $p : E \to E$ by

$$p(f) := \inf\{S(g) : g \in H,\ f^+ \leq g\}.$$

It is straightforward to check that p is a sublinear map, and that for each $f \in H$ one has

$$S(f) \leq S(f^+) \leq S(g)$$

whenever g belongs to G with $f^+ \leq g$. This implies that $S \leq p$ on G. Hence, by the Hahn-Banach Theorem, S has an extension $T : E \to F$. Moreover, if $0 \leq f \in E$, then

$$-T(f) = T(-f) \leq p(-f) = 0,$$

yielding that T is also a positive operator. Now define $\overline{\mu} : 2^X \to E$ by

$$\overline{\mu}(A) := T(\chi_A).$$

Obviously, $\overline{\mu}$ is a signed extension of μ. Since T is positive, it follows that $\overline{\mu}$ is a finitely additive measure. This completes the proof. $\square$

References

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