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UNIFORM DIFFERENTIABILITY

Abstract

The concept of uniform differentiability is introduced to characterize sequences of McShane and Henstock equi-integrable functions.

1 Introduction

In [14], the McShane derivative is introduced. It deviates from the ordinary definition by using McShane interval-point pairs. The same derivative is also called strong derivative in [2, 6, 14]. Chew in [5] characterized the Lebesgue and Bochner integrals using strong derivatives together with inner variation and Lusin condition.

Equi-integrability is relatively well-known (see [13, 15]). In [13], a convergence theorem is proved for a sequence of McShane integrable functions based on equi-integrability and this convergence theorem is equivalent to the Vitali convergence theorem.

In this paper, we introduce the concept of uniformly strongly differentiable functions, and investigate the McShane and Henstock integrals in terms of equi-integrability and uniform strong differentiability.

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Let us revisit the following definitions. See [9, 10, 11, 13].

Definition 1. Let δ be a positive function on a closed interval $[a, b]$. A tagged division $D = \{([u, v], \xi)\}$ of $[a, b]$ is said to be a *McShane δ-fine division of $[a, b]$ if $[u, v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$ and $\xi \in [a, b]$ for every $([u, v], \xi) \in D$.* A function $f : [a, b] \to \mathbb{R}$ is said to be *McShane integrable* to a real number A on $[a, b]$ if for each $\epsilon > 0$, there exists $\delta(\xi) > 0$ on $[a, b]$ such that whenever $D = \{([u, v], \xi)\}$ is a McShane δ-fine division of $[a, b]$, we have

$$\left|(D) \sum f(\xi)(v - u) - A\right| < \epsilon.$$

If we choose $\xi \in [u, v]$ for every interval-point pair in any δ-fine division $D = \{[u, v]; \xi\}$, we obtain the definition of the *Henstock* integral. We use the symbols

$$(\mathcal{M}) \int_a^b f \text{ and } (\mathcal{H}) \int_a^b f$$

to denote the McShane integral and Henstock integral of f on $[a, b]$, respectively.

Definition 2. A sequence $\{f_n\}$ of McShane (resp. Henstock) integrable functions on $[a, b]$ is *equi-integrable* on $[a, b]$ if for any $\epsilon > 0$, there exists $\delta(\xi) > 0$ such that

$$\left|(D) \sum f_n(\xi)(v - u) - (\mathcal{M}) \int_a^b f_n\right| < \epsilon, \quad \text{for all } n$$

$$\left(\text{resp. } \left|(D) \sum f_n(\xi)(v - u) - (\mathcal{H}) \int_a^b f_n\right| < \epsilon, \quad \text{for all } n\right)$$

whenever $D = \{([u, v], \xi)\}$ is a McShane (resp. Henstock) δ-fine division of $[a, b]$.

2 McShane integral

In this section, we shall characterize the McShane equi-integrability in terms of uniform strong differentiability.

Definition 3. Let $\{F_n\}$ be a sequence of functions defined on $[a, b]$. We say that $\{F_n\}$ is *uniformly strongly differentiable at $x \in [a, b]$*, with f_n being the strong derivative of F_n, if for each $\epsilon > 0$, there exists $\delta(x) > 0$ such that for each n, we have

$$|F_n(v) - F_n(u) - f_n(x)(v - u)| < \epsilon|v - u|$$

whenever $[u, v] \subset (x - \delta(x), x + \delta(x))$. $\{F_n\}$ is said to be *uniformly strongly differentiable on* $X \subseteq [a, b]$ if it is uniformly strongly differentiable at every $x \in X$. We denote the difference $F_n(v) - F_n(u)$ by $F_n(u, v)$.

Consider the following example.

Example 4. Let $F_n(x) = \frac{1}{n+1}x^{n+1}$ and $f_n(x) = x^n$ for $x \in [0, a]$ with $a < 1$. Given $\epsilon > 0$, define $\delta : [0, a] \to \mathbb{R}^+$ by $\delta(x) = \frac{1}{2}\epsilon(1 - a)$. Let n be any positive integer and $([u, v], x)$ be any interval-point pair such that $[u, v] \subset (x - \delta(x), x + \delta(x))$. Applying the Mean Value Theorem to $g_n(x) = x^{n+1}$ defined on $[0, a]$, we can find $w \in (u, v)$ such that

$$g_n'(w) = \frac{g_n(v) - g_n(u)}{v - u} = \frac{v^{n+1} - u^{n+1}}{v - u}$$

that is,

$$(n+1)w^n = \frac{v^{n+1} - u^{n+1}}{v - u}.$$

Since $x, w \in (x - \delta(x), x + \delta(x))$, we have

$$|w^n - x^n| \leq |w - x| \cdot \sum_{k=1}^{n} |w|^{n-k}|x|^{k-1} < 2\delta(x) \cdot \sum_{k=1}^{\infty} \left(1^{n-k} \cdot a^{k-1}\right) = \frac{2\delta(x)}{1 - a}.$$

Hence,

$$\begin{aligned}
\Big|F_n(u, v) - f_n(x)(v - u)\Big| &= \left|\frac{v^{n+1}}{n+1} - \frac{u^{n+1}}{n+1} - x^n(v - u)\right| \\
&= \frac{|v - u|}{n+1} \cdot \left|\frac{v^{n+1} - u^{n+1}}{v - u} - x^n(n+1)\right| \\
&= \frac{|v - u|}{n+1} \cdot \Big|(n+1)w^n - (n+1)x^n\Big| \\
&= |v - u| \cdot \big|w^n - x^n\big| \\
&< |v - u| \cdot \frac{2\delta(x)}{1 - a} = |v - u| \cdot \frac{2}{1 - a} \cdot \frac{\epsilon(1 - a)}{2} \\
&< \epsilon|v - u|.
\end{aligned}$$

Therefore, $\{F_n\}$ is uniformly strongly differentiable on $[0, a]$ for $a < 1$. ◀

As seen in the above example, $\{F_n\}$ is uniformly strongly differentiable on the compact interval $[0, 1]$ except at the point 1.

We now introduce the concept of "covering relation", see [12].

Definition 5. Let $\mathcal{I}$ be the collection of all closed interval in $\mathbb{R}$ and $E \subseteq \mathbb{R}$. A *covering relation* β on E is a subset of $\mathcal{I} \times E$ with the property that for each $x \in E$ there exists $I \in \mathcal{I}$ such that $(I, x) \in \beta$.

Note that x is not necessarily in I; whence, we call β as *McShane covering relation on* E. If $x \in I$, then we call β as *Henstock covering relation on* E. A covering relation β is said to be *fine at a point* x if for each $\delta(x) > 0$, there exists a interval-point pair $([y, z], x) \in \beta$ such that $[y, z] \subset (x - \delta(x), x + \delta(x))$. β is a *fine covering relation on a set* E if it is fine at each point of E. Henstock refers to a fine covering relation on E as an *inner covering* of E.

We now give the definition of inner variation zero.

Definition 6. Let E be a set of real numbers. For a covering relation β on E, we define $Var(\beta) = \sup \sum_{(I,x)\in P} |I|$, where the supremum is taken over all partial divisions $P \subseteq \beta$. Let $IV(E) = \inf Var(\beta)$, where the infimum is taken over all McShane (resp. Henstock) inner coverings β on E. We say that E is of *McShane* (*resp. Henstock*) *inner variation zero* if $IV(E) = 0$.

In $\mathbb{R}^1$, Henstock inner variation zero is equivalent to measure zero, see [4]. However, a set of McShane inner variation zero may not be of measure zero, as was pointed out by Henstock in [7, p.136]. Example 12 below shows the Cantor set P of positive measure to be of McShane inner variation zero, as was also implicitly proved in [14].

Using the above definition, it is easy to check the following lemma.

Lemma 7. *A set $E \subset \mathbb{R}$ is of McShane* (*resp. Henstock*) *inner variation zero if and only if for each $\epsilon > 0$ there exists a McShane* (*resp. Henstock*) *inner covering β_0 of E such that for each partial division $P = \{([u, v], \xi)\}$ with $P \subseteq \beta_0$, we have*

$$(P) \sum |v - u| < \epsilon.$$

The following result was proved in [8].

Theorem 8. *If f is McShane integrable on $[a, b]$, then its primitive F has strong derivative $F^*(x)$ and $F^*(x) = f(x)$ everywhere except on a set of McShane inner variation zero.*

We now state and prove our main results.

Theorem 9. *If $\{f_n\}$ is McShane equi-integrable on $[a, b]$ with primitives F_n, then $\{F_n\}$ is uniformly strongly differentiable on $[a, b] \setminus S$, where S is a set of McShane inner variation zero.*

Proof: Let S be the subset of $[a,b]$ where $\{F_n\}$ is not uniformly strongly differentiable. Then for each $x \in S$, there exists $\eta(x) > 0$ such that for each $\delta(x) > 0$, there exist m and a McShane δ-fine interval-point pair $([u,v],x)$ such that

$$\Big|F_m(u,v) - f_m(x)(v-u)\Big| \geq \eta(x) \cdot |v-u|. \tag{1}$$

Now, fix $k \in \mathbb{N}$. Let

$$S_{m,k} = \big\{x \in S : \text{ there exists } m \text{ such that (1) holds and } \eta(x) \geq \tfrac{1}{k}\big\}.$$

Then $S = \bigcup_{m,k} S_{m,k}$. To show that S is of McShane inner variation zero, it suffices to show that each $S_{m,k}$ is of McShane inner variation zero.

Fix m and k. Let $\epsilon > 0$. Since $\{f_n\}$ is equi-integrable, there exists $\delta(\xi) > 0$ such that for each n and for any McShane δ-fine division $D = \{([u,v],\xi)\}$ of $[a,b]$, we have

$$(D)\sum \Big|F_n(u,v) - f_n(\xi)(v-u)\Big| < \frac{\epsilon}{k}.$$

Let $\mathcal{C}_{m,k}(\epsilon)$ be the set of all interval-point pairs $([u,v],x)$ such that $([u,v],x)$ is McShane δ-fine and $\Big|F_m(u,v) - f_m(x)(v-u)\Big| \geq \frac{1}{k}|v-u|$. Then $\mathcal{C}_{m,k}(\epsilon)$ is a McShane inner covering of $S_{m,k}$. Let $P = \{([u,v],x)\}$ be any McShane δ-fine partial division with $P \subset \mathcal{C}_{m,k}$. Then for any $([u,v],x) \in P$, we have

$$\Big|F_m(u,v) - f_m(x)(v-u)\Big| \geq \tfrac{1}{k} \cdot |v-u|.$$

Thus,

$$\frac{\epsilon}{k} > (P)\sum \Big|F_m(u,v) - f_m(\xi)(v-u)\Big| \geq \tfrac{1}{k} \cdot (P)\sum |v-u|,$$

that is,

$$(P)\sum |v-u| < \epsilon.$$

Thus, $S_{m,k}$ is of McShane inner variation zero. ■

In [11], it is shown that if $f_k : I \to \mathbb{R}$ are McShane integrable over I with $f_k(t) \uparrow f(t) \in \mathbb{R}$ for every $t \in I$ and $\sup_k \int_I f_k < \infty$, then $\{f_k\}$ is equi-integrable over I. It is also shown that if $f_k, g : I \to \mathbb{R}$ are McShane integrable over I with $|f_k| \leq g$ on I and $f_k \to f$ pointwise on I, then $\{f_k\}$ is equi-integrable over I. Thus, we have the following corollaries.

Corollary 10. *Let $\{f_n\}$ be an increasing sequence of McShane integrable functions on $[a,b]$ with primitives F_n and $f_n \to f$ pointwisely on $[a,b]$. If $\sup\left\{(\mathcal{M})\int_a^b f_n : n \in \mathbb{N}\right\} < \infty$, then $\{F_n\}$ is uniformly strongly differentiable on $[a,b]$ except on a set of McShane inner variation zero.*

An analogous result also holds for a decreasing sequence.

Corollary 11. *Let $\{f_n\}$ be a sequence of McShane integrable functions on $[a,b]$ with primitives F_n and $f_n \to f$ pointwisely on $[a,b]$. If $g : [a,b] \to \mathbb{R}$ is McShane integrable on $[a,b]$ and $|f_n - f_m| \le g$ for each n,m, then $\{F_n\}$ is uniformly strongly differentiable on $[a,b]$ except on a set of McShane inner variation zero.*

We refer to the preceding two corollaries being the Monotone Convergence Theorem and the Dominated Convergence Theorem, respectively.

Example 12. Let $0 < \alpha < 1$. A Cantor set $P \subset [0,1]$ can be constructed as follows (see [3]): From the interval $[0,1]$, remove an open interval, centered at $\frac{1}{2}$ whose length is $\frac{1}{2}\alpha$. Thus, leaving two closed intervals (each of length $\frac{1}{2} - \frac{1}{4}\alpha$). Let P_1 be the union of the two closed intervals. At the second stage, we shall remove from P_1 two further open intervals (each of length $\frac{1}{8}\alpha$), one from each of the two closed intervals, leaving P_2 consisting of four closed intervals (each of length $\frac{1}{4} - \frac{3}{16}\alpha$). We proceed inductively and at the nth stage, we are left with P_n consisting of 2^n closed intervals (each of length $\frac{1}{2^n} - \frac{2^n-1}{2^{2n}}\alpha$). We define P as the intersection of all the P_n's. The set P is nowhere dense (see [3]). Define the sequence $\{f_n\}$ by

$$f_n(x) = \begin{cases} 1, & x \in P_n, \\ 0, & x \notin P_n. \end{cases}$$

Then $\{f_n\}$ is a decreasing sequence of McShane integrable functions on $[0,1]$ and $f_n \to f$ pointwisely on $[0,1]$, where f is the function defined by

$$f(x) = \begin{cases} 1, & x \in P, \\ 0, & x \notin P. \end{cases}$$

Moreover, $(\mathcal{M})\int_0^1 f_n(x)\,dx = 1 - \alpha + \dfrac{\alpha}{2^n}$ for each n. Thus,

$$\lim_{n\to\infty} (\mathcal{M})\int_0^1 f_n(x)\,dx = 1 - \alpha < \infty.$$

Hence, by Corollary 10, the sequence of primitives $\{F_n\}$ of f_n is uniformly strongly differentiable on $[0,1]$ except on a set S of McShane inner variation zero. Next, we will show that S is indeed P.

Let S be the set of all $x \in [0,1]$ such that $\{F_n\}$ is not uniformly strongly differentiable at x. Note that $x \in S$ if and only if there exists $\eta(x) > 0$ such that for each $\delta(x) > 0$, there exist $m \in \mathbb{N}$ and a McShane δ-fine interval-point pair $([u,v], x)$ such that

$$\Big|F_m(u,v) - f_m(x)(v-u)\Big| \geq \eta(x) \cdot |v-u|.$$

If $x \in P$, then $x \in P_n$ for each n; so $f_n(x) = 1$ for each n. Take $\eta(x) = \frac{1}{2}$ and consider any $\delta(x) > 0$. Since P is nowhere dense in $[0,1]$, the interval $(x-\delta(x), x+\delta(x))$ has a subinterval (u,v) containing no points of P. Note that x may not be in (u,v). Hence,

$$\varnothing = (u,v) \cap \Big(\bigcap_{n=1}^{\infty} P_n\Big) = \bigcap_{n=1}^{\infty} \Big(P_n \cap (u,v)\Big).$$

Since $P_{n+1} \subset P_n$ for each n, there exists $m \in \mathbb{N}$ such that $P_m \cap (u,v) = \varnothing$. Thus, $f_m(t) = 0$ for each $t \in (u,v)$. Therefore,

$$\begin{aligned} \Big|F_m(u,v) - f_m(x)(v-u)\Big| &= \Big|(\mathcal{M})\int_u^v f_m(t)dt - 1\cdot(v-u)\Big| \\ &= |v-u| \\ &\geq \eta(x)\cdot|v-u|. \end{aligned}$$

This implies that $x \in S$. Hence, $P \subseteq S$.

Now, we shall prove that $S \subseteq P$. Let $x \in S$ and suppose $x \in [0,1] \setminus P$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $x \notin P_N$. Since $P_{n+1} \subset P_n$ for each n, it follows that $x \notin P_n$ for each $n \geq N$. Define $\delta(x) > 0$ such that $(x-\delta(x), x+\delta(x))$ contains no points in P_N. Thus, $(x-\delta(x), x+\delta(x))$ contains no points in P_n for each $n \geq N$ and so, $f_n(x) = 0$ for each $n \geq N$. If $[u,v] \subset (x-\delta(x), x+\delta(x))$, then $[u,v]$ contains no points in P_n for each $n \geq N$. Hence, for each $n \geq N$, $F_n(u,v) = 0$ implying that

$$\big|F_n(u,v) - f_n(x)(v-u)\big| = 0 < \epsilon \cdot |v-u|.$$

Suppose $k < N$. To show that F_k is strongly differentiable on $[0,1] \setminus P$, let $x_0 \in [0,1] \setminus P$. Let m be the least positive integer such that $x_0 \notin P_m$. If $m \leq k$, then $f_k(x_0) = 0$. Define $\delta_k(x_0) > 0$ such that $(x_0 - \delta_k(x_0), x_0 + \delta_k(x_0)) \cap P_k = \varnothing$. Thus, for each $[u,v] \subset (x_0 - \delta_k(x_0), x_0 + \delta_k(x_0))$

$$\big|F_k(u,v) - f_k(x_0)(v-u)\big| = 0 < \epsilon \cdot |v-u|.$$

On the other hand, if $m > k$, then $x_0 \in P_k$. Moreover, x_0 is not an endpoint of P_k. Thus, $f_k(x_0) = 1$. Define $\delta_k(x_0) > 0$ such that $(x_0 - \delta_k(x_0), x_0 + \delta_k(x_0)) \subset P_k$. Hence, for each $[u, v] \subset (x_0 - \delta_k(x_0), x_0 + \delta_k(x_0))$, we have $f_k(t) = 1$ for each $t \in [u, v]$ and

$$\begin{aligned} \left|F_k(u,v) - f_k(x_0)(v-u)\right| &= \left|(\mathcal{M})\int_u^v f_k(t)dt - f_k(x_0)(v-u)\right| \\ &= \left|(\mathcal{M})\int_u^v dt - (v-u)\right| \\ &= 0 \\ &< \epsilon \cdot |v-u|. \end{aligned}$$

In either case, we have

$$\left|F_k(u,v) - f_k(x_0)(v-u)\right| < \epsilon \cdot |v-u|.$$

Thus, F_k is strongly differentiable on $[0,1]\setminus P$ for each $k < N$. Consequently, $\{F_n\}$ is uniformly strongly differentiable on $[0,1]\setminus P$. This is impossible because $x \in S$. Hence, $x \in P$; that is, $S \subseteq P$. Accordingly, $S = P$. ◀

Next, we formulate the converse. We start with the following definition.

Definition 13. Let $\{F_n\}$ be a sequence of functions defined on $[a, b]$ and $X \subset [a, b]$. $\{F_n\}$ is said to be *uniformly McShane* (*resp. Henstock*) *AC*(X) if for each $\epsilon > 0$, there exist $\eta > 0$ and $\delta(\xi) > 0$ on X such that for any McShane (resp. Henstock) δ-fine partial division $P = \{([u, v], x)\}$ with $x \in X$ and $(P)\sum |v - u| < \eta$, we have

$$(P)\sum \left|F_n(v) - F_n(u)\right| < \epsilon, \quad \text{for all } n.$$

In Theorem 9, we defined the set S as follows: $x \in S$ if and only if there exists $\eta(x) > 0$ such that for each $\delta(x) > 0$, there exist m and a McShane δ-fine interval-point pair $([u, v], x)$ such that

$$\left|F_m(u,v) - f_m(x)(v-u)\right| \geq \eta(x) \cdot |v-u|. \tag{2}$$

The idea of the following proof follows that of a single function used in [4, 5]. In the following result, let $\beta(\eta, \delta)$ denotes the collection of all McShane δ-fine interval-point pair $([u, v], x)$ with $x \in S$ such that inequality (2) holds.

Theorem 14. *Let $\{f_n\}$ be a sequence of McShane integrable functions on $[a, b]$ and $\lim_{n\to\infty} f_n(x) = f(x)$ pointwisely on $[a, b]$. If the sequence $\{F_n\}$ of primitives of f_n is uniformly strongly differentiable on $[a, b]\setminus S$ and suppose the following conditions hold:*

(*i*) *for each* $\epsilon > 0$*, there exists* $\delta(x) > 0$ *on* S *such that for any McShane* δ*-fine partial division* $P = \{([u,v],x)\}$ *with* $P \subseteq \beta(\epsilon,\delta)$*, we have*

$$(P)\sum |v-u| < \epsilon, \qquad \textit{and}$$

(*ii*) $\{F_n\}$ *is uniformly McShane* $AC(S)$*.*

Then $\{f_n\}$ *is equi-integrable on* $[a,b]$*.*

Proof: Let $\epsilon > 0$. Since $\{F_n\}$ is uniformly strongly differentiable on $[a,b]\setminus S$, for each $x \in [a,b]\setminus S$ there exists $\delta_0(x) > 0$ such that whenever $([u,v],x)$ is a McShane δ_0-fine interval-point pair with $x \in [a,b]\setminus S$, we have

$$|F_n(v) - F_n(u) - f_n(x)(v-u)| \quad < \quad \epsilon \cdot |v-u|, \quad \text{for each } n. \tag{3}$$

Since $f_n \to f$ pointwisely on $[a,b]$, so for each $x \in [a,b]$, $\{f_n(x)\}$ is bounded. For each $i \in \mathbb{N}$, let

$$S_i = \{x \in S : \ i-1 \le |f_n(x)| < i, 1 \text{ for all } \ n\}.$$

Then $S = \bigcup_{i=1}^{\infty} S_i$. Since $\{F_n\}$ is uniformly McShane $AC(S)$, $\{F_n\}$ is also uniformly McShane $AC(S_i)$ for each i. Hence, for each i, there exist $\eta_i > 0$ and $\delta_i(x) > 0$ on S_i such that for any McShane δ_i-fine partial division $P_i = \{([u,v],x)\}$ with $x \in S_i$ and $(P_i)\sum |v-u| < \eta_i$, we have

$$(P_i)\sum \big|F_n(v) - F_n(u)\big| \quad < \quad \frac{\epsilon}{2^i} \quad \text{ for each } n.$$

For each i, let $\mu_i = \min\left\{\frac{\epsilon}{i2^i}, \eta_i\right\}$. Since $S_i \subseteq S$ for each i, condition (*i*) also holds. Thus, there exists a positive function $\delta_i'(x) \le \delta_i(x)$ on S_i such that for any McShane δ_i'-fine partial division $P_i = \{([u,v],x)\}$ with $P_i \subseteq \beta(\epsilon,\delta_i)$, we have

$$(P_i)\sum |v-u| < \mu_i \le \frac{\epsilon}{i2^i}.$$

Note that if $x \in S$, then there exists i such that $x \in S_i$. Define $\delta : [a,b] \to \mathbb{R}$ by

$$\delta(x) = \begin{cases} \delta_0(x) & , \text{ if } x \in [a,b]\setminus S, \\ \min\{\delta_0(x), \delta_i(x), \delta_i'(x)\} & , \text{ if } x \in S_i. \end{cases}$$

Let $D = \{([u_j,v_j],x_j)\}_{j=1}^{r}$ be any McShane δ-fine division of $[a,b]$. Let

$$\begin{aligned} D_1 &= \{([u_j,v_j],x_j) \in D : x_j \notin S\}, \\ D_2 &= \{([u_j,v_j],x_j) \in D : x_j \in S \ \text{ and } ([u_j,v_j],x_j) \text{ satisfies (3)}\} \quad \text{ and} \\ D_3 &= \{([u_j,v_j],x_j) \in D : x_j \in S \ \text{ and } ([u_j,v_j],x_j) \in \beta(\epsilon,\delta)\}. \end{aligned}$$

Then, for each n

$$
\begin{aligned}
&(D_1 \cup D_2) \sum \left|F_n(v_j) - F_n(u_j) - f_n(x_j)(v_j - u_j)\right| \\
&\qquad < (D_1 \cup D_2) \sum \left(\epsilon \cdot |v_j - u_j|\right) \\
&\qquad \leq \epsilon \cdot (b - a). \qquad (4)
\end{aligned}
$$

Note that D_3 is a McShane δ-fine partial division with $D_3 \subseteq \beta(\epsilon, \delta)$ and $x_j \in S$. So D_3 can be expressed as the union of some McShane δ_i'-fine partial divisions $P_i = \{([u, v], x)\}$ with $P_i \subseteq \beta(\epsilon, \delta_i)$ and $(P_i) \sum |v - u| < \mu_i$. Thus, for each n

$$
\begin{aligned}
(D_3) \sum \left|F_n(v_j) - F_n(u_j)\right| &= \sum \left[(P_i) \sum \left|F_n(v) - F_n(u)\right|\right] \\
&< \sum \frac{\epsilon}{2^i} \\
&\leq \epsilon
\end{aligned}
$$

and

$$
\begin{aligned}
(D_3) \sum \left(|f_n(x_j)| \cdot |v_j - u_j|\right) &= \sum \left[(P_i) \sum \left(|f_n(x_j)| \cdot |v - u|\right)\right] \\
&\leq \sum \left(i \cdot (P_i) \sum |v - u|\right) \\
&< \sum \left(i \cdot \mu_i\right) \\
&\leq \sum \left(i \cdot \frac{\epsilon}{i2^i}\right) \\
&\leq \epsilon.
\end{aligned}
$$

Thus, for each n

$$
\begin{aligned}
&(D_3) \sum \left|F_n(v_j) - F_n(u_j) - f_n(x_j)(v_j - u_j)\right| \\
&\qquad \leq (D_3) \sum \left|F_n(v_j) - F_n(u_j)\right| + (D_3) \sum \left(|f_n(x_j)| \cdot |v_j - u_j|\right) \\
&\qquad < 2\epsilon. \qquad (5)
\end{aligned}
$$

Hence, by (4) and (5), for each n

$$\begin{aligned}\left|\sum_{j=1}^{r} f_n(x_j)(v_j-u_j)-(\mathcal{M})\int_a^b f_n\right| & \\ \leq \quad & (D_1\cup D_2)\sum \left|f_n(x)(v-u)-F_n(v)+F_n(u)\right| \\ & \qquad +(D_3)\sum \left|f_n(x)(v-u)-F_n(v)+F_n(u)\right| \\ < \quad & \epsilon\cdot(b-a)+2\epsilon \\ = \quad & \epsilon(b-a+2).\end{aligned}$$

This shows that $\{f_n\}$ is equi-integrable on $[a,b]$. ■

3 Henstock Integral

Now we consider similar results for the Henstock integral.

Definition 15. Let $\{F_n\}$ be a sequence of functions defined on $[a,b]$. We say that $\{F_n\}$ is *uniformly differentiable* at $x\in[a,b]$, with f_n being the derivative of F_n, if for each $\epsilon>0$ there exists $\delta(x)>0$ such that for each n, we have

$$|F_n(u,v)-f_n(x)(v-u)|<\epsilon|v-u|$$

whenever $x\in[u,v]\subset(x-\delta(x),x+\delta(x))$. $\{F_n\}$ is said to be *uniformly differentiable on* $X\subseteq[a,b]$ if it is uniformly differentiable at every $x\in X$.

By considering, in the proof of Theorem 9, interval-point pairs $([u,v],\xi)$ with $\xi\in[u,v]$, the following result holds for the Henstock integral.

Theorem 16. *Let $\{f_n\}$ be a sequence of Henstock integrable functions on $[a,b]$ with primitives $\{F_n\}$. If $\{f_n\}$ is equi-integrable on $[a,b]$, then $\{F_n\}$ is uniformly differentiable on $[a,b]\setminus S$, where S is a set of Henstock inner variation zero.*

As pointed out earlier, in $\mathbb{R}^1$, Henstock inner variation zero is equivalent to measure zero. The sequence $\{f_n\}$ defined in Example 12 is uniformly differentiable on $[0,1]$ except on a set of measure zero.

The result of Theorem 14 for Henstock integrable functions is proved in [1].

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