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Tan Soon Boon, National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, Singapore 637616, Republic of Singapore.

Toh Tin Lam, National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, Singapore 637616, Republic of Singapore.

THE ITÔ-HENSTOCK STOCHASTIC DIFFERENTIAL EQUATIONS

Abstract

In this paper, we study the stochastic integral equation with its stochastic integral defined using the Henstock approach, or commonly known as the generalized Riemann approach, instead of the classical Itô integral, which we shall call it the Itô-Henstock integral equation. Our aim is to prove the existence of solution of the Itô-Henstock integral equation using the well known method used in the existence theorem of the ordinary differential equation, namely the Picard's iteration method.

1 Introduction.

In the study of stochastic calculus, it is well known and often emphasized in texts that the Riemann approach, which uses the uniform meshes, cannot be used to define the stochastic integral which has integrator that is of unbounded variation and highly oscillatory integrands. The deficiency of Riemann approach is due to the uniform meshes used in the Riemann sums which is unable to handle highly oscillatory integrators and integrands. However, this shortcoming had been overcome by R. Henstock and J. Kurzweil in the late 1950s when they independently introduced the Riemann-type integral that uses non-uniform meshes in the study of classical (non-stochastic) integral. It turns out that this integral, known as the Henstock integral, is more general than the classical Riemann integral and the Lebesgue integral (see [4], [7]).

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The Henstock approach which makes use of non-uniform meshes, has been successfully used in giving an alternative definition to the the classical stochastic integral ([2], [8], [9], [10], [11]), known as the Itô-Henstock integral. The major advantage of this approach has been its explicitness and intuitiveness in giving a direct definition of the integral rather than the classical non-explicit L^2-procedure thus making it easier for more people to understand. The work of Toh and Chew ([2], [8]) had shown that the Itô-Henstock integral encompasses the classical stochastic integral. The Henstock approach has also been used to characterize stochastic integrable processes (see [8]), to derive an integration-by-part formula for stochastic integral (see [9]) and also the Itô's Formula (see [10]).

In this paper, we shall extend the Itô-Henstock integral theory to prove the existence of solutions to the stochastic differential equation of the form

$$dX_t = f(t, X_t)dB_t, \quad X_0 = \phi$$

where B_t is a Brownian motion and the initial value X_0 is a random variable. The equation above can be written in the integral form,

$$X_t = X_0 + \int_a^t f(s, X_s)dB_s. \tag{1}$$

In the classical case, the integral in (1) is understood as the classical Itô integral. In this paper, we shall define the integral in (1) as the Itô-Henstock integral, an extension on the existing theory developed by Toh and Chew ([8], [9], [10], [11]) and prove the existence theorem for the equation.

2 Preliminaries.

Let $\mathbb{R}$ denote the set of real numbers and

$$[a, b] = \{x \in \mathbb{R} : a \le x \le b\},$$
$$(a, b] = \{x \in \mathbb{R} : a < x \le b\}.$$

Definition 2.1. *The triple $(\Omega, \mathcal{F}, P)$ consisting of a sample space Ω, the σ-algebra $\mathcal{F}$ of subsets of Ω and a probability measure P defined on $\mathcal{F}$ is known as a probability space.*

Definition 2.2. *A filtration is a family $\{\mathcal{F}_t\}_{t>0}$ of increasing sub-σ-algebras of $\mathcal{F}$ (i.e $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$ for all $0 \le t < s < \infty$). When the probability space $(\Omega, \mathcal{F}, P)$ is complete, the filtration is said to satisfy the usual conditions if*

- $\mathcal{F}_s = \bigcap_{t>s} \mathcal{F}_t$ *for all* $s \leq 0$ *(the filtration is right-continuous);*
- *all* $A \in \mathcal{F}$ *with* $P(A) = 0$ *are contained in* $\mathcal{F}_0$ *(all null-set of* $\mathcal{F}$ *belong to* $\mathcal{F}_0$*).*

The probability space together with its family of increasing sub-σ*-algebras denoted by* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ *is called a standard filtering space.*

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A real-valued function $X : \Omega \to \mathbb{R}$ is called $\mathcal{F}$-measurable or random variable, if for all $a \in \mathbb{R}$, $\{\omega \in \Omega : X(\omega) \leq a\} \in \mathcal{F}$.

We further define $E(X)$ to be $\int_\Omega X dP$, for any random variable X.

A family of random variable $\{X_t, t \in I\}$, where $I \subset \mathbb{R}$ is an interval, defined on a probability space $(\Omega, \mathcal{F}, P)$ and indexed by a parameter t where t takes all possible values of I is called a stochastic process.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a standard filtering space and $I \subset \mathbb{R}$ be an interval. The stochastic process X_t is said to be $\{\mathcal{F}_t\}$-adapted if for all $t \in I$, the random variable X_t is $\mathcal{F}_t$-measurable.

2.1 Brownian Motion.

Definition 2.3. *Let* $(\Omega, \mathcal{F}, P)$ *be a complete probability space with a filtration* $\{\mathcal{F}_t\}_{t>0}$ *satisfying the usual conditions. A canonical Brownian motion* $B = \{B_t, t \geq 0\}$ *is a* $\{\mathcal{F}_t\}$*-adapted stochastic process with the following properties:*

1. $B(0) = 0$ *(starting from the origin 0);*
2. *for all* $0 \leq s_1 < t_1 \leq s_2 < t_2 : (B_{t_1} - B_{s_1})$ *and* $(B_{t_2} - B_{s_2})$ *are independent (Independent increments property);*
3. *for all* $0 \leq s < t : B_t - B_s$ *is* $N(0, t-s)$*-distributed (Normal distribution with mean 0 and variance* $t - s$*);*
4. *it has continuous sample paths.*

Some of the standard properties of Brownian motion are: (a) for any $s, t \geq 0$, $E[B_s B_t] = \min\{s, t\}$, (b) for any $t \geq s$, $E[B_s] = E[E[B_t|\mathcal{F}_s]] = E[B_t]$, (c) for any $a \leq u < v \leq s < t < b$, $E[(B_t - B_s)(B_v - B_u)] = 0$.

It is also well-known, see for example Friedman [3], that a canonical Brownian motion is a martingale. In fact, it is a square-integrable martingale with $E(B_t^2) = t$, see property (a) defined above.

2.2 The L^2-space and $\mathcal{L}^2$-space.

Let $L^2(\Omega, \mathcal{F}, P)$ or, if no ambiguity is possible, let L^2 be the space of all $\mathcal{F}$-measurable random variables $X \in \Omega$ defined on the probability space $(\Omega, \mathcal{F}, P)$ such that

$$\|X\|_{L^2}^2 = \int_\Omega |X(\omega)|^2 dP.$$

We denote by $\mathcal{L}^2$ the space of all $\{\mathcal{F}_t\}$-adapted processes X_t for $0 \le t \le T$ defined on the probability space $(\Omega, \mathcal{F}, P)$ such that

$$\|X_t\|_{\mathcal{L}^2}^2 = \int_0^T \int_\Omega |X(t,\omega)|^2 dP dt$$

and using the notation that $E[\cdot] = \int_\Omega \cdot dP$, we have

$$\|X_t\|_{\mathcal{L}^2}^2 = \int_0^T E|X(t,\omega)|^2 dt < \infty.$$

It can be easily shown that $\|\cdot\|_{\mathcal{L}^2}$ is the norm on $\mathcal{L}^2$-space and the space is complete (see [1]).

2.3 Itô-Henstock Integral.

Definition 2.4. *Let $D = \{((\xi_i, v_i], \xi_i)\}_{i=1}^n$ be a finite collection of interval-point pairs of $[a,b]$.*

1. *D is said to be a partial division of $[a,b]$ if $\{(\xi_i, v_i]\}_{i=1}^n$ are disjoint subintervals of $[a,b]$.*
2. *Let δ be a positive function on $[a,b]$. Then an interval-point pair $((\xi, v], \xi)$ is said to be δ-fine belated if $(\xi, v] \subset (\xi, \xi + \delta(\xi)]$ whenever $(\xi, v] \subset [a,b]$ and $\xi \in [a,b]$.*

We call D a δ-fine belated partial division of $[a,b]$ if D is a partial division of $[a,b]$ and for each i, $((\xi_i, v_i], \xi_i)$ is δ-fine belated.

In addition, if $\bigcup_{i=1}^n (\xi_i, v_i]) = (a,b]$, then D is a full division of $(a,b]$. We note that such a δ-fine belated full division may not exist, for example take $\delta(\xi) = (b-\xi)/2$. The point b is not covered by any finite collection of δ-fine belated intervals. However, by virtue of Vitali's covering theorem (see [4]) which states that if a closed interval $[a,b]$ is covered by a collection of open intervals, we can always make the part of $[a,b]$ that is not covered arbitrarily small.

Definition 2.5. *Given $\eta > 0$, a δ-fine belated partial division D is said to be a (δ, η)-fine belated partial division of $[a.b]$ if it fails to cover $[a, b]$ by at most a set of Lebesgue measure η, that is*

$$|b - a - (D)\sum(v - \xi)| \leq \eta.$$

Definition 2.6 (Itô-Henstock Integral). *(Toh and Chew [10, Definition 3])*
Let $f = \{f_t : t \in [a, b]\}$ be a process adapted to the standard filtering space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. Then f is said to be Itô-Henstock integrable (IH) on $[a, b]$ with respect to the Brownian motion B, if there exist an $A \in L^2$ such that for any $\epsilon > 0$, there exists a positive function $\delta > 0$ on $[a, b]$ and a positive number $\eta > 0$ such that for any (δ, η)-fine belated partial division $D = \{(((\xi_i, v_i], \xi_i) : i = 1, 2, \ldots, n\}$ of $[a, b]$, we have

$$E\left(\sum_{i=1}^{n} f_{\xi_i}[B_{v_i} - B_{\xi_i}] - A\right)^2 < \epsilon.$$

It is not difficult to check that A is unique up to a set of probability measure zero, whenever it exists (see Toh and Chew [9]). We call A the Itô-Henstock integral of f and denote A by $(IH)\int_a^b f_t dB_t$.

As in the classical integration theory, the standard properties of integrals, such as the additivity of the integral, integrability over subinterval and Cauchy criterion, hold true for the Itô-Henstock integral. It can also be easily shown that the classical Itô Isometry can be extended to the Itô-Henstock integral presented below.

Theorem 2.7 (Itô-Henstock Isometry). *(Toh and Chew [11, Theorem 17])*
For $f \in \mathcal{L}^2$ and $t \in [a, b]$,

$$E((IH)\int_a^b f(t, \omega)dB_t)^2 = \int_a^b E(f(t, \omega)^2)dt.$$

Lastly, in this section we append the Itô Formula and the Mean Convergence Theorem in Itô-Henstock context as they are needed for subsequent proofs.

Theorem 2.8 (Itô's Formula (Henstock's version)). *(Toh and Chew [10, Theorem 13])*
Let $F : \mathbb{R} \to \mathbb{R}$ be a function whose second order partial derivatives are continuous. Suppose that

(i) $F_2(t, B_t)$ *is Itô-Henstock integrable;*

(ii) $E(F_2(t, B_t))^2$ *is integrable over* $[a, b]$*;*

(iii) $E(F_{2,2}(t, B_t))^2$ *is integrable over* $[a, b]$*.*

Then for almost all $\omega \in \Omega$*, we have*

$$F(b, B_b) - F(a, B_a) = (R) \int_a^b \left[F_1(t, B_t) + \frac{1}{2} F_{2,2}(t, B_t) \right] dt + \int_a^b F_2(t, B_t) dB_t.$$

Theorem 2.9 (**Mean Convergence Theorem**). *(Toh and Chew [11, Theorem 15])*
Let $f^{(n)}$*,* $n = 1, 2, \ldots,$ *be a sequence of IH-integrable processes on* $[a, b]$ *and* f *be a process on* $[a, b]$ *adapted to the standard filtering space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ *such that*

(i) for almost all $t \in [a, b]$*,* $E(f_t^{(n)} - f_t)^2 \to 0$ *as* $n \to \infty$*;*

(ii) $E(\int_a^b (f^{(n)} - f^{(m)})_t dB_t)^2 \to 0$ *as* $n, m \to \infty$*.*

Then f *is* IH*-integrable on* $[a, b]$ *and*

$$E(\int_a^b (f^{(n)} - f)_t dB_t)^2 \to \infty$$

as $n \to \infty$*.*

3 Existence and Uniqueness Theorems.

3.1 The setting of equations.

Consider a standard filtering space denoted by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ for $t \in [a, b]$, B_t a canonical Brownian motion (see Definition 2.3) and a stochastic differential equation of the form

$$dX_t = f(t, X_t) dB_t, \quad X_0 = \phi$$

where $t \in [a, b]$ and the initial value X_0 is a random variable.

Let $f(t, x)$ be a measurable function on $t \in [a, b]$ and $x \in \mathbb{R}$. The above stochastic differential equation can be written as the stochastic integral equation below

$$X_t = X_0 + (IH) \int_a^t f(s, X_s) dB_s \tag{2}$$

where $X_0 = \phi \in L^2$ is a given initial condition which is $\mathcal{F}_0$-measurable, X_t is a stochastic process and $f(s, X_s)$ is Itô-Henstock integrable with respect to the Brownian motion B_t.

Definition 3.1. *A stochastic process X_t, $a \leq t \leq b$, is called a solution of the stochastic integral equation (2), if it satisfies the following conditions:*

1. *$f(t, X_t) \in \mathcal{L}^2$ for $t \in [a, b]$;*
2. *X_0 is $\mathcal{F}_a$-measurable with $E(X_0) \leq \infty$, i.e., $X_0 \in L^2$;*
3. *X_t is continuous and $\mathcal{F}_t$-measurable for all $t \in [a, b]$;*
4. *For each $t \in [a, b]$ and $X_t \in \mathcal{L}^2$, $f(t, X_t)$ is $\mathcal{F}_t$-measurable.*

Definition 3.2. *(Klebaner [5]) A measurable function $g(t, x)$ on $[a, b] \times \mathbb{R}$ is said to satisfy the Lipschitz condition in x, if there exists a constant $C_1 > 0$ such that*

$$|g(t, x) - g(t, y)| \leq C_1|x - y|, \quad \textit{for all} \quad a \leq t \leq b, \quad x, y \in \mathbb{R}.$$

Definition 3.3. *(Klebaner [5]) A measurable function $g(t, x)$ on $[a, b] \times \mathbb{R}$ is said to satisfy the linear growth condition in x, if there exists a constant $C_2 > 0$ such that*

$$|g(t, x)| \leq C_2(1 + |x|), \quad \textit{for all} \quad a \leq t \leq b, \quad x \in \mathbb{R}.$$

The linear growth condition above guarantees that the solution does not "explode" in a finite time interval.

3.2 Existence of solution.

Theorem 3.4. ***(Existence Theorem)***
Let $f(t, x)$ be a measurable function on $[a, b] \times \mathbb{R}$ satisfying the Lipschitz condition (Definition 3.2) and linear growth condition (Definition 3.3). Suppose X_0 is an $\mathcal{F}_0$-measurable random variable with $E(X_0)^2 < \infty$. Then the stochastic integral equation (2)

$$X_t = X_0 + (IH)\int_a^t f(s, X_s)dB_s$$

has a continuous solution X_t.

The proof of the existence theorem is modeled after the existence proof for ordinary differential equations based on Picard's iteration procedure.

Define a sequence $\{X_t^{(k)}\}_{k=0}^{\infty}$ of stochastic processes by setting $X_t^{(0)} = X_0$ and, for $k \geq 0$,

$$X_t^{(k+1)} = X_0 + (IH)\int_a^t f(s, X_s^{(k)})dB_s. \tag{3}$$

The following lemmas are required to prove the Existence Theorem 3.4.

Lemma 3.5. *The sequence $\{X_t^{(k)}\}_{k=0}^{\infty}$ belongs to the space $\mathcal{L}^2$.*

Proof. We proof the lemma by induction.

It is clear that $X_t^{(0)} \in \mathcal{L}^2$. We now make the inductive assumption for $X_t^{(k)} \in \mathcal{L}^2$ and show that $X_t^{(k+1)} \in \mathcal{L}^2$.

Applying the linear growth condition (Definition 3.3),

$$\begin{aligned}
\|f(t, X_t^{(k)})\|_{\mathcal{L}^2}^2 &= \int_a^b E|f(t, X_t^{(k)})|^2 dt \\
&\leq \int_a^b C_2(1 + E|X_t^{(k)}|^2)dt \\
&\leq \int_a^b C_2 dt + \int_a^b C_2 E|X_t^{(k)}|^2 dt \\
&\leq C_2(b-a) + C_2\|X_t^{(k)}\|_{\mathcal{L}^2}^2 \\
&< \infty.
\end{aligned}$$

We therefore have $f(t, X_t^{(k)}) \in \mathcal{L}^2$.

Also, by the Itô-Henstock Isometry (Theorem 2.7), we have

$$E[(IH)\int_a^t |f(s, X_s^{(k)})|^2 dB_s] = E[\int_a^t |f(s, X_s^{(k)})|^2 ds] < \infty. \tag{4}$$

Moreover,

$$\begin{aligned}
\|X_t^{(k+1)}\|_{\mathcal{L}^2}^2 &\leq 2\left(\int_a^b E|X_0|^2 dt + \int_a^b E|(IH)\int_a^t f(s, X_s^{(k)})dB_s|^2 dt\right) \\
&\leq 2\left(\|X_0\|_{L^2}^2 + (b-a)\|f(s, X_s^{(k)})\|_{\mathcal{L}^2}^2\right) \\
&< \infty
\end{aligned}$$

thereby completing our proof. □

Lemma 3.6. *For $k \geq 0$ and the sequence $\{X_t^{(k)}\}_{k=0}^{\infty} \subset \mathcal{L}^2$, we have*

$$\|X_t^{(k+1)} - X_t^{(k)}\|_{\mathcal{L}^2}^2 \leq \frac{\left(C_2^{k+1}(1 + \|X_t^0\|_{\mathcal{L}^2}^{k+2})(t-a)^{k+1}(b-a)^{k+1}\right)}{(k!)}. \quad (5)$$

Proof. The proof can by carried out by induction as in Lemma 3.5, hence we omit it. □

Now, we shall proceed to prove Theorem 3.4.

Proof of Theorem 3.4. Let $m > n \geq 0$, we have

$$\begin{aligned}
\|X_t^{(m)} - X_t^{(n)}\|_{\mathcal{L}^2}^2 &= \|X_t^{(m)} - X_t^{(m-1)} + X_t^{(m-1)} - X_t^{(n)}\|_{\mathcal{L}^2}^2 \\
&= \|X_t^{(m)} - X_t^{(m-1)} + X_t^{(m-1)} - X_t^{(m-2)} + \dots \\
&\quad + X_t^{(n+1)} - X_t^{(n)}\|_{\mathcal{L}^2}^2 \\
&= \|\Sigma_{k=n}^{m-1} X^{(k+1)} - X^{(k)}\|_{\mathcal{L}^2}^2 \\
&\leq \Sigma_{k=n}^{m-1} \|X^{(k+1)} - X^{(k)}\|_{\mathcal{L}^2}^2 \\
&\leq \Sigma_{k=n}^{m-1} \left(\frac{C_2^{(k+1)}(1 + \|X_t^0\|_{\mathcal{L}^2}^{(k+2)})(t-a)^{(k+1)}(b-a)^{(k+1)}}{(k!)} \right).
\end{aligned}$$

It is easy to check that by the ratio test, the series on the right-hand side of the inequality above is convergent. Thus we have

$$\|X_t^{(m)} - X_t^{(n)}\|_{\mathcal{L}^2}^2 \to 0 \quad as\ m,\ n \to \infty. \quad (6)$$

Therefore $\{X_t^{(k)}\}_{k=0}^{\infty}$ is a Cauchy sequence in $\mathcal{L}^2$.

Since $\mathcal{L}^2$ is complete, i.e. every Cauchy sequence is convergent, $\{X_t^{(k)}\}_{k=0}^{\infty}$ is convergent in $\mathcal{L}^2$.

Define $X_t = \lim_{k\to\infty} X_t^{(k)}$, where the limit is in $\mathcal{L}^2$.

Hence, X_t is continuous and since $X_t \in \mathcal{L}^2$, X_t is $\mathcal{F}_t$-measurable for all t.

The next step is to show that the sequence $\{f(t, X_t^{(k)})\}_{k=0}^{\infty}$ converges. By the Lipschitz condition (Definition 3.2),

$$\begin{aligned}
\|f(t, X_t^{(k)}) - f(t, X_t^{(k-1)})\|_{\mathcal{L}^2}^2 &= \int_a^b E|f(t, X_t^{(k)}) - f(t, X_t^{(k-1)})|^2 dt \\
&\leq C_1^2 \int_a^b E|X_t^{(k)} - X_t^{(k-1)}|^2 dt \\
&= C_1^2 \|X_t^{(k)} - X_t^{(k-1)}\|_{\mathcal{L}^2}^2.
\end{aligned}$$

Since by equation (6), we have $\|X_t^{(m)} - X_t^{(n)}\|_{\mathcal{L}^2} \to 0$ as $n, m \to \infty$. We have

$$\|f(t, X_t^{(k)}) - f(t, X_t^{(k-1)})\|_{\mathcal{L}^2}^2 \to 0 \quad \text{as} \quad k \to \infty. \tag{7}$$

Therefore, $\{f(t, X_t^{(k)})\}_{k=0}^{\infty}$ converges in $\mathcal{L}^2$.

Define $f(t, X_t) = \lim_{k\to\infty} f(t, X_t^{(k)})$, where the limit is in $\mathcal{L}^2$.

Now, from the Itô-Henstock Isometry (Theorem 2.7) and (7) above, we have

$$\begin{aligned}
&E\left(|(IH)\int_a^t f(s, X_s^{(k)})dB_s - (IH)\int_a^t f(s, X_s^{(k-1)}dB_s|^2\right) \\
&= \int_a^t E\left(|f(s, X_s^{(k)}) - f(s, X_s^{(k-1)}|^2\right) ds \\
&\to 0 \quad \text{as} \quad k \to \infty.
\end{aligned}$$

Therefore, by the Mean Convergence Theorem (Theorem 2.9), $f(t, X_t)$ is Itô-Henstock integrable and we define

$$(IH)\int_a^t f(s, X_s)dB_s = \lim_{k\to\infty} (IH)\int_a^t f(s, X_s^{(k)})dB_s$$

where the limit is in L^2.

We conclude that for all $t \in [a, b]$, we have

$$X_t = X_0 + (IH)\int_a^t f(s, X_s)dB_s$$

i.e. X_t satisfies equation (2). □

3.3 Uniqueness Theorem.

Theorem 3.7 (Uniqueness Theorem)**.** *Let $f(t, x)$ be a measurable function on $[a, b] \times \mathbb{R}$ satisfying the Lipschitz condition (Definition 3.2). Suppose X_0 is an $\mathcal{F}_0$-measurable random variable with $E(X_0)^2 < \infty$. Then the stochastic integral equation (2)*

$$X_t = X_0 + (IH)\int_a^t f(s, X_s)dB_s$$

has a unique solution X_t up to probability measure zero.

To prove Theorem 3.7, we need Gronwall's inequality presented below.

Lemma 3.8. ***(Gronwall's inequality)***
Let $M \geq 0$ be a constant, and let $u(t)$ and $v(t)$ be real-valued nonnegative continuous functions such that

$$u(t) \leq M + \int_{t_0}^{t} u(s)v(s)ds, \qquad t_0 \leq t < T \quad (where \quad T \leq \infty).$$

Then

$$u(t) \leq M \exp(\int_{t_0}^{t} v(s)ds), \qquad t_0 \leq t < T.$$

Proof. A proof of the Gronwall's inequality can be found in Kuo [6, page 188]. □

Proof of Theorem 3.7. Let X_t and Y_t be two continuous solutions to the stochastic integral equation (2), we shall prove that $X_t = Y_t$.

From the representation given by the stochastic integral equation (2), we know that the difference of any two solutions can be written as

$$X_t - Y_t = (IH) \int_a^t (f(s, X_s) - f(s, Y_s))\, dB_s.$$

Taking expectations, we have

$$\begin{aligned} E(|X_t - Y_t|^2) &= E\left((IH) \int_a^t (f(s, X_s) - f(s, Y_s))\, dB_s\right)^2 \\ &= \int_a^t E\left[f(s, X_s) - f(s, Y_s)\right]^2 ds \\ &\leq C_1^2 \int_a^t E[X_s - Y_s]^2 ds. \end{aligned}$$

Let $u(t) = E(|X_t - Y_t|^2)$, $v(s) = C_1^2$ and $M = 0$. By the Gronwall's inequality (Lemma 3.8), $u(t) = E(|X_t - Y_t|^2) = 0$ for all $t \in [a, b]$ and we have $X_t - Y_t = 0$ almost surely for each $t \in [a, b]$. Hence X_t and Y_t are the same continuous stochastic process completing the proof of the theorem. □

4 Solution of Stochastic Differential Equation.

In this section, we will show an example of a stochastic differential equation (or rather a stochastic integral equation) satisfy the Existence Theorem 3.4.

Consider the following stochastic differential equation

$$dX_t = X_t dB_t, \quad X_0 = 0 \tag{8}$$

or in integral form,

$$X_t = X_0 + (IH)\int_0^s X_t dB_t$$

for $t \in [0, s]$.

To solve this equation, we let $F(t,x) = e^x e^{-\frac{1}{2}}$. Then we have

$$\begin{aligned} F_1(t,x) &= -\frac{1}{2}e^x e^{-\frac{1}{2}} = -\frac{1}{2}F(t,x) \\ F_2(t,x) &= e^x e^{-\frac{1}{2}} = F(t,x) \\ F_{2,2}(t,x) &= e^x e^{-\frac{1}{2}} = F(t,x). \end{aligned}$$

Hence

1. $F_2(t, B_t) = e^{B_t} e^{-\frac{1}{2}}$ and

   $$E(F_2(t,B_t))^2 = E(e^{B_t}e^{-\frac{1}{2}})^2 = 1.$$

   Thus $E(F_2(t,B_t))^2$ is bounded over $[0,s]$ and hence it is integrable on $[0,s]$, showing that $F_2(t,B_t)$ is Itô-Henstock integrable on $[0,s]$.

2. Similarly, for $F_{2,2}(t,B_t) = e^{B_t}e^{-\frac{1}{2}}$, $E\left(F_{2,2}(t,B_t)\right)^2$ is bounded over $[0,s]$ and thus $F_{2,2}(t,B_t)$ is integrable on $[0,s]$.

Since the conditions of the Itô's Formula (Henstock's version) Theorem 2.8 are met, we apply the theorem to get

$$\begin{aligned} dX_t &= dF(t,B_t) \\ &= \left(F_1(t,B_t) + \frac{1}{2}F_{2,2}(t,B_t)\right)dt + F_2(t,B_t)dB_t \\ &= \left(-\frac{1}{2}F(t,B_t) + \frac{1}{2}F(t,B_t)\right)dt + F(t,B_t)dB_t \\ &= F(t,B_t)dB_t \\ &= X_t dB_t. \end{aligned}$$

Next we show that X_t is indeed an stochastic process, we have

$$E\int_0^s |X_t|dt = \int_0^s Ee^{B_t}e^{-\frac{1}{2}}dt = \int_0^s dt = s < \infty$$

which proves that X_t belongs to $\mathcal{L}^2$.

We note that X_t is continuous since $e^{B_t}e^{-\frac{1}{2}}$ is continuous.

Hence, the conditions of Definition 3.1 are satisfied. Therefore, $X_t = e^{B_t}e^{-\frac{1}{2}}$ is the solution to the stochastic differential equation (8).

5 Conclusion.

In this paper, we extended the Itô-Henstock integral theory on our study of stochastic differential equations. Modeled after the classical existence theorem proof of the ordinary differential equation, using Picard's iteration, we established an existence theorem for a stochastic differential equation under the Itô-Henstock approach. However, we note that the conditions imposed on the equation in this paper, namely the Lipschitz condition and linear growth condition, may be too strict for practical purpose. We shall next look into more relaxed conditions and establishing an existence theorem in a more general setting for a larger class of stochastic differential equations defined in the Itô-Henstock context, which will appear as a paper in the future.

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