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ON THE PROBLEM OF CHARACTERIZING DERIVATIVES

Abstract

We observe how a slight and even natural change in the Kurzweil-Henstock integral leads to a characterization of derivatives. We will also argue that the only way to characterize derivatives is by using some object or procedure which is at least as complicated as an integral.

A function $f : R \to R$ is called a derivative if there exists a function $F : R \to R$ such that for all $x \in R$, $f(x) = F'(x)$. A longstanding problem in real analysis is to characterize derivatives ([10]).

Any problem which simply asks us to characterize something is, of course, vague. After all, there are always trivial characterizations. For example,

"f is a derivative if and only if it has a primitive F."

That is, to see if f is a derivative, just integrate it then differentiate it and see if you get f again. Presumably this does not count as a solution to the problem. But what would? Several characterizations have been proposed. Some of these are mentioned below. What would make one of them interesting and/or non-trivial and/or useful enough to be recognized as a solution? The standards are not clear. It is generally believed to be one of those things which will be recognized when it is seen, and also generally believed to be something which hasn't been seen yet. But if anything does seem clear about the standards, it is that we should rule out integrating f as a method of solution.

Derivatives have some nice structural properties. For example, they are both Baire-one and Darboux. So it is natural to ask (see [1], [10]) whether such structural properties might provide a characterization. It is also true that certain types of characterizations, for example, characterizations in terms of associated sets, are impossible (see [1]).

Descriptive set theoretic results by Mazurkiewicz ([7]) and by Dougherty and Kechris ([5]) show that the set of derivatives is sufficiently complicated

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to rule out many natural (Borel) types of characterizations. However, such results do not render the problem impossible, since what is complicated in one context may be simple and even natural in another. Indeed, many of the basic concepts of analysis are inherently non-Borel.

In this note, we would like to assert that perhaps no "satisfactory" solution is possible, since any characterization must either explicitly or implicitly contain integration.

To be more precise, consider the following three questions:

1. Given a Baire-one function f, is f a derivative?
2. Given a derivative f and a number L, is $\int_0^1 f(x)dx = L$?
3. Given a derivative f and a number L, is $\int_0^1 f(x)dx \geq L$?

The first is our characterization problem. The third is a form of the primitive problem. That is, any method of answering 3) immediately yields a method of determining (to any desired accuracy) what the primitive is. In general, this is a complicated procedure.

The second problem is asking for an equivalent definition for the definite integral of a derivative. Instead of being asked for the value of the integral, we are only asked to check whether a certain "guess" is correct.

We will observe below that any solution to the first problem must already contain a solution to the second problem. It is in this sense, then, that we can't "know" which functions are derivatives without already having "knowledge" of what their integrals are! The argument would be somewhat stronger if we could show that any solution to 1) gives a solution to 3).

This seems to severely restrict the possibilities for a characterization. We still might have:

1. A characterization of derivatives in terms of their primitives (like the trivial one mentioned above).
2. A characterization of derivatives using some sort of integration procedure.
3. A characterization of the set of derivatives in terms of some other complicated construct (ie. a relative characterization). For example, we might define the set of derivatives in terms of itself.

We might also have a combination or disguised version of one of these. None of these types of characterizations are the kind which analysts have been searching for, yet it is hard to imagine how anything else is possible!? In section three we will give an example of each of these types, including a simple characterization making a natural change in the definition of Kurzweil-Henstock integrability.

1 Using a Characterization of Derivatives to Define an Integral

Suppose we are given a derivative f and a number L and we wish to know whether $\int_0^1 f dx = L$, but we are only allowed to ask questions of the form "Is $g(x)$ a derivative?"

Assume without loss of generality that $f(0) = f(1) = 0$. For each positive integer n, let $a_n = 1 - \frac{1}{n}$, $m_n = \frac{1}{a_{n+1} - a_n} = n(n+1)$, and let $h_n(x)$ denote the line with slope m_n mapping a_n to 0 and a_{n+1} to 1. Define

$$g_L(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ f(h_n(x)) & \text{if } x \in [a_n, a_{n+1}] \\ L & \text{if } x \geq 1. \end{cases}$$

Note that at the points $x = a_n$ there is no discrepancy since $f(0) = f(1) = 0$.

Theorem 1. *Let f be a derivative with $f(0) = f(1) = 0$. Then $\int_0^1 f(x)dx = L$ if and only if $g_L(x)$ is a derivative, where $g_L(x)$ is defined as above.*

PROOF. Let F denote the antiderivative of f with $F(0) = 0$, so that $\int_0^1 f(x)dx = F(1)$. Fix L and let $g(x) = g_L(x)$ be defined as above. Define $G(x)$ as follows:

$$G(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ a_n F(1) + \frac{1}{m_n} F(h_n(x)) & \text{if } x \in [a_n, a_{n+1}] \\ F(1) + L(x-1) & \text{if } x \geq 1. \end{cases}$$

We will show below that G is well-defined and continuous. We will also show that $G'(x) = g(x)$ at each $x \neq 1$. From this it follows that g is a derivative if and only if it is the derivative of G. The third claim is that the derivative of G from the left is $F(1)$ and the derivative from the right is $g(1) = L$. From this it follows that $G' = g$ if and only if $L = F(1)$. Combining these proves the theorem. It remains only to verify the three claims.

First, notice that if $x = a_{n+1}$ then G is defined twice. The first definition, using $[a_n, a_{n+1}]$ gives

$$\begin{aligned} G(a_{n+1}) &= a_n F(1) + \frac{1}{m_n} F(h_n(a_{n+1})) \\ &= a_n F(1) + (a_{n+1} - a_n) F(1) = a_{n+1} F(1)\,, \end{aligned}$$

while the second, using $[a_{n+1}, a_{n+2}]$ gives

$$\begin{aligned} G(a_{n+1}) &= a_{n+1} F(1) + \frac{1}{m_{n+1}} F(h_{n+1}(a_{n+1})) \\ &= a_{n+1} F(1) + \frac{1}{m_{n+1}} F(0) = a_{n+1} F(1)\,, \end{aligned}$$

and the two agree. Similarly, at $a_1 = 0$ we get $G(a_1) = 0 \cdot F(1) + \frac{1}{m_1} F(0) = 0$ which agrees with $G(0) = 0$. Also, since F is bounded on $[0, 1]$ and $\lim_{n\to\infty} \frac{1}{m_n} = 0$ and $\lim_{n\to\infty} a_n = 1$, we get $\lim_{x\to 1^-} G(x) = F(1) = G(1)$ and so G is continuous.

If $x < 0$ then $G'(x) = 0 = g(x)$. Also, the derivative from the left at $x = 0$ is $0 = g(0)$. If $x \in [a_n, a_{n+1}]$, then

$$\begin{aligned} G'(x) &= \frac{1}{m_n} F'(h_n(x)) \cdot h_n'(x) \\ &= \frac{1}{m_n} f(h_n(x)) m_n = f(h_n(x)) = g(x). \end{aligned}$$

Similarly, the derivative from the right at a_n and from the left at a_{n+1} both agree with the function g. If $x > 1$ then $G'(x) = L$ and $g(x) = L$. So it only remains to show that the derivative of G from the left at $x = 1$ is $F(1)$. Let $x \in [a_n, a_{n+1}]$. We compute

$$\begin{aligned} \frac{G(1) - G(x)}{1 - x} &= \frac{F(1) - a_n F(1) - \frac{1}{m_n} F(h_n(x))}{1 - x} \\ &= F(1) \frac{1 - a_n}{1 - x} - (a_{n+1} - a_n) F(h_n(x)) \\ &= F(1) + F(1) \frac{x - a_n}{1 - x} - (a_{n+1} - a_n) F(h_n(x))\,. \end{aligned}$$

so, if B is a bound for F on $[0,1]$, then

$$\begin{aligned}
\left|\frac{G(1)-G(x)}{1-x}-F(1)\right| &\leq B\frac{x-a_n}{1-x}+B(a_{n+1}-a_n)\\
&\leq B\frac{a_{n+1}-a_n}{1-a_{n+1}}+B(a_{n+1}-a_n)\\
&= B(a_{n+1}-a_n)\left(\frac{1}{1-a_{n+1}}+1\right)\\
&= B\left(\frac{1}{n(n+1)}\right)(n+1+1)\,,
\end{aligned}$$

which approaches zero as $n \to \infty$. □

2 Examples

An example of the first type of characterization (in terms of the primitive) was given by Neugebauer [8]. A simplified version of this characterization goes as follows.

Recall that a result of Gleyzal [6] says that a function is Baire-one if and only if it is the limit of an interval function, $H(I)$. That is, H maps closed intervals to reals and $f(x)$ is the limit of $H(I)$ as $|I| \to 0$ with $x \in I$. Let us call an interval function "balanced" if $H(I)|I|+H(J)|J| = H(K)|K|$ whenever the interval K is partitioned by the intervals I and J. That is, whenever $K = I \cup J$ and $I \cap J$ contains only one point. Functions which are the limits of balanced interval functions are exactly the derivatives.

To see this, suppose for example that f is a derivative with a primitive F. Define $H(I)$ to be the difference quotient of F over the interval I and H is immediately seen to have the desired properties. In the other direction, suppose that $f(x)$ is the limit of such an interval function. Let $F(x) = xH([0,x])$. Then,

$$\frac{F(x+h)-F(x)}{h} = \frac{(x+h)H([0,x+h])-xH([0,x])}{h}.$$

Using that H is balanced, this is

$$= \frac{hH([x,x+h])}{h} = H([x,x+h]),$$

and then from the limiting property of H it follows that $F'(x) = f(x)$.

This characterization makes a nice analogy between Baire-one functions and derivatives. (Neugebauer's original version was designed to draw a similar analogy between derivatives and Baire-one Darboux functions.) Yet it is easy to see that the function H is just a disguised form of the primitive function.

For an example of the second type, consider the definition of Kurzweil-Henstock integrability:

Definition 1. Let I be a closed interval and let $I_1, \ldots, I_n$ be a partition of I and let $x_1, \ldots, x_n$ be a sequence of points such that for each i, $x_i \in I_i$. Then the system of intervals and points is called a "tagged" partition of I. If f is any function on I then the tagged partition yields a "Riemann sum", given by $\sum_{i=1}^n f(x_i)|I_i|$. If $\delta(x)$ is a positive function defined on I each $|I_i| < \delta(x_i)$ then the tagged partition is called "δ-fine". Positive functions $\delta(x)$ are often called "gauge" functions.

Definition 2. A function $f(x)$ is KH-integrable if and only if, $(\forall I)$ $(\forall \epsilon > 0)(\exists \delta : R \to R^+)$(any two δ-fine tagged partitions of I have Riemann sums which differ by less than $\epsilon|I|$). The KH-integral is then defined to be the limit of the corresponding Riemann sums as $\epsilon \to 0$.

The importance of this definition is summarized by the following theorem. The proof is provided for future reference. Note that if I is the interval $[a, b]$ then we use $F(I)$ to abbreviate the difference $F(b) - F(a)$.

Theorem 2. (Kurzweil-Henstock) *Derivatives are KH-integrable.*

PROOF. Let $f(x)$ be the derivative of a function $F(x)$. Given $\epsilon > 0$ there is a $\delta(x) > 0$ such that for any interval I containing x with $|I| < \delta(x)$ we have $F(I)|I|$ within $\frac{\epsilon}{2}$ of $f(x)$. Let I be an interval, partitioned into the subintervals $I_1, \ldots, I_n$ and let x_i be chosen in each I_i so that the resulting tagged partition is δ-fine. Then each $F(I_i)$ is within $\frac{\epsilon}{2}|I_i|$ of $f(x)|I_i|$. Summing over i we get that $F(I)$ is within $\frac{\epsilon}{2}|I|$ of $\sum_{i=1}^n f(x_i)|I_i|$ which, of course, is the Riemann sum. It follows that any two such Riemann sums are within $\epsilon|I|$ of each other. □

The quantifier $(\forall I)$ is usually left out of the previous definition because if a function is integrable over one interval, then it is provably integrable over every subinterval. But here we leave it in so as to emphasize the often overlooked fact that the gauge δ not only depends on x and ϵ but is also allowed to depend on the interval which is being integrated! This dependence may even seem a bit unnatural since we usually think of the gauge as a local property of the function f. Who cares if we are integrating from 0 to 1 or 0 to 10? With this motivation, suppose that we demand that the gauge be interval independent. Then we get a characterization of derivatives:

Proposition 1. A function f is a derivative if and only if $(\forall \epsilon > 0)(\exists \delta : R \to R^+)(\forall I)$(any two δ-fine tagged partitions have Riemann sums which differ by less than $\epsilon|I|$).

Proof. The $\Rightarrow$ direction is exactly like the proof given above that derivatives are integrable. One simply notices that the choice of δ comes from the definition of derivative and has nothing to do with the interval being integrated. For the other direction, notice that if f satisfies the right side then it is KH-integrable. Let F(x) be its integral. Given $\epsilon > 0$ there is a corresponding gauge $\delta : R \to R^+$. Let I be an interval containing x with $|I| < \delta(x)$. Then I with x as a tag, is itself a δ-fine partition of I, with Riemann sum $f(x)|I|$. This must differ from the Riemann sum of any other such tagged partition by at most $\epsilon|I|$, and hence differs from the integral F(I) by at most $\epsilon|I|$. Therefore, $f(x)$ differs from $F(I)/|I|$ by at most ϵ. □

An example of the third type is provided by Preiss and Tartaglia [9], who give, using the Axiom of Choice, an interesting characterization of derivatives in terms of the set of derivatives, a sort of circular characterization. Their result says that f is a derivative if and only if for each set of real numbers E, there is a derivative g such that $\{x|f(x) \in E\} = \{x|g(x) \in E\}$. Despite the circularity, this characterization really seems to say something. If a function fails to be a derivative, then it is prevented from being one solely because it has the wrong kind of inverse image on some set.

This characterization has been recently improved by Ciesielski ([2]). There is a collection of sets A of cardinality no larger than than of the reals, and a property D such that a Borel function f is a derivative if and only if the inverse image of every set in A has property D. This makes it look remarkably similar to familiar characterizations of continuity, etc. Here $D = \{f^{-1}(S)|S \in A, f$ is a derivative$\}$. Moreover, given that f is already known to be Darboux and Baire-one, the collection A can be just the translations of a single set.

3 Conclusion

The characterization problem for derivatives may be impossible for the structural kind of characterization we would like. But that doesn't mean that we shouldn't keep searching. Perhaps we will find ones which are nontrivial, interesting, perhaps even useful. All of the characterizations given so far are in terms of more complicated objects, be it the primitive, the gauge functions, or the set of derivatives itself. We have suggested that this may be unavoidable.

One direction, which might be more appealing, is to give some sort of "first return" characterization. The idea would be to make the more complicated object just a countable sequence of real numbers, and the test as to whether a function is a derivative would only depend on the limiting properties of the original function on this sequence. There seems to be good hope for this. Baire-one and Baire-one Darboux functions have been similarly characterized in terms of such limits (see [3], [4]). At least on the surface, this type of characterization might appear to be a more structural.

Another possibility, suggested by Brian Thomson, is to investigate characterizations of functions which are not exact derivatives, but which come close, for example, functions which, except on a countable set, are the derivative of their integral.

References

[1] A. Bruckner, *Differentiation of Real Functions* CRM monograph series, vol.5, American Mathematical Society, Providence, 1994.

[2] K. Ciesielski, "Characterizing derivatives by preimages of sets", to appear.

[3] U.B. Darji, M.J. Evans, and R.J. O'Malley, "First return path systems: differentiability, continuity, and orderings", Acta. Math. Hungar., 66 (1995), 83–103.

[4] U.B. Darji, M.J. Evans, and R.J. O'Malley, "A first return characterization of Baire-one functions", Real. Anal. Exch. 19 (1993–94), 510–515.

[5] R. Dougherty and A. Kechris, "The complexity of antidifferentiation", Adv. Math. 88 no.2 (1991), 145–169.

[6] A. Gleyzal, "Interval functions", Duke Math. J. 8 (1941), 223–230.

[7] S. Mazurkiewicz, "Uber die Menge der differenzierbaren Funktionen", Fund. Math. 27 (1936), 244–249.

[8] C. Neugebauer, "Darboux functions of Baire class 1 and derivatives", Proc. Amer. Math. Soc. 13 (1962), 838–843.

[9] D. Preiss and M.Tartaglia, "On characterizing derivatives", Proc. Amer. Math. Soc. 123 (1995) 2417–2420.

[10] W. H. Young, "A note on the property of being a differential coefficient", Proc. Lond. Math. Soc. 9 (1911), 360–368.