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Tomasz Natkaniec,* Institute of Mathematics, University of Gdańsk, Wita Stwosza 57, 80-952 Gdańsk, Poland. email: mattn@mat.ug.edu.pl

AN EXAMPLE OF A QUASI-CONTINUOUS HAMEL FUNCTION

Abstract

We say that $f : \mathbb{R} \to \mathbb{R}$ is a Hamel function if f, considered as a subset of $\mathbb{R}^2$, is a Hamel basis of $\mathbb{R}^2$. For a Cantor set $C \subset \mathbb{R}$ we construct a quasi-continuous Hamel function such that $f \restriction (\mathbb{R} \setminus C)$ is of Baire class one.

1 Introduction.

Let us establish some of terminology to be used. By $\mathbb{R}$ and $\mathbb{Q}$ we denote the sets of all reals and rationals, respectively. The symbol $|A|$ stands for the cardinality of a set A. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$. Ordinal numbers are identified with the set of their predecessors. No distinction is made between a function and its graph. The symbol $\operatorname{rng}(f)$ denotes the range of f. We say that $C \subset \mathbb{R}$ is a Cantor set if C is homeomorphic with the ternary Cantor set (i.e., C is perfect, nowhere dense and bounded).

A function $f : \mathbb{R} \to \mathbb{R}$ is quasi-continuous (in the sense of Kempisty) at a point $x_0 \in \mathbb{R}$ if $\operatorname{int}(U \cap f^{-1}(V)) \neq \emptyset$ for all open neighbourhoods U of x_0 and V of $f(x_0)$. f is quasi-continuous if it is quasi-continuous at each $x \in \mathbb{R}$. Recall that each quasi-continuous function $f : \mathbb{R} \to \mathbb{R}$ is pointwise discontinuous, i.e., the set $C(f)$ of all continuity points of f is dense in $\mathbb{R}$, and consequently f has the Baire property. (See e.g. [8].)

We will consider $\mathbb{R}^n$, $n < \omega$, as a linear space over the field $\mathbb{Q}$. For $A \subset \mathbb{R}^n$ the symbol $\operatorname{LIN}(A)$ denotes the linear span of A. Any basis of $\mathbb{R}^n$ over $\mathbb{Q}$ will be referred as a Hamel basis. We say that $f : \mathbb{R} \to \mathbb{R}$ is a *Hamel*

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function if f, considered as a subset of $\mathbb{R}^2$, is a Hamel basis of $\mathbb{R}^2$. A function $f\colon A \to \mathbb{R}$, where $A \subset \mathbb{R}$ is called *linearly independent function* if f is a linearly independent subset of $\mathbb{R}^2$.

For $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, the symbol $\mathrm{LC}(f)$ denotes the set of all sums $\sum_{i=1}^{k} q_i f(x_i)$ such that: $k < \omega$, $q_i \in \mathbb{Q}$ and $x_i \in A$ for $i < k$, and $\sum_{i=1}^{k} x_i = 0$. Notice that $\mathrm{LC}(f)$ is always a linear subspace of $\mathbb{R}$ [9].

If V is a linear space (over $\mathbb{Q}$) and W is a subspace of V, then the symbol $\mathrm{codim}_V(W)$ denotes the codimension of W in V (i.e., the dimension of the quotient space V/W).

The class of Hamel functions has been introduced by Krzysztof Płotka in his Ph.D. Dissertation and studied in [9]–[11], [12], [6], [5], and [2]. In particular, it is known that every Hamel function is continuous on no non-degenerate interval [9, Fact 2.3(iii)]. Moreover, since Hamel basis can not be Borel set, no Hamel function is Borel measurable. (See [6, Remark 5.7(2)].) In the first result of this note we show that for every non-degenerate interval $J \subset \mathbb{R}$ there is a Hamel function f such that $f\restriction J$ is of the Baire class one. (As usually, we denote the first Baire class by B_1.) This answers a question posed recently by I. Recław (oral communication). In the second theorem we construct a quasi-continuous Hamel function. This solves Problem 5.3 from [6]. Our example is pointwise discontinuous thus Baire measurable, and can (or not) be Lebesgue measurable. We expand here a method of construction of measurable Hamel functions introduced in [2].

2 Main Results.

Lemma 1. [3, Lemma 1] *Let $I = [0, 1]$ and C be a Cantor set. There exists a strictly increasing quasi-continuous injection $f : I \to C$. This means, in particular, that f is of the first Baire class.*

Corollary 2. *If $I \subset \mathbb{R}$ is a non-degenerate interval and C is a Cantor set then there is a strictly increasing quasi-continuous injection $f : I \to C$.*

Lemma 3. [12, Lemma 2] *Let $H_1, H_2 \subset \mathbb{R}$ be Hamel bases. If f_0 is a bijection between $\mathbb{R} \setminus H_1$ and H_2, then the function $f = f_0 \cup (H_1 \times \{0\})$ is a Hamel function.*

Theorem 4. *For every non-degenerate interval $I \neq \mathbb{R}$ there exists a Hamel function $f : \mathbb{R} \to \mathbb{R}$ such that $f\restriction I$ is in the first Baire class.*

Proof. Let H_2 be a Hamel basis which contains a perfect set C and such that $|H_2 \setminus C| = \mathfrak{c}$. (See e.g. [4, Theorem XI.7.2].) By Corollary 2 there exists a quasi-continuous injection $f_0 : I \to C$ which is of the first Baire class. Since

$\operatorname{int}(\mathbb{R}\setminus I) \neq \emptyset$, it contains a Hamel basis. (See e.g. [4, Corollary IX.3.2], p. 214.) Fix a Hamel basis $H_1 \subset \mathbb{R}\setminus I$ and a notice that $|\mathbb{R}\setminus(I\cup H_1)| = \mathfrak{c} = |H_2\setminus \operatorname{rng}(f_0)|$. Let $f_1 : \mathbb{R} \setminus (I \cup H_1) \to H_2 \setminus \operatorname{rng}(f_0)$ be a bijection and $f_2 = H_1 \times \{0\}$. Then Lemma 3 implies that $f = f_0 \cup f_1 \cup f_2$ is a Hamel function. □

Theorem 5. *Let J be an open interval. If $f : J \to \mathbb{R}$ is a derivative, then it is not linearly independent function.*

PROOF. We repeat the proof of [9, Fact 2.3(iii)]. Fix $a \in J$ and $h > 0$ with $(a-h, a+h) \subset J$. Let $g : [0, h) \to \mathbb{R}$ be defined by $g(x) = f(a-x)+f(a+x)$. Notice that g is a derivative. Indeed, if F is a primitive of f then $G : x \mapsto -F(a - x) + F(a + x)$ is a primitive for g. Hence g is Darboux. Now, two cases are possible.

Case 1. g is constant on $[0, h)$. Then for any $x \in (0, h)$ we have

$$\langle a - x, f(a - x)\rangle + \langle a + x, f(a + x)\rangle = \langle 2a, g(x)\rangle = \langle 2a, g(0)\rangle = 2\langle a, f(a)\rangle$$

thus f is not linearly independent.

Case 2. g is not constant. Then $\operatorname{rng}(g)$ is a non-degenerate interval and therefore there are $x_1, x_2 \in (0, h)$ and $q_1, q_2 \in \mathbb{Q} \setminus \{0\}$ such that $x_1 \neq x_2$ and $g(x_i) = g(a) + q_i = 2f(a) + q_i$ for $i = 1, 2$. But then

$$\langle a - x_i, f(a - x_i)\rangle + \langle a + x_i, f(a + x_i)\rangle = \langle 2a, g(x_i)\rangle$$

for $i = 1, 2$, and

$$q_2\langle 2a, g(x_1)\rangle - q_1\langle 2a, g(x_2)\rangle = 2(q_2 - q_1)\langle a, f(a)\rangle,$$

which implies that f is not linearly independent. □

Problem 1. *Does there exist a Hamel function $f : \mathbb{R} \to \mathbb{R}$ such that $f\restriction J$ is Darboux Baire one for some non-degenerate interval J?*

Lemma 6. [9, Example 2.2] *If $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ is an injection and* $\operatorname{rng}(f)$ *is linearly independent, then f is a linearly independent function.*

Theorem 7. [5, Theorem 6] *Suppose $A \subset \mathbb{R}$ spans $\mathbb{R}$ and $f : A \to \mathbb{R}$ is a linearly independent function. Then f is extendable to a Hamel function iff* $\operatorname{codim}_{\mathbb{R}^2}(\mathrm{LIN}(f)) = |\mathbb{R} \setminus A|$.

Lemma 8. [5, Lemma 6] *Suppose $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ is a linearly independent function. Then* $\operatorname{codim}_{\mathrm{LIN}(A)\times\mathbb{R}}(\mathrm{LIN}(f)) \geq \operatorname{codim}_{\mathbb{R}}\mathrm{LC}(f)$.

Corollary 9. *Let $A \subset \mathbb{R}$ span $\mathbb{R}$, $H \subset \mathbb{R}$ be a Hamel basis and $f : A \to H$ be an injection. If $|\mathbb{R} \setminus A| = \mathfrak{c} = |H \setminus \operatorname{rng}(f)|$, then f is extendable to a Hamel function on $\mathbb{R}$.*

PROOF. Lemma 6 implies that f is a linearly independent function. Since $|H\setminus \mathrm{rng}(f)| = \mathfrak{c}$, $|\mathrm{codim}_{\mathbb{R}}\mathrm{LC}(f)| = \mathfrak{c}$ and Lemma 8 yields $\mathrm{codim}_{\mathbb{R}^2}(\mathrm{LIN}(f)) = \mathfrak{c}$. Thus $\mathrm{codim}_{\mathbb{R}^2}(\mathrm{LIN}(f)) = |\mathbb{R} \setminus A|$, and by Theorem 7, f is extendable to a Hamel function. $\square$

Theorem 10. *Let $C \subset \mathbb{R}$ be a Cantor set. There exists a quasi-continuous Hamel function $f : \mathbb{R} \to \mathbb{R}$ with $f{\restriction}(\mathbb{R} \setminus C) \in B_1$.*

PROOF. Let $\{I_n : n < \omega\}$ be a sequence of all open intervals with rational end-points and let $\{P_{n,m} : n, m < \omega\}$ be a sequence of perfect sets such that

1. if $P_{i,j} \cap P_{n,m} \neq \emptyset$ then $\langle i, j\rangle = \langle n, m\rangle$;
2. $\bigcup_m P_{n,m} \subset I_n$ for each $n < \omega$;
3. $P = \bigcup_n \bigcup_m P_{n,m}$ is a linearly independent set [7, Theorem 1]. (See also [1, Lemma 3.3].)

Let H be a Hamel basis with $P \subset H$. We may assume that $|H \setminus P| = \mathfrak{c}$. Let $\mathcal{J}$ be the family of all components of the set $\mathbb{R} \setminus C$ and let $\langle \mathcal{J}_n\rangle_n$ be a partition of $\mathcal{J}$ with $C \subset \mathrm{cl}(\bigcup \mathcal{J}_n)$ for each n. Let $\mathcal{J}_n = \{J_{n,m} : m < \omega\}$. For $n, m < \omega$ let $f_{n,m} : J_{n,m} \to P_{n,m}$ be a quasi-continuous increasing injection as in Corollary 2, hence $f_{n,m} \in B_1$. Let $f_0 = \bigcup_{n,m<\omega} f_{n,m}$. Then f_0 is an injection from $\mathbb{R}\setminus C$ into P. By Lemma 6, f_0 is a linearly independent function. Clearly $\mathbb{R}\setminus C$ spans $\mathbb{R}$ and $|H\setminus \mathrm{rng}(f_0)| = \mathfrak{c} = |C|$, thus Corollary 9 yields that f_0 can be extended to a Hamel function $f : \mathbb{R} \to \mathbb{R}$. Obviously, $f{\restriction}(\mathbb{R} \setminus C)$ is of the first Baire class. We will verify that f is quasi-continuous at each point $x \in \mathbb{R}$. This is clear for $x \in \mathbb{R} \setminus C$. Now, fix $x \in C$ and $\delta, \varepsilon > 0$. There are $n, m < \omega$ for which $f(x) \in I_n \subset (f(x) - \varepsilon, f(x) + \varepsilon)$ and $J_{n,m} \subset (x - \delta, x + \delta)$. Then $f(J_{n,m}) = \mathrm{rng}(f_{n,m}) \subset P_{n,m} \subset I_n$. $\square$

3 Some Final Remarks.

Notice that the function f constructed above is pointwise discontinuous and therefore it has the Baire property. Moreover, f is different from a Baire one function on the set C. Thus, if C has Lebesgue measure null then f is Lebesgue measurable. On the other hand, if C has positive measure, then an easy modification of the construction of f gives a non-measurable quasi-continuous Hamel function.

Corollary 11. *There exists a quasi-continuous Hamel function which is not Lebesgue measurable.*

Proof. Let H and P be as in the proof of Theorem 10. Let $C \subset \mathbb{R}$ be a Cantor set of positive measure. Let $\langle B_0, B_1, B_2 \rangle$ be a partition of C onto Bernstein-like sets sets, hence non-measurable and of size $\mathfrak{c}$ each. Let $\langle H_0, H_1, H_2 \rangle$ be a partition of $H \setminus P$ such that $|H_0| = |H_1| = |H_2| = \mathfrak{c}$ and the distance between H_0 and H_1 is equal to $\varepsilon > 0$. Now, let $A = \mathbb{R} \setminus B_2$ and $f : A \to H$ be an injection such that:

1. f is defined on $\mathbb{R} \setminus C$ as in the proof of Theorem 10;
2. $f{\restriction}B_i$ is a bijection between B_i and H_i for $i = 0, 1$.

By Corollary 9, f is extendable to a Hamel function. Let U be the $\varepsilon/2$-neighbourhood of H_0 and $D = (f{\restriction}C)^{-1}(U)$. Then $B_0 \subset D$ and $B_1 \subset \mathbb{R} \setminus D$, hence D is not measurable, so $f{\restriction}C$ is non-measurable and therefore f is non-measurable too. $\square$

Problem 2. *Does there exist a quasi-continuous Hamel function having Darboux property?*

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