# 6. Superstable Theories

In Section 5.6 we defined the notion of a basis of a type $p$ (relative to a set) and posed several questions on the behavior of the corresponding dimension function (including its well-definedness). We proved in Section 5.6.3 that on the class of weight 1 types dimension is well-defined, and nonorthogonality is the same as domination equivalence. In Section 5.6.4 we showed that in a superstable theory every type is domination equivalent to a finite product of weight 1 types. For a full-featured dimension theory, though, we need an additional property (additivity) which may fail for a weight 1 type (see Remark 5.6.7). In the second section of this chapter we develop the theory of a class of weight 1 types called the regular types in an arbitrary superstable theory. A regular type in a superstable theory satisfies the additivity property missing for weight 1 types (Proposition 6.3.2) and every weight 1 type is domination equivalent to a regular type.

This well-behaved dimension theory is at the heart of the solution of such problems as Morley's Conjecture for countable first-order theories (mentioned in the Preface). Regular types will be used to characterize the models of a "bounded" t.t. theory in Section 7.1.1.

Before turning to regular types we develop two notions of rank which are used in virtually every study of superstable theories.

## 6.1 More Ranks

Many of the properties proved for t.t. theories relied heavily on the existence of Morley rank; i.e., the fact that every type has Morley rank $< \infty$. The family of $\Delta-$ranks served to define the forking dependence relation but, because it is a family of ranks instead of a single rank, it is missing many of properties of an ordinal-valued rank. Here we define two ranks which exist in superstable theories and provide a sharper measure of the complexity of formulas and types with respect to the forking relation.

**Throughout the section we assume any mentioned theory to be stable.**

**Definition 6.1.1.** *(i) In a stable theory we define the rank $U(-)$ on complete types by the following recursion. For $p$ a complete type and $\alpha$ an ordinal,*

*$U(p) \geq \alpha$ if for all $\beta < \alpha$ there is a forking extension $q$ of $p$ such that $U(q) \geq \beta$. We write $U(p) = \alpha$ and say the $U-$*rank of *p* is *$\alpha$ if $U(p) \geq \alpha$ and $U(p) \not\geq \alpha + 1$. If $U(p) \geq \alpha$ for all $\alpha$ we write $U(p) = \infty$ and say the $U-$rank of $p$ does not exist.*

*(ii) For consistent formulas $\varphi(x)$ and $\alpha$ an ordinal or $-1$ the relation $R^\infty(\varphi)$ is defined by the following recursion.*

(1) $R^\infty(\varphi) = -1$ *if $\varphi$ is inconsistent;*
(2) $R^\infty(\varphi) = \alpha$ *if*

$$\{\, p \in S_n(\mathfrak{C}) :\ \varphi \in p \text{ and } \neg\psi \in p \text{ for all formulas } \psi \text{ with } R^\infty(\psi) < \alpha \,\}$$

*is nonempty and has cardinality $< |\mathfrak{C}|$.*

*The relation $R^\infty(\varphi) = \alpha$ is read* the $\infty-$rank of $p$ is $\alpha$. *When $\varphi$ is consistent and $R^\infty(\varphi) \neq \alpha$ for all ordinals $\alpha$ we write $R^\infty(\varphi) = \infty$ and say that the $\infty-$rank of $p$ does not exist. For $p$ an arbitrary type, $R^\infty(p) =$* $\inf\{\, R^\infty(\varphi) : \varphi \text{ is implied by } p \,\}$.

When the $U-$rank of every complete type exists it gives a direct measure of forking: When $p \subset q$ are complete types, $U(p) = U(q)$ if and only if $q$ is a nonforking extension of $p$ (see the exercises). Thus, when the $U-$rank of every complete type exists it is natural to define $U(p)$, for $p \in S(\mathfrak{C})$, to be $U(p \restriction A)$, where $A$ is any set over which $p$ does not fork (the rank is the same over any such $A$). Be aware that $U-$rank is only defined on *complete* types (a point we will expand on later).

As the reader can see, the definition of $\infty-$rank is very similar to the definition of Morley rank, except that here the appropriate set of types is only required to have cardinality $< |\mathfrak{C}|$ rather than $< \aleph_0$. Many of the most basic properties of Morley rank extend to $\infty-$rank with the same proofs:

**Lemma 6.1.1.** *Let $T$ be a complete theory, $p$ an $n-$type and $\alpha$ an ordinal.*

*(i) If $p \in S(\mathfrak{C})$ then $R^\infty(p) = 0$ if and only if $p$ is algebraic.*

*(ii) $R^\infty(p) = \alpha$ if and only if* $\{\, q \in S_n(\mathfrak{C}) :\ p \subset q \text{ and } R^\infty(q) = \alpha \,\}$ *is nonempty and has cardinality $< |\mathfrak{C}|$.*

*(iii) If $R^\infty(p) = \alpha$ there is a $q \in S_n(\mathfrak{C})$ such that $q \supset p$ and $R^\infty(q) = \alpha$.*

*(iv) $R^\infty(p) \geq \alpha$ if and only if, for all $\beta < \alpha$ and all $\varphi$ implied by $p$,* $\{\, q \in S_n(\mathfrak{C}) :\ \varphi \in q \text{ and } R^\infty(q) \geq \beta \,\}$ *and has cardinality $< |\mathfrak{C}|$.*

*(v)*

$$R^\infty(\varphi) \text{ is the least ordinal } \alpha \text{ such that } \{\, p \in S_n(\mathfrak{C}) :\ \varphi \in p \text{ and } R^\infty(p) \geq \alpha \,\} \text{ has cardinality } < |\mathfrak{C}|. \tag{6.1}$$

In Shelah's terminology, $R^\infty(-) = R(-, L, \infty)$.

The basic existence properties are proved in

**Lemma 6.1.2.** *Let $T$ be a stable theory.*

*(i) Then $T$ is superstable if and only if for every formula $\varphi$, $R^\infty(\varphi) < \infty$.*

*(ii) For every complete type $p$, $U(p) \leq R^\infty(p)$.*

*Proof.* The proofs of both parts rely on

*Claim.* For $p \in S(A)$, $q \supset p$ and $\alpha$ an ordinal, if $R^\infty(p) = \alpha$, then

$$q \text{ does not fork over } A \iff R^\infty(q) = \alpha.$$

Without loss of generality, $q \in S(\mathfrak{C})$. Suppose $R^\infty(q) = \alpha$ and let $Q \subset S(\mathfrak{C})$ be the set of extensions on $p$ of $\infty$−rank $\alpha$. Then $Q$ has cardinality $< |\mathfrak{C}|$. Moreover, every conjugate over $A$ of $q$ is in $Q$, so $q$ does not fork over $A$ by Lemma 5.1.13. Now suppose that $q$ does not fork over $A$. There is an extension $r$ of $p$ in $S(\mathfrak{C})$ having $\infty$−rank $\alpha$, which we just proved does not fork over $A$. All nonforking extensions of $p$ in $S(\mathfrak{C})$ are conjugate over $A$ (by Corollary 5.1.8(ii)), hence $r$ and $q$ are conjugate. Therefore, $R^\infty(q) = \alpha$, proving the claim.

(i) First suppose each formula has $\infty$−rank. By the claim, each element of $S(\mathfrak{C})$ does not fork over a finite set, hence $T$ is superstable.

Now suppose there is a formula $\varphi$ which does not have $\infty$−rank. Then there is a complete type $p$ over a set $A$ such that $R^\infty(p) = \infty$. The nonsuperstability of $T$ will follow from

*Claim.* If $p \in S(A)$ and $R^\infty(p) = \infty$ there is a forking extension $q$ of $p$ with $R^\infty(q) = \infty$.

Suppose, towards a contradiction, that there is no such $q$. Let $\alpha = \sup\{ R^\infty(q) : q \text{ is a forking extension of } p\}$. Consider $Q = \{q \in S(\mathfrak{C}) : q \supset p \text{ and } R^\infty(q) \geq \alpha + 1\}$. Then $Q$ is nonempty and contains only nonforking extensions of $p$. Thus, $|Q| < |\mathfrak{C}|$. By Lemma 6.1.1(v), $R^\infty(p) < \infty$, a contradiction which proves the claim.

Proceeding with the main body of the proof, iterated use of the preceding claim generates an infinite chain $p_0 \subset p_1 \subset \ldots \subset p_i \subset \ldots$ of complete types such that $p_{i+1}$ is a forking extension of $p_i$, for all $i$. This proves that $T$ is not superstable, completing the proof of (i).

(ii) We need to show that $U(p) \geq \alpha \implies R^\infty(p) \geq \alpha$, for all complete types $p$ and ordinals $\alpha$. Using the first claim this becomes an easy induction which is relegated to the exercises.

It follows quickly from the first claim in the proof of the lemma that in a superstable theory any formula $\varphi$ has $\leq 2^{|T|}$ extensions in $S(\mathfrak{C})$ of the same $\infty$−rank; i.e., the bound in the definition of $\infty$−rank can be taken to be $2^{|T|}$ instead of an arbitrary cardinal $< |\mathfrak{C}|$.

*Remark 6.1.1.* It is actually the case that a stable theory $T$ is superstable if and only if $R^\infty(x = x) < |T|^+$. This, however, is significantly harder to prove than (i) of the lemma (see [She90, II]. Since this refined bound has few uses we will not prove it here.

*Example 6.1.1.* (A formula of $\infty-$rank 1 which does not have Morley rank) Let $T$ be the theory of infinitely many refining equivalence relations with only infinite classes such that $E_0$ has one class and $E_{i+1}$ splits each $E_i$ class into two classes. Then, the formula $x = x$ has $2^{\aleph_0}$ many nonalgebraic completions in $S(\mathfrak{C})$, hence $R^\infty(x = x) = 1$. Every nonalgebraic consistent formula in $x$ has continuum many extensions in $S(\mathfrak{C})$, hence there is no formula of Morley rank $> 0$.

**Definition 6.1.2.** *(i) A formula of* $\infty-$*rank* 1 *is also known as a* weakly minimal formula.

*(ii) A set defined by a weakly minimal formula in some model is called a* weakly minimal set.

*(iii) A type having a unique nonalgebraic completion over* $\mathfrak{C}$ *is called a* minimal type.

*(iv) The set of realizations of a minimal type in some model is called a* minimal set.

*Remark 6.1.2.* A weakly minimal formula has $\leq 2^{|T|}$ many nonalgebraic completions over $\mathfrak{C}$. A complete type is minimal if and only if it is stationary and has $U-$rank 1. A formula $\varphi$ over $A$ is weakly minimal if and only if every nonalgebraic completion of $\varphi$ over $acl(A)$ is minimal.

A strongly minimal formula is both a weakly minimal formula and a minimal type.

In some superstable theories $U-$rank and $\infty-$rank agree on the complete types, however there are relatively simple examples where $U-$rank and $\infty-$rank differ.

*Example 6.1.2.* (Where $U-$rank and $\infty-$rank agree) Consider the theory $T_0$ of a single equivalence relation $E$ with infinitely many infinite classes and no finite classes. Up to conjugacy, there are three types in $S_1(\mathfrak{C})$: those containing $x = a$, for some $a$; the nonalgebraic types containing $E(x, a)$, for some $a$; and those containing $\neg E(x, a)$, for all $a$. These types have $U-$rank, $\infty-$rank and Morley rank 0, 1, and 2, respectively. Similarly, these ranks agree on $S_n(\mathfrak{C})$.

(Where $U-$rank and $\infty-$rank differ) Now we will define a theory consisting of "infinitely many disjoint copies of $T_0$". The language contains unary predicates $U_i$ and binary predicates $E_i$, for $i < \omega$. The axioms for $T$ say that the $U_i$'s are pairwise disjoint and $E_i$ defines a copy of $T_0$ on the elements satisfying $U_i$. Certainly, $T$ is superstable (in fact, $\omega-$stable) and quantifier eliminable. Let $q \in S(\emptyset)$ be the unique $1-$type containing $\neg U_i(x)$, for all $i < \omega$. Then $U(q) = 1$ and $R^\infty(q) = 2$. (Any formula in $q$ is consistent with some $U_i$, hence has $\infty-$rank 2.)

There are identifiable properties of a theory which guarantee that $U-$rank and $\infty-$rank agree on all complete types. Isolating fairly broad classes of theories in which this is true is a difficult matter which is relegated to Section 7.2.

For now we simply remark that Morley rank and $\infty$−rank agree in uncountably categorical and $\aleph_0$−categorical, $\aleph_0$−stable theories, and Morley rank and $\infty$−rank agree with $U$−rank on the complete types in such theories.

**Definition 6.1.3.** *A map $R$ which associates an ordinal with a complete type $p$ is called a* notion of rank *if it satisfies:*

(1) *$R(p) = R(fp)$, for any $f \in \mathrm{Aut}(\mathfrak{C})$.*
(2) *If $q \supset p$ is complete, $R(q) \le R(p)$.*
(3) *For each $A$ containing the domain of $p$ there is a $q \in S(A)$ containing $p$ with $R(q) = R(p)$.*
(4) *If $p \in S(B)$ there is a finite $B' \subset B$ such that $R(p) = R(p \restriction B')$.*
(5) *There is a cardinal $\lambda$ such that any type has at most $\lambda$ extensions of the same rank over $\mathfrak{C}$.*

*Let $R$ be a notion of rank.*

- *$R$ is called* connected *if $R(p) = \alpha$ and $\beta \le \alpha$ implies the existence of a complete type $q \supset p$ with $R(q) = \beta$.*
- *$R$ is called* continuous *if whenever $p \in S(A)$ and $R(p) = \alpha$ there is a $\varphi \in p$ such that $R(q) \le \alpha$ for all $q \in S(A)$ containing $\varphi$.*

Properties (1)–(5) are exactly what is needed of an ordinal-valued function on types to induce a freeness relation on the universe (see Definition 3.3.1). Morley rank, $\infty$−rank and the $\Delta$−ranks are all continuous. A unique property of $U$−rank is that there are superstable theories in which it is *not* continuous. It follows that there is no notion of rank defined via formulas (as was $\infty$−rank) which agrees with $U$−rank on complete types in every superstable theory. (The example given above where $U$−rank and $\infty$−rank differ is also a counter-example to the continuity of $U$−rank.) On the other hand, $U$−rank is connected (see the exercises) while $\infty$−rank is not. (The type $q$ in Example 6.1.2 has $\infty$−rank 2 and every forking extension has $\infty$−rank 0.)

It is tempting to extend $U$−rank to the incomplete types in a superstable theory $T$ with the rule:

$$U(p) = \sup\{\, U(q) : \ q \supset p, \ q \in S(\mathfrak{C}) \,\},$$

where $p$ is an arbitrary type in $T$. But there are choices for $T$ and $p$ for which this supremum is not attained; i.e., there is no $q \in S(\mathfrak{C})$, $q \supset p$, such that $U(q) = \sup\{\, U(q) : \ q \supset p, \ q \in S(\mathfrak{C}) \,\}$. This seriously limits the usefulness of $U$−rank on incomplete types. There is one general setting, however, in which it is appropriate to speak of the $U$−rank of an incomplete type. Suppose that $G$ is a superstable group (i.e., a stable group $\bigwedge$−definable in a superstable theory). Proposition 5.3.1 says that the action of $G$ on the generics in $S^G(G)$ is transitive, hence all generics have the same $U$−rank $\alpha$ (see the exercises). For $p$ the type defining $G$ we then define $U(p)$ to be $\alpha$ (which is the supremum of $\{\, U(q) : \ q \supset p$ and $q$ complete$\}$).

That $U$−rank fails to be continuous is offset by the fact that it satisfies some additivity properties reminiscent of dimension in a strongly minimal set. If $D$ is strongly minimal and $\bar{a}$, $\bar{b}$ are tuples from $D$, then $\dim(\bar{a}\bar{b}) = \dim(\bar{a}/\bar{b}) + \dim(\bar{b})$. We will see momentarily that $U$−rank satisfies a corresponding identity when all the relevant ranks are finite. When some of the types have infinite $U$−rank the irregularities of ordinal addition create problems. For example, if the sequences $\bar{a}$ and $\bar{b}$ from $D$ are independent, then $\dim(\bar{a}/\bar{b}) = \dim(\bar{a})$ and $\dim(\bar{b}/\bar{a}) = \dim(\bar{b})$, so $\dim(\bar{a}\bar{b}) = \dim(\bar{a}) + \dim(\bar{b})$ and this is also $\dim(\bar{b}/\bar{a}) + \dim(\bar{a}) = \dim(\bar{b}) + \dim(\bar{a})$. Rewriting this with $U$−rank replacing dimension results in the identity: $U(\bar{a}) + U(\bar{b}) = U(\bar{b}) + U(\bar{a})$ for any two independent tuples in a superstable theory. Since addition of infinite ordinals is noncommutative this may not hold. We can, however, obtain useful inequalities if we replace ordinal addition by the *natural sum* (or *Hessenberg sum*) on ordinals (which is commutative). The *natural sum* of two ordinals $\alpha$ and $\beta$, denoted $\alpha \oplus \beta$, is defined recursively by the clause:

$$\alpha \oplus \beta = \sup(\{\, \alpha \oplus \beta' + 1 : \beta' < \beta \,\} \cup \{\, \alpha' \oplus \beta + 1 : \alpha' < \alpha \,\}).$$

The important features of $\oplus$ are:

(1) $\oplus$ agrees with $+$ when the ordinals are finite.
(2) $\oplus$ is commutative.
(3) If $\beta' < \beta$, $\alpha \oplus \beta' < \alpha \oplus \beta$.

We prove

**Proposition 6.1.1 (Additivity).** *If $T$ is superstable, then for all $a$, $b$ and $A$,*

$$U(a/A \cup \{b\}) + U(b/A) \leq U(ab/A) \leq U(a/A \cup \{b\}) \oplus U(b/A).$$

*Proof.* Without loss of generality, $A = \emptyset$. We prove by induction that for $\alpha = U(a/b) + U(b)$, $U(ab) \geq \alpha$. When $\alpha$ is a limit ordinal this follows quickly by induction. Suppose $\alpha = \beta + 1$. Then $U(b)$ must be a successor, say $\gamma + 1$. By the connectedness of $U$−rank there is a set $B$ such that $U(b/B) = \gamma$. Without loss of generality, $a$ is independent from $B$ over $b$. Since $\delta = U(a/b)$ is an ordinal such that $\delta + \gamma + 1 = \beta + 1$ and $\delta = U(a/B \cup \{b\})$, $\delta + \gamma = U(a/B \cup \{b\}) + U(b/B) = \beta$. By induction, $U(ab/B) \geq \beta$. Since $ab$ depends on $B$ over $\emptyset$, $U(ab) \geq \beta + 1$, completing the proof of this part of the proposition.

Next, we prove by induction that $U(ab) \geq \alpha \implies U(a/b) \oplus U(b) \geq \alpha$. Again, it suffices to consider the case $\alpha = \beta + 1$. Thus, there is a set $B$ on which $ab$ depends such that $U(ab/B) = \beta$. Then $b$ depends on $B$ or $a$ depends on $B$ over $b$ and $U(a/B \cup \{b\}) \oplus U(b/B) \geq \beta$ (by induction). These dependencies imply that $U(a/B \cup \{b\}) < U(a/b)$ or $U(b/B) < U(b)$, respectively. In either case, $U(a/B \cup \{b\}) \oplus U(b/B) < U(a/b) \oplus U(b)$. Thus, $U(a/b) \oplus U(b) \geq \beta + 1$, completing the proof.

Simply because $+$ and $\oplus$ agree on finite ordinals,

**Corollary 6.1.1.** *Suppose that $T$ is superstable and $U(b/A)$ and $U(a/A \cup \{b\})$ are finite. Then*
*(i) $U(ab/A) = U(a/A \cup \{b\}) + U(b/A)$ and*
*(ii) $U(a/A) - U(a/A \cup \{b\}) = U(b/A) - U(b/A \cup \{a\})$.*

The second part of the corollary can be viewed as a strong form of the Symmetry Lemma. It says that $a$ depends on $b$ the same amount that $b$ depends on $a$. In the following discussion we generalize this corollary to types of infinite $U-$rank.

While the above definition of $\oplus$ makes it easy to prove properties of the operation, it makes it difficult to compute particular values. The following equivalent definition shows how to perform the operation. Given an ordinal $\alpha$ there are ordinals $\beta_1 > \ldots > \beta_k$ and natural numbers $n_1, \ldots, n_k$ such that $\alpha = \omega^{\beta_1} \cdot n_1 + \omega^{\beta_2} \cdot n_2 + \ldots + \omega^{\beta_k} \cdot n_k$. This expression is unique (when each $n_i$ is nonzero) and is called the *Cantor normal form* of $\alpha$. Given two ordinals $\alpha$ and $\alpha'$, expand them as $\alpha = \omega^{\beta_1} \cdot n_1 + \ldots + \omega^{\beta_k} \cdot n_k$ and $\alpha' = \omega^{\beta_1} \cdot n_1' + \ldots + \omega^{\beta_k} \cdot n_k'$, where $\beta_1 > \ldots > \beta_k$ and $n_i$, $n_i' < \omega$. Then

$$\alpha \oplus \alpha' = \omega^{\beta_1} \cdot (n_1 + n_1') + \ldots + \omega^{\beta_k} \cdot (n_k + n_k').$$

(This is the definition given in [Hau62]. The equivalence of the two definitions is left to the reader.)

Ordinals of the form $\omega^\beta$ act as "limit points" of the operation $\oplus$ in the sense that $\alpha, \alpha' < \omega^\beta \implies \alpha \oplus \alpha' < \omega^\beta$. Generalizing, we write $\alpha \ll \omega^{\alpha_1} \cdot m_1 + \ldots + \omega^{\alpha_l} \cdot m_l$ (where $\alpha_1 > \ldots > \alpha_l$) if $\alpha < \omega^{\alpha_l}$. Then, $\alpha, \alpha' \ll \beta \implies \alpha \oplus \alpha' \ll \beta$. As the notation suggests $\alpha \ll \beta$ is one way of formally saying that $\alpha$ is much less than $\beta$.

**Lemma 6.1.3.** *Suppose that $T$ is superstable and $p$, $q$ are complete types such that $U(q) \ll U(p)$. Then $p$ is orthogonal to $q$.*

*Proof.* Without loss of generality, $p$ and $q$ are stationary. Suppose, to the contrary, that $p$ is nonorthogonal to $q$. Let $A$ be a set on which both $p$ and $q$ are based such that there are $a$ realizing $p|A$ and $b$ realizing $q|A$ which are dependent over $A$. Let $U(p) = \beta + \omega^\alpha \cdot n$, where $\beta = 0$ or $\omega^\alpha \ll \beta$, and $U(q) = \gamma \ll U(p)$ (equivalently $\gamma < \omega^\alpha$). By Proposition 6.1.1, $\beta + \omega^\alpha \cdot n = U(a/A) \leq U(ab/A) \leq U(a/A \cup \{b\}) \oplus U(b/A)$. Since $U(a/A \cup \{b\}) < \beta + \omega^\alpha \cdot n$ and $\gamma < \omega^\alpha$, $U(a/A \cup \{b\}) \oplus U(b/A) = \beta + \omega^\alpha \cdot (n-1) + \gamma'$ for some $\gamma' < \omega^\alpha$. This contradicts the inequality and the stated value of $U(a/A)$.

Some applications of the additivity of $U-$rank are most easily stated through the following equivalence relation. For ordinals $\alpha$, $\beta$ and $\gamma$ we write $\beta \approx_\alpha \gamma$ if there are $\beta'$, $\gamma' < \omega^\alpha$ such that $\beta + \beta' = \gamma + \gamma'$. Intuitively, ordinals are $\approx_\alpha$ if they are "$=$ modulo $\omega^\alpha$". The key ingredient in the proof of the following corollary is:

$$\beta,\ \gamma < \omega^{\alpha+1} \implies \beta + \gamma \approx_\alpha \beta \oplus \gamma. \tag{6.2}$$

(This follows quickly from the fact that an ordinal $< \omega^{\alpha+1}$ can be written as $\omega^\alpha \cdot n + \epsilon$, where $\epsilon < \omega^\alpha$.) Furthermore, we can do basic arithmetic "modulo $\omega^\alpha$" on ordinals. Given $\beta$, $\gamma$, $\delta$, $\epsilon < \omega^{\alpha+1}$,

$$\beta \approx_\alpha \gamma \text{ and } \delta \approx_\alpha \epsilon \implies \beta + \delta \approx_\alpha \gamma + \epsilon,$$

and

$$\beta + \gamma \approx_\alpha \delta + \gamma \implies \beta \approx_\alpha \delta.$$

**Corollary 6.1.2.** *Suppose that $T$ is superstable and both $U(a)$ and $U(b)$ are $< \omega^{\alpha+1}$. Then*
*(i) $U(ab) \approx_\alpha U(a/b) + U(b)$ and*
*(ii) $U(a) \approx_\alpha U(a/b) + \omega^\alpha \cdot m \iff U(b) \approx_\alpha U(b/a) + \omega^\alpha \cdot m$.*

*Proof.* (i) This follows immediately from Proposition 6.1.1 and (6.2).
(ii) It suffices to show the $\implies$ direction. Suppose that $U(a) \approx_\alpha U(a/b) + \omega^\alpha \cdot m$. Two applications of (i) and (6.2) show that $U(a/b)+U(b) \approx_\alpha U(b/a)+U(a)$. Substituting yields,

$$U(a/b) + U(b) \approx_\alpha U(b/a) + U(a/b) + \omega^\alpha \cdot m$$

and cancelation gives the desired equation $U(b) \approx_\alpha U(b/a) + \omega^\alpha \cdot m$.

**Historical Notes.** The rank $R^\infty(-)$ is defined in [She90, II] as $R(-, L, \infty)$ and Lemma 6.1.2(i) is Theorem 3.14 of that chapter. Lascar's $U-$rank is defined and developed in [Las76], where Proposition 6.1.1 is proved.

**Exercise 6.1.1.** Let $T$ be superstable and $p \subset q$ complete types. Show that $U(p) = U(q)$ if and only if $q$ is a nonforking extension of $p$.

**Exercise 6.1.2.** Prove Lemma 6.1.2(ii).

**Exercise 6.1.3.** Suppose that $T$ is superstable and $a \in acl(A \cup \{b\})$. Then $R^\infty(a/A) \leq R^\infty(ab/A) = R^\infty(b/A)$ and $U(a/A) \leq U(ab/A) = U(b/A)$.

**Exercise 6.1.4.** Suppose that $T$ is superstable, $\varphi$ is formula over $A$ and $R^\infty(\varphi) \geq k$, where $k < \omega$. Prove that there is a $p \in S(A)$ containing $\varphi$ with $U(p) \geq k$. (HINT: Use induction.)

**Exercise 6.1.5.** Prove the connectedness of $U-$rank.

**Exercise 6.1.6.** Show that when $T$ is stable and $a$ and $b$ are interalgebraic over $A$, $U(a/A) = U(b/A)$.

**Exercise 6.1.7.** Show that in a superstable group $G$ all generic types have the same $U-$rank.

**Exercise 6.1.8.** Let $T$ be a superstable theory and $E$ a definable equivalence relation in $T$. Let $a$ be an element and $b$ the name for the $E$-class of $a$. Supposing that $U(a) = 5$ and $U(a/b) = 2$, compute $U(b)$.

**Exercise 6.1.9.** Give an example of a superstable theory containing a complete type of $U$-rank $\omega$.

## 6.2 Geometrical Matters: A Dichotomy Theorem

Throughout Chapter 4 we saw how detailed information about the pregeometries in an uncountably categorical theory effected the overall structure of the theory. For example, when an uncountably categorical theory contains a locally modular strongly minimal set, the theory is 1-based. Here we prove (Theorem 6.2.1), saying that many superstable theories contain minimal sets which are locally modular. Basic consequences of this theorem will be stated without proof.

Let $D$ be a minimal set, $\bigwedge$-definable over a set $A$ in a stable universal domain $\mathfrak{C}$. Let $c\ell(-)$ be $acl(- \cup A) \cap D$. As with strongly minimal sets, $(D, c\ell)$ is a homogeneous pregeometry. As usual, when $P$ is a property of a pregeometry and $p$ is a minimal type we say that $p$ has property $P$ when $p(\mathfrak{C})$ has property $P$.

**Theorem 6.2.1 (Dichotomy Theorem).** *Let $T$ be superstable and $D$ a minimal set which is not locally modular. Then $D$ is strongly minimal.*

This will be proved with several component results. First we reduce the problem to considering only minimal sets which are weakly minimal; i.e., minimal sets of $\infty$-rank 1.

**Lemma 6.2.1.** *Let $T$ be superstable and $D$ a minimal set which is nontrivial. Then $D$ is weakly minimal.*

*Proof.* Without loss of generality, $D = p(\mathfrak{C})$ for a stationary type $p \in S(\emptyset)$. Since $D$ is nontrivial we can also assume that there is $\{a, b, c\} \subset D$ which is pairwise independent but dependent. Let $\varphi(x, y, z) \in tp(abc)$ be such that, for all $a', b', c'$,

$$\models \varphi(a', b', c') \implies a' \in acl(b', c') \text{ and } b' \in acl(a', c').$$

Let $\alpha = R^\infty(p)$ and $\theta \in p$ a formula of $\infty$-rank $\alpha$. Let $\psi(x)$ be the formula $d_p y(\exists z(\varphi(x, y, z)))$ (see the notation on page 224). Then $\psi(x) \in p$ (since $\models \psi(a)$).

*Claim.* $\psi$ is weakly minimal.

It suffices to show that for any $a' \in \psi(\mathfrak{C})$, $U(a') \leq 1$. (If $R^\infty(\psi) \geq 2$, there is a $q \in S(\emptyset)$ containing $\psi$ such that $U(q) \geq 2$ by Exercise 6.1.4.) Choose $a'$ satisfying $\psi$. By the definition of $\psi$ there are $b'$ and $c'$ such that $b' \in D$, $b' \downarrow a'$, and $\models \varphi(a', b', c') \wedge \theta(c')$. Since $b' \in acl(a', c')$, $\alpha = R^\infty(b'/a') \leq R^\infty(c'/a') \leq R^\infty(c') \leq \alpha$. Thus, $c'$ is independent from $a'$ over $\emptyset$. Since $a' \in acl(b', c')$, $U(a'/c') \leq U(b'/c') \leq 1$. Thus, $U(a') \leq 1$, proving the claim and the lemma.

**Definition 6.2.1.** *Let $D$ be a minimal set, $\bigwedge-$definable over $\emptyset$ in a superstable theory. A* plane curve *in $D$ is a minimal subset of $D^2$.*

*We call $D$* linear *if whenever $p(\mathfrak{C})$ is a plane curve in $D$ and $c$ is interalgebraic with $Cb(p)$, $U(c) \leq 1$.*

*Remark 6.2.1.* Let $D$ be a minimal set in a superstable theory, $\bigwedge-$definable over $\emptyset$, and $D'$ a strongly minimal set over $\emptyset$ in a t.t. theory. Plane curves in $D$ behave very much like plane curves in $D'$. The principle difference is that the canonical parameter of a plane curve in $D'$ is an element of $(D')^{eq}$ and the canonical base of plane curve in $D$ is an infinite subset of $D^{eq}$. However, if $X = p(\mathfrak{C})$ is a plane curve in $D$ and $C = Cb(p)$, there is a $c \in Cb(p)$ such that $c$ is interalgebraic with $C$. The resemblance between the two types of plane curves is even stronger. There is a $c \in Cb(p)$ such that, letting $q = tp(a/c)$ for $a \in X$, any $b \in D^2$ realizing $q$ is in $X$. (This is left for the reader to prove.) It follows that $Cb(p) = dcl(c)$. Thus, we will call $c$ a canonical base of $X$ and identify $X$ with the element $c$.

**Lemma 6.2.2.** *Let $D$ be a minimal set in a superstable theory, $\bigwedge-$definable over $\emptyset$. The following are equivalent.*

(1) *$D$ is locally modular,*
(2) *$D/A$ is locally modular, for some set $A$,*
(3) *$D$ is linear.*

*Proof.* This is proved just like the corresponding result for strongly minimal sets, Lemma 4.2.4.

*Proof of Theorem 6.2.1.* Let $D$ be a minimal set, $\bigwedge-$definable over $\emptyset$, which is not locally modular. By the preceding lemma there is a plane curve $X \subset D^2$ such that for $c$ a canonical base of $X$, $U(c) > 1$. By the connectedness of $U-$rank there is a set $A$ such that $U(c/A) = 2$. Working over $A$, $X$ is a plane curve in $D/A$ with canonical base $c$, so we may as well assume that $U(c/\emptyset) = 2$.

An element of $X$ has the form $a = (a_0, a_1) \in D^2$, where $a_0 \in acl(c, a_1)$ and $a_1 \downarrow c$. We will prove that $r = tp(a/c)$ has Morley rank 1. It follows that $tp(a_1/c)$ has Morley rank 1, hence $D$ has Morley rank 1 (since $a_1 \downarrow c$), proving the theorem. The formula in $tp(a/c)$ of Morley rank 1 is found with a compactness argument applied to the following theory. Below a formula

$\psi(x, y)$ is called "provably algebraic in $y$" if for any $b$, $\psi(x, b)$ is algebraic; i.e., $\models \forall y \exists^{<n} x \psi(x, y)$, for some $n$. Given a realization $d$ of $tp(c)$, let $r_d$ denote the conjugate of $r$ over $d$.

The lemmas proved earlier about families of plane curves in strongly minimal sets are true for minimal sets after $\dim(-)$ is replaced by $U(-)$. (This uses that $U$−rank satisfies the identities in Corollary 6.1.1.) Since $r$ defines a plane curve whose canonical base is $c$, for any $d \neq c$ realizing $tp(c)$, $r(x) \cup r_d(x)$ is algebraic. By compactness, there are formulas $\sigma(x, y)$ $\psi(x, yz)$ such that

- $\psi(x, yz)$ is provably algebraic in $yz$,
- for any $d$ realizing $tp(c)$, if $d \neq c$, $\sigma(x, c) \wedge \sigma(x, d)$ implies $\psi(x, cd)$.

Since $D$ is weakly minimal, $X$ is weakly minimal and we can choose $\sigma$ so that

$$\begin{aligned}&\sigma(x, c) \text{ is weakly minimal and for all} \\ &a' \in \sigma(\mathfrak{C}, c),\ \ a' \not\perp c \ \text{ and } \ R^{\infty}(a') \leq 2.\end{aligned} \tag{6.3}$$

Combining this with the generalization of Lemma 4.2.6 yields

$$\begin{aligned}\text{for any } \ a' \in \sigma(\mathfrak{C}, c) \setminus acl(c),\ U(a') = R^{\infty}(a') = 2 \text{ and}& \\ U(c/a') = U(c) - 1 = 1.&\end{aligned} \tag{6.4}$$

With these pieces in place we can prove

*Claim.* $\sigma(x, c)$ has Morley rank 1.

We need to show that the formula $\sigma(x, c)$ has finitely many nonalgebraic completions in $S(acl(c))$. Let $d$ be a realization of $stp(c)$ which is independent from $c$ over $\emptyset$. Since $\sigma(x, c) \wedge \sigma(x, d)$ implies $\psi(x, cd)$ and $\psi(x, cd)$ is algebraic it suffices to show

(∗) any nonalgebraic element of $S(acl(c))$ containing $\sigma(x, c)$ is realized in $\sigma(\mathfrak{C}, c) \cap \sigma(\mathfrak{C}, d)$.

Let $a' \in \sigma(\mathfrak{C}, c) \setminus acl(c)$. By (6.4), $U(a') = 2$ and $U(c/a') = 1$. Choose $d'$ realizing $stp(c/a')$ which is independent from $c$ over $a'$. We show that $d'$ is independent from $c$ with the following $U$−rank calculation. $U(d'ca') = U(d'c/a') + U(a') = U(d'/ca') + U(c/a') + U(a') = 1 + 1 + 2 = 4$. Also, $U(d'ca') = U(a'/d'c) + U(d'c) = U(a'/d'c) + U(d'/c) + U(c) = 0 + U(d'/c) + 2$. We conclude that $U(d'/c) = 2$, hence $d'$ is independent from $c$. Since $d'$ and $d$ have the same strong type over $\emptyset$ and both are independent from $c$, $stp(d'/c) = stp(d/c)$. Let $f$ be an automorphism of $\mathfrak{C}$ which fixes $acl(c)$ and maps $d'$ to $d$. Then $f(a')$ is a realization of $stp(a'/c)$ in $\sigma(\mathfrak{C}, c) \cap \sigma(\mathfrak{C}, d)$. This proves (∗), hence the claim and the theorem.

In Theorem 4.3.1 we showed that an uncountably categorical theory $T$ has a locally modular strongly minimal set if and only if $T$ is 1−based. This extends the simplicity of the pregeometry on a strongly minimal set to the simplicity of the entire universal domain with respect to forking dependence.

The following generalizes Theorem 4.3.1.

**Theorem 6.2.2.** *Let $T$ be a superstable theory such that for each complete type $p$*

– *$U(p) < \omega$, and*
– *if $U(p) = 1$, $p$ is locally modular.*

*Then $T$ is $1-$based.*

We will not prove this theorem here. All of the key ideas are found in the proof of Theorem 4.3.1. The proof used the fact that the universal domain of an uncountably categorical theory is asm-constructible. We simply need a concept corresponding to "almost strongly minimal set" which is appropriate for minimal sets. The "semi-minimal sets" fill the gap. (See [Bue86] or [Pil]. There is also an exposition of this material in [Bue93].)

The theme underlying Theorem 6.2.1 is that geometrical and topological complexity cannot coexist. (A minimal set over $acl(A)$ which is not strongly minimal is a relatively complicated point in the Stone space topology on $S(acl(A))$.) The application which best displays Theorem 6.2.1 and Theorem 6.2.2 is the following corollary. We need to borrow a definition and results from later sections. A stable theory $T$ is *unidimensional* if all nonalgebraic stationary types are nonorthogonal (Definition 7.1.1). In a unidimensional theory $T$ every complete type has finite $U-$rank (by Corollary 7.2.1) and there is no strongly minimal set unless $T$ is uncountably categorical. (See Examples 7.1.1 and 7.1.2.) Combining these facts with the Theorems 6.2.1 and 6.2.2 proves

**Corollary 6.2.1.** *Let $T$ be a superstable unidimensional theory which is not uncountably categorical. Then $T$ is $1-$based.*

Many of the results in Chapter 4 have faithful generalizations to superstable theories in which each type has finite $U-$rank. There is also a "geometrical theory" surrounding regular types (instead of minimal types). This material is expounded in [Pil].

The proof of Vaught's conjecture for superstable theories of finite rank depended heavily on the results in this section.

**Historical Notes.** Theorems 6.2.1 and 6.2.2 are due to Buechler, see [Bue85a] and [Bue86], respectively.

## 6.3 Regular Types

In this section we prove the facts about regular types outlined in the introduction of the chapter.

**We assume throughout the section that any theory mentioned is stable.**

**Definition 6.3.1.** *The nonalgebraic stationary type $p$ is* regular *if for any set $A$ over which $p$ is based, any extension of $p|A$ which forks over $A$ is orthogonal to $p$.*

It is clear that any strongly minimal type, in fact minimal type is regular. As (iii) of the next lemma says, one of the most basic properties of a regular type is that forking dependence is transitive on its set of realizations.

**Lemma 6.3.1.** *(i) For $p$ a nonalgebraic stationary type the following are equivalent:*

(1) *$p$ is regular;*
(2) *for some set $A$ on which $p$ is based, any extension of $p|A$ which forks over $A$ is orthogonal to $p$;*
(3) *for some set $A$ on which $p$ is based,*

$$\text{when } a, b \text{ realize } p|A,\ b \mathop{\smile\!\!\!\!\!\!\mid}_{A} C \text{ and } a \mathop{\not\smile\!\!\!\!\!\!\mid}_{A \cup C} b,\quad a \mathop{\smile\!\!\!\!\!\!\mid}_{A} C.$$

*(ii) If $p \in S(A)$ is a nonalgebraic stationary type, where $|A| < \kappa(T)$, and $M \supset A$ is an a-model, then $p$ is regular if $p$ is orthogonal to every forking extension of $p$ in $S(M)$.*

*(iii)* (Transitivity) *Let $p \in S(A)$ be a regular type, $\{a\} \cup \{ b_i : i \in I \} \subset p(\mathfrak{C})$ and $C$ a set.*

$$\text{If } a \mathop{\not\smile\!\!\!\!\!\!\mid}_{A} \{ b_i : i \in I \} \text{ and } b_i \mathop{\not\smile\!\!\!\!\!\!\mid}_{A} C \text{ (for } i \in I) \text{, then } a \mathop{\not\smile\!\!\!\!\!\!\mid}_{A} C.$$

*Proof.* (i) The equivalence of (2) and (3) is simply a matter of rewording the relevant definitions, while (1) $\Longrightarrow$ (2) is trivial.

To prove (2) $\Longrightarrow$ (1) suppose (2) holds with $A$ is in the statement, and $B$ is a set on which $p$ is based. Let $q = tp(c/C)$ be a stationary type which is a forking extension of $p|B$ and assume to the contrary that $q \not\perp p$. Since $p$ is based on $B$ any conjugate of $q$ over $B$ also satisfies these conditions. Thus, we can assume $C \cup \{c\}$ to be independent from $A$ over $B$. In particular, $c$ realizes the nonforking extension of $p|B$ over $A \cup B$; i.e., $p|(A \cup B)$. Thus, $q|(A \cup C)$ is a forking extension of the types: $p|B$, $p|(A \cup B)$ and $p|A$. By (2) $q$ is orthogonal to $p|A$, contradicting the assumption that $q \not\perp p$.

(ii) This is left to the reader in the exercises.

(iii) Assume to the contrary that $a$ is independent from $C$ over $A$ and, without loss of generality, $C \supset A$. By the transitivity of independence, $a$ depends on $\{ b_i : i \in I \}$ over $C$. Thus, there are $J \subset I$ and $j \in I \setminus J$ such that $a$ is independent from $D = C \cup \{ b_i : i \in J \}$ over $C$ and $a$ depends on $b_j$ over $D$. However, the strong type of $b_j$ over $D$ is a forking extension of $p|A$, hence is orthogonal to $p$. Since $stp(a/D)$ is parallel to $p$ this contradicts the dependence of $a$ and $b_j$ over $D$ to prove (iii).

As promised, dimension is well-defined on regular types:

**Proposition 6.3.1.** *Every regular type has weight* 1.

*Proof.* We must show, given a regular type $p$ based on $A$, $pwt(p|A) = 1$. For notational simplicity, take $A = \emptyset$. Suppose, towards a contradiction, that $a$ is a realization of $p$ and there are sets $B$ and $C$ such that $B \perp C$, $a \not\perp B$ and $a \not\perp C$. Without loss of generality, $B$ is the universe of an $a$−model. Let $I \subset B$ be an infinite indiscernible set whose average type over $B$ is $tp(a/B)$. Since $a \not\perp B$, $Av(I/B) \neq p|B$, hence $\{a\} \cup I$ must be dependent over $\emptyset$. Let $J$ be a minimal subset of $I$ such that $\{a\} \cup J$ is dependent. Since $\{a\} \cup I$ is indiscernible (see Lemma 5.1.17), any proper subset of $\{a\} \cup J$ is independent, hence a Morley sequence in $p$. Let $b \in J$ and $J' = J \setminus \{b\}$. By Lemma 6.3.1(iii),

$$b \underset{J'}{\not\perp} a \text{ and } a \not\perp C \implies b \underset{J'}{\not\perp} C.$$

This contradicts the independence of $B$ and $C$ to prove the proposition.

Following is the canonical example of a weight 1 type which is not regular.

*Example 6.3.1.* Let $M$ be the direct sum of $\aleph_0$ copies of the group $(\mathbb{Z}_4, +)$. (This is also Example 3.5.1(iii).) We will show that the generic type of $M$ has weight 1, but is not regular. Since $M$ is simply a module over $\mathbb{Z}$ its quantifier eliminability down to the positive primitive formulas implies (after a little work):

- $T = Th(M)$ is categorical in every infinite cardinal and has Morley rank 2.
- $M$ contains a unique strongly minimal subset definable over 0, namely $2M =$ the elements of order 2 in $M$.
- $M$ and $2M$ are connected.

Let $p \in S(M)$ be the unique generic type and $q \in S(M)$ the generic type of $2M$. Since $q$ is strongly minimal it is regular. To prove that $p$ has weight 1 it suffices (by Lemma 5.6.4(iii) and Remark 5.6.6) to show that $p \triangleleft q$. Since every model of this theory is an $a$−model we need only show that when $b$ is a realization of $q$ and $N$ is the prime model over $M \cup \{b\}$, $p$ is realized in $N$ (see Proposition 5.6.4). Let $a$ be any element of $N$ such that $2a = b$. An analysis of the possibilities for $tp(a/M)$ using the $pp$−elimination of quantifiers leads to the conclusion that $a$ realizes $p$ as desired.

Now let $a$ and $b$ be independent realizations of $p$ and $a' = a + 2b$. Then, $a \not\perp a'$, $b \not\perp \{a, a'\}$ and $a \perp b$. Combining this with Lemma 6.3.1(ii), shows that $p$ is not regular.

By Corollary 5.6.4, dimension is well-defined on a collection of realizations of a regular type. Furthermore, nonorthogonality is the same as domination equivalence on regular types (see Corollary 5.6.5). Combining previous facts concerning weight 1 types and the transitivity of dependence on the realizations of a regular type results in the following critical additivity property of dimension.

**Proposition 6.3.2 (Additivity of Dimension).** *Let $T$ be stable, $M \subset N$ $a-$models, $A \subset M$ a set of cardinality $< \kappa(T)$ and $p \in S(A)$ a regular type. Then,*

$$\dim(p, N) = \dim(p, M) + \dim(p|M, N).$$

*Proof.* Let $I$ be a basis for $p$ in $M$ and $J$ a basis for $p|M$ in $N$. It suffices to show that $I \cup J$ is a basis for $p$ in $N$. If $a$ is an arbitrary element of $p(N)$, $\{a\} \cup J$ depends on $M$ over $A$. By Lemma 5.4.2, $tp(ab/p(M) \cup A) \models tp(ab/M)$ for any finite $b \subset J$. Thus, $\{a\} \cup J$ depends on $p(M)$ over $A$, in fact $\{a\} \cup J$ depends on $I$ over $A$ by Lemma 6.3.1(ii). This proves that $I \cup J$ is a basis for $p$ in $N$, as required.

**Corollary 6.3.1.** *If $p \in S(A)$ is regular and $C \supset B$ are subsets of $p(\mathfrak{C})$, then* $\dim(C/A) = \dim(C/A \cup B) + \dim(B/A)$.

*Proof.* Left to the exercises.

The proof of the proposition contains a proof of

**Corollary 6.3.2.** *Let $M$ be an $a-$model, $A \subset M$ of cardinality $< \kappa(T)$, $p \in S(A)$ a regular type and $I$ a basis for $p$ in $M$. Then, given a Morley sequence $J$ in $p$ which depends on $M$ over $A$, $J$ depends on $I$ over $A$.*

The collection of regular types provide us with a class on which dimension in $a-$models is particularly well-behaved:

**Proposition 6.3.3.** *Let $T$ be superstable and $p \in S(A)$ a regular type, where $A$ is finite.*

*(i) If $M \supset A$ is $a-$prime over a finite set, then $p$ has dimension $\aleph_0$ in $M$.*

*(ii) If $I$ is an infinite Morley sequence in $p$ and $M$ is $a-$prime over $A \cup I$, then $I$ is a basis for $p$ in $M$.*

*(iii) For any $a-$model $M \supset A$, if $B \subset M$ is finite and $q \in S(B)$ is a regular type nonorthogonal to $p$, then* $\dim(p, M) = \dim(q, M)$.

*(iv) For any $\kappa \geq \lambda(T)$ there is an $a-$model $M \supset A$ of cardinality $\kappa$ such that* $\dim(p, M) = \kappa$ *and $\dim(q, M) = \aleph_0$ for any regular type $q$ over a finite subset which is orthogonal to $p$.*

*(v) Let $M_0$ be an $a-$model of cardinality $\lambda \geq \lambda(T)$ and $X \subset S(M_0)$ a set of regular types over $M_0$ such that there is a regular type $q \in S(M_0)$ orthogonal to every element of $X$. Then, there is an $a-$model $M$ of cardinality $\lambda$ containing $M_0$ such that* $\dim(r, M) = 0$, *for all $r \in X$, and* $\dim(q, M) = \lambda$ *for any regular type $q$ over a finite subset of $M$ such that $q \perp r$ for all $r \in X$.*

*Proof.* (i) Suppose $M \supset A$ is $a-$prime over the finite set $B$. Recall from Theorem 5.5.2(iii) that when $M$ is $a-$prime over a set it does not contain an uncountable set of indiscernibles over that set. Let $I$ be a basis for $p$ in $M$. There is a finite $J \subset I$ such that $I$ is independent from $B$ over $J \cup A$. Then

$I \setminus J$ is Morley sequence in $p|(A \cup B)$, which hence has cardinality $\aleph_0$ since $M$ is $a$-prime over $A \cup B$.

(ii) This follows from the proof of Theorem 5.5.2(iii), which is relatively easy.

(iii) Let $N \subset M$ be an $a$-prime model over $A \cup B$. Since $p$ and $q$ are nonorthogonal regular types, they are domination equivalent. By Corollary 5.6.3, $\dim(p|N, M) = \dim(q|N, M)$. From (i) we conclude that $\dim(p, N) = \dim(q, N) = \aleph_0$, so (ii) follows from the additivity of dimension (Proposition 6.3.2).

(iv) Let $I$ be a Morley sequence in $p$ of cardinality $\kappa$ and $M$ an $a$-prime model over $A \cup I$. Let $q$ be a regular type over a finite subset $B$ of $M$. Let $J \subset I$ be a countably infinite set with $stp(B/A \cup J)$ $a$-isolated. Thus, there is $N \subset M$ an $a$-prime model over $A \cup J$ containing $B$. By Corollary 5.5.3, $M$ is $a$-prime over $N \cup I$. By (i), $J$ is a basis for $p$ in $N$ and by Corollary 6.3.2, $I \setminus J = I'$ is a Morley sequence in $p|N$. Since $M$ is $a$-prime over $N \cup I'$, every element of $M \setminus N$ depends on $I'$ over $N$ (by Corollary 5.6.1). Thus, every type over $N$ realized in $M \setminus N$ is nonorthogonal to $p$. In particular, $q|N$ is omitted in $M$, so the additivity of dimension implies that $\dim(q, M) = \dim(q, N) = \aleph_0$.

(v) We define an elementary chain of $a$-models, $M_0 \subset M_1 \subset \ldots \subset M_n \subset \ldots$, each of cardinality $\lambda$ so that the union $M$ has the desired properties. Suppose $M_n$ has been defined and let $Q_n = \{q \in S(M_n) :$ $q$ is a regular type orthogonal to every $r \in X\}$. Then $Q_n$ is nonempty and $|Q_n| \leq \lambda$ (since $T$ is $\lambda$-stable). Let $I$ be an $M_n$-independent set of cardinality $\lambda$ so that for each $q \in Q_n$, $I$ contains a Morley sequence in $q$ of cardinality $\lambda$ and $tp(I/M_n) \perp r$, for all $r \in X$. Let $M_{n+1}$ be an $a$-prime model over $M \cup I$. All of the desired properties are easily verified.

*Remark 6.3.1.* The relationships between the dimensions of nonorthogonal regular types when these dimensions are finite is a complicated matter. Consider, for example, a model $M$ of an uncountably categorical theory and complete strongly minimal (hence regular) types $p$, $q$ over a finite set $A$. If $p$ and $q$ are modular, then $\dim(p, M) = \dim(q, M)$. (See Corollary 4.3.5.) Using deeper results from geometrical stability theory it is possible to show that $\dim(p, M) = \dim(q, M)$ when both $p$ and $q$ are locally modular and nonmodular. (This was proved by Hrushovski and Laskowski in [Las88].) The relationship between $\dim(p, M)$ and $\dim(q, M)$ when $p$ and $q$ are nonorthogonal regular types over a finite set $A \subset M$ and $M$ is a model is related to how the nonorthogonality of $p$ and $q$ is witnessed. As stated in Remark 5.6.2 this is a deep problem in geometrical stability theory.

The usefulness of regular types in superstable theories depends heavily on the following existence result.

**Lemma 6.3.2 (Existence).** *Suppose that $T$ is superstable, $M \subset N$ are $a$-models, $A \subset M$ is finite and $q \in S(A)$ is realized in $N \setminus M$. Then, there is a regular type $p \in S(M)$ containing $q$ which is realized in $N$.*

*Proof.* Let $p \in S(M)$ be an extension of $q$ of least $U-$rank which is realized in $N \setminus M$. We will show that $p$ is regular. Let $B \subset M$ be a finite set containing $A$ on which $p$ is based. Assuming, to the contrary, that $p$ is not regular there is a finite set $C$, $B \subset C \subset M$ and an $r \in S(C)$ which is a forking extension of $p|B$ nonorthogonal to $p$ (see Lemma 6.3.1(ii)). We contradict the minimality assumption on $U(p)$ as follows. Enlarging $C$ if necessary we can assume by the nonorthogonality of the relevant types that there are $a$ realizing $p|C$ and $b$ realizing $r$ which are dependent over $C$. Since $N$ is an $a-$model realizing $p$ we can require that $a \in N$ realizes $p$ and $b$ is in $N$. Since $a$ depends on $b$ over $C$ and $a$ is independent from $M$ over $C$, $b$ must be in $N \setminus M$. Since $U(r) < U(p)$ and $r \supset q$ this contradicts the minimality assumption on $U(p)$. We conclude that $p$ is regular.

In the proof of the lemma we also show

**Corollary 6.3.3.** *For $T$ a superstable theory, $M$ an $a-$model and $p \in S(M)$, the following are equivalent.*

(1) *$p$ is regular.*
(2) *There is an $a-$model $N \supset M$ realizing $p$ and a finite set $A \subset M$ such that whenever $b \in N \setminus M$ realizes $p \restriction A$, $b$ realizes $p$.*

This corollary can be useful in verifying that certain types are regular. Take, for example, the theory of an equivalence relation $E$ with infinitely many infinite classes and no finite classes. Then, $T$ is $\omega-$stable and $\omega-$categorical, so every model is an $a-$model. To verify that the unique $q \in S_1(\emptyset)$ is regular let $M$ be any model, $a$ a realization of $q|M$ and $N$ the prime model over $M \cup \{a\}$. Our knowledge of the models of this theory tells us that every element of $N \setminus M$ is $E-$equivalent to $a$, hence not $E-$equivalent to any element of $M$. That is, every element of $N \setminus M$ realizes $q|M$. We conclude that $q$ is regular by the corollary.

The proof of the lemma also shows that for $T$ a 1−sorted superstable theory, every regular type in $T^{eq}$ is nonorthogonal to a regular 1−type in the sort of $T$ (see the exercises).

Concerning possible extensions of Lemma 6.3.2, it is true that for any two distinct *models* $M \subset N$ of a superstable theory there is a regular type over $M$ realized in $N$. However, the proof of this more general result, found in [SB89], is considerably more difficult than the one above and requires more sophisticated machinery. We will not reproduce the proof here since the above result together with the existence lemma for strongly regular types in t.t. theories proved later, are sufficient to prove a high percentage of the existing results. The hypothesis that $T$ is superstable in the lemma is necessary, though. There is an example of a countable stable theory having distinct $\aleph_1-$saturated models $M \subset N$ with no regular type over $M$ realized in $N$.

**Corollary 6.3.4.** *Given a weight* 1 *type $p$ in a superstable theory there is a regular type $q$ domination equivalent to $p$.*

*Proof.* Without loss of generality, $p \in S(M)$, for $M$ some $a$−model. Let $a$ be a realization of $p$ and $N$ an $a$−prime model over $M \cup \{a\}$. By Lemma 6.3.2 there is a regular type $q \in S(M)$ realized in $N$. This forces $p$ and $q$ to be nonorthogonal, hence $p \sqsubseteq q$, as needed.

**Corollary 6.3.5.** *Let $T$ be superstable, $M \subset N$ $a$−models and $C \subset N$ a maximal $M$−independent set such that, for each $c \in C$, $tp(c/M)$ is regular. Then $N$ is dominated by $C$ over $M$.*

*Proof.* Combine the claim in the proof of Theorem 5.6.1 with the preceding corollary.

The following decomposition theorem allows us to deal only with regular types in many settings. Recall from Corollary 5.6.7 that every stationary type in a superstable theory has finite weight. This can be extended to:

**Corollary 6.3.6 (Decomposition Theorem).** *If $T$ is superstable and $p$ is a stationary type there are regular types $q_1, \ldots, q_n$, for $n = wt(p)$, such that $p \sqsubseteq q_1 \otimes \ldots \otimes q_n$. In fact, if $M$ is an $a$−model on which $p$ is based, $a$ realizes $p|M$ and $N$ is $a$−prime over $M \cup \{a\}$, $N$ is also $a$−prime over $M \cup \{b\}$, where $b = \{b_0, \ldots, b_n\}$ is a maximal $M$−independent sequence of elements realizing regular types over $M$.*

*Proof.* Given a stationary type $p$, by Theorem 5.6.1 there are weight 1 types $r_1 \ldots, r_n$ such that $p \sqsubseteq r_1 \otimes \ldots \otimes r_n$. From Corollary 6.3.4 we get, for each $1 \leq i \leq n$, a regular type $q_i$ domination equivalent to $r_i$. Using Remark 5.6.3, $p$, $r_1 \otimes \ldots \otimes r_n$ and $q_1 \otimes \ldots \otimes q_n$ are domination equivalent.

The second part of the corollary follows from the claim in the proof of Theorem 5.6.1 and Corollary 6.3.4.

Any sequence $\{q_0, \ldots, q_{n-1}\}$ of regular types such that $p \sqsubseteq \bigotimes_{i<n} q_i$ is called a *regular decomposition of p*. Part (i) of the following corollary addresses the uniqueness of a regular decomposition. The other parts of the corollary indicate the degree to which this decomposition theorem reduces orthogonality and domination on all stationary types to considering only regular types.

**Corollary 6.3.7.** *Let $T$ be superstable.*

*(i) If $\{q_0, \ldots, q_{n-1}\}$ and $\{q'_0, \ldots, q'_{n'-1}\}$ are regular decompositions of $p$ and $p'$, respectively, then $p' \lhd p$ if and only if there is a one-to-one map of indices $j < n'$ into indices $i_j < n$ such that $q'_j \sqsubseteq q_{i_j}$ for all $j < n'$. Thus, $p \sqsubseteq p'$ if and only if there is a bijection $j \mapsto i_j$ such that $q'_j \sqsubseteq q_{i_j}$ for all $j < n'$.*

*(ii) Given stationary types $p$, $p'$, $r$ and $r'$, if $r \sqsubseteq r'$ then $p' \otimes r' \lhd p \otimes r$ implies $p' \lhd p$.*

*(iii) For any stationary $p$ and $p'$, $p \not\perp p'$ if and only if there is a regular $q$ nonorthogonal to both $p$ and $p'$.*

*Proof.* (i) Suppose that $M \subset N$ are $a$–models and $C$ is an $M$–independent set of realizations of regular types over $M$ such that $N$ is dominated by $C$ over $M$. Let $X$ be an equivalence class of the regular types over $M$ with respect to domination equivalence and $C_X$ the elements $c$ of $C$ with $tp(c/M) \in X$.

*Claim.* $C_X$ is a basis for $D = \{ d \in N : tp(d/M) \in X \}$.

Since $C$ is $M$–independent and every element of $C \setminus C_X$ realizes a type over $M$ orthogonal to the elements of $X$, $tp(C/M \cup C_X)$ is orthogonal to each element of $X$. For $d \in D$,

$$\text{since} \quad d \underset{M}{\not\perp} C \text{ and } d \underset{M}{\perp} C_X, \; d \underset{M \cup C_X}{\not\perp} C,$$

a contradiction from which we conclude that $d$ depends on $C_X$ over $M$, as claimed.

Now suppose that $p$, $p'$, the $q_i$'s and $q_i'$'s are as hypothesized, all are based on the $a$–model $M$, $p' \lhd p$, $a$ realizes $p|M$ and $N$ is the $a$–prime model over $M \cup \{a\}$. Let $c = (c_0, \ldots, c_{n-1})$ and $c' = (c_0', \ldots, c_{n'-1}')$ be realizations of $\bigotimes_{i<n} q_i|M$ and $\bigotimes_{j<n'} q_j'|M$, respectively, in $N$. Since $N$ is $a$–prime over $M \cup \{c\}$, $c$ is a basis for the realizations of the regular types over $M$ in $N$. The argument in the first paragraph shows that we can partition the $c_i$'s and $c_i'$'s into equivalence classes with respect to the nonorthogonality of their types and the dimension of a class of $c_i'$'s is $\leq$ the dimension of the corresponding class of $c_i$'s. This indicates how to define the one-to-one map required in the statement.

The other direction of the biconditional is clear.

Parts (ii) and (iii) follow quickly from (i).

Applications of this decomposition in terms of regular types will be seen in Section 7.1.

### 6.3.1 Rank Considerations

**We assume every theory mentioned in this subsection to be superstable.** In Lemma 6.1.3 we showed that widely different $U$–ranks imply the orthogonality of the two types. It is a short step from there to

**Lemma 6.3.3.** *If $p$ is a stationary complete type of $U$–rank $\omega^\alpha$ for some $\alpha$ then $p$ is regular.*

*Proof.* Since $\beta < \omega^\alpha \implies \beta \ll \omega^\alpha$ any forking extension of $p$ is orthogonal to $p$ by Lemma 6.1.3.

We prove in the next proposition that the types of $U$–rank $\omega^\alpha$ are the canonical regular types. Properties of dependence on these special regular types are occasionally easier to prove.

**Proposition 6.3.4.** *If $p$ is a stationary complete type of weight 1 and $U(p) = \gamma + \omega^\alpha \cdot n$ where $\omega^\alpha \ll \gamma$ or $\gamma = 0$, then*

- *there is a set $A$ on which $p$ is based,*
- *an $a$ realizing $p|A$, and*
- *a $c \in acl(A \cup \{a\})$ with $U(c/A) = \omega^\alpha$.*

*A fortiori, $p$ is nonorthogonal to a type of $U$–rank $\omega^\alpha$.*

The proposition says, for instance, that when (in addition) $U(p)$ is finite $p$ is nonorthogonal to a type of $U$–rank 1. Dependence on the set of realizations of a $U$–rank 1 type is algebraic dependence which is particularly easy to study. This reduction to types of $U$–rank 1 is often helpful when working in the finite rank context. Before turning to the main body of the proof we prove a general result allowing us to consider only the case $\gamma = 0$.

**Lemma 6.3.4.** *If $U(c) = \omega^{\alpha_1} \cdot n_1 + \ldots + \omega^{\alpha_k} \cdot n_k$ where $\alpha_1 > \ldots > \alpha_k$ and $1 \leq l \leq k$, there is a $c' \in acl(c)$ such that $U(c/c') = \omega^{\alpha_1} \cdot n_1 + \ldots + \omega^{\alpha_l} \cdot n_l$ and $U(c') = \omega^{\alpha_{l+1}} \cdot n_{l+1} + \ldots + \omega^{\alpha_k} \cdot n_k$.*

*Proof.* First consider the case $l = 1$ and simplify the notation by writing $U(c) = \omega^\alpha \cdot n + \beta$, where $\beta < \omega^\alpha$.

By the connectivity of $U$–rank there is a $d$ such that $U(c/d) = \omega^\alpha \cdot n$. Without loss of generality, $d \in Cb(c/d)$.

*Claim.* $U(d) < \omega^\alpha$.

Choosing $\{c_0, \ldots, c_m\}$ to be a Morley sequence in $stp(c/d)$ on which this strong type is based, $d \in acl(c_0, \ldots, c_m)$. By repeated application of Corollary 6.1.2, $U(c_0 \cdots c_m d) = U(c_0 \cdots c_m) \approx_\alpha \omega^\alpha \cdot n(m+1) \approx_\alpha U(c_0 \cdots c_m/d)$. Since $d \in acl(c_0, \ldots, c_m)$, $U(d)$ is certainly $< \omega^{\alpha+1}$. Thus, Corollary 6.1.2 says that $U(d) \approx_\alpha 0$; i.e., $U(d) < \omega^\alpha$.

Let $c'$ be an element of $C = Cb(d/c)$ such that $d$ is independent from $c$ over $c'$. Then $c' \in acl(c)$ and $C \subset acl(c')$. Since $c$ is independent from $d$ over $c'$, $U(c/c') = U(c/c'd) \leq U(c/d) = \omega^\alpha \cdot n$. As in the proof of the claim, $c'$ is algebraic in a Morley sequence in $stp(d/c)$ and $U(c') < \omega^\alpha$. It is easy to verify that $\delta' \ll \delta \implies \delta + \delta' = \delta \oplus \delta'$. Thus, the additivity of $U$–rank implies $U(c/c') + U(c') = U(cc') = U(c)$; i.e., $\omega^\alpha \cdot n + U(c') = \omega^\alpha \cdot n + \beta$. Since $\beta < \omega^\alpha$ we conclude that $U(c') = \beta$, as desired.

Applying the $l = 1$ case repeatedly yields elements $c_1, \ldots, c_{k-1}$ such that, letting $c_0 = c$, $c_{i+1} \in acl(c_i)$, $U(c_i/c_{i+1}) = \omega^{\alpha_{i+1}} \cdot n_{i+1}$ and $U(c_{i+1}) = \omega^{\alpha_{i+2}} \cdot n_{i+2} + \ldots + \omega^{\alpha_k} \cdot n_k$. The argument used at the end of the previous paragraph shows that $U(c/c_i) = \omega^{\alpha_1} \cdot n_1 + \ldots + \omega^{\alpha_i} \cdot n_i$ whenever $1 \leq i < k$. Thus, the conditions of the lemma are satisfied by taking $c' = c_l$.

*Proof of Proposition 6.3.4.* By Lemma 6.3.4 there is a set $B$ on which $p$ is based, $a$ realizing $p|B$ and $a' \in acl(A \cup \{a\})$ with $U(a'/A) = \omega^\alpha \cdot n$. Since

$wt(a'/A) = 1$ it suffices to prove the proposition for $stp(a'/A)$ instead of $p$, so without loss of generality, $U(p) = \omega^\alpha \cdot n$.

Choose a set $A$ and element $b$ such that for some $a$ realizing $p|A$, $a$ depends on $b$ over $A$, $U(a/A \cup \{b\}) \geq \omega^\alpha \cdot (n-1)$ and $U(b/A)$ is minimal with respect to these restrictions. Without losing this minimality assumption we can assume that $b \in Cb(a/A \cup \{b\})$ (since the canonical base is algebraic in $A \cup \{b\}$). Without loss of generality, $A = \emptyset$.

*Claim.* $U(b) = \omega^\alpha$.

We first show that $U(b) \approx_\alpha \omega^\alpha$. Let $a_0, \ldots, a_k$ be a Morley sequence in $stp(a/b)$ on which this strong type is based. Then $U(a_i/b) \approx_\alpha \omega^\alpha \cdot (n-1)$ for each $i$. If $U(a_i/a_0) \approx_\alpha \omega^\alpha \cdot n$, then $a_i \perp a_0$. The data: $a_i$ realizes $p|a_0$, $U(b/a_0) < U(b)$ and $U(a_i/a_0 b) \approx_\alpha \omega^\alpha \cdot (n-1)$ contradict the above minimality assumption on $U(b/A)$. From Corollary 6.1.2 and $U(a_i/a_0) \approx_\alpha \omega^\alpha \cdot (n-1)$ (for $1 \leq i \leq k$) we get $U(a_0 \cdots a_k b) = U(a_0 \cdots a_k) \approx_\alpha \omega^\alpha \cdot n + \omega^\alpha \cdot (n-1)k$. Since $U(a_0 \cdots a_k/b) \approx_\alpha \omega^\alpha \cdot (n-1)(k+1)$, the relation $U(b) \approx_\alpha \omega^\alpha$ follows from that same corollary.

Now suppose, towards a contradiction, that $U(b) = \omega^\alpha + \beta$ for some $\beta > 0$. By Lemma 6.3.4 there is an element $b' \in acl(b)$ with $U(b/b') = \omega^\alpha$ and $U(b') = \beta$. Since $\beta \ll U(a)$, $a \perp b'$. The properties: $a$ realizes $p|b'$, $a$ depends on $b$ over $b'$, and $U(a/A \cup \{b, b'\}) \geq \omega^\alpha \cdot (n-1)$, contradict the minimality assumption on $U(b/A)$, completing the proof of the claim.

Now we reverse the roles of $a$ and $b$ to find $c$. First notice that $U(b/a) \approx_\alpha 0$. Choose $c$ interalgebraic with $Cb(b/a)$ and $\{b_0, \ldots, b_m\}$ a Morley sequence in $stp(b/a)$ in which $c$ is algebraic. Since $wt(a) = 1$ and $a$ depends on each $b_i$, $b_i \not\perp b_0$ for all $i$, hence $U(b_i/b_0) < \omega^\alpha$. Since $U(b_i/c) \approx_\alpha 0$ repeating the argument in the previous paragraph shows that $U(c) \approx_\alpha \omega^\alpha$. Reasoning as at the end of the proof of the claim yields a $d \in acl(c)$ such that $a$ realizes $p|d$, $c \in acl(ad)$ and $U(c/d) = \omega^\alpha$. This proves the proposition.

In the exercises the reader is asked to show that we can take $A$ to be any $a$-model $M$ on which $p$ is based. It is interesting to note that we use the weight 1 hypothesis only near the end where we prove that $\{b_0, \ldots, b_m\}$ cannot be pairwise independent.

**Corollary 6.3.8.** *If $p$ is a nonalgebraic complete type of finite $U$-rank, then $p$ is nonorthogonal to a minimal type.*

**Historical Notes.** Regular types were developed by Shelah in [She90, V]. Our treatment of their properties follows Makkai [Mak84] to some degree. The results on $U$-rank in the subsection are by Lascar [Las84].

**Exercise 6.3.1.** Prove: For $T$ a 1-sorted superstable theory, every regular type in $T^{eq}$ is nonorthogonal to a regular 1-type in the sort of $T$.

**Exercise 6.3.2.** Give an example of a regular type of Morley rank $\omega$ (in some t.t. theory).

**Exercise 6.3.3.** Prove Lemma 6.3.1(ii).

**Exercise 6.3.4.** Prove Corollary 6.3.1.

**Exercise 6.3.5.** Write out a direct proof of Corollary 6.3.3.

**Exercise 6.3.6.** For $p$ a type in a superstable theory let $q$ be a stationary type of least $\infty-$rank nonorthogonal to $p$. Show that $q$ is regular.

**Exercise 6.3.7.** Show that we can take $A$ to be any $a-$model on which $p$ is based in Proposition 6.3.4.

## 6.4 Strongly Regular Types

The dimension theory for regular types and the class of $a-$models in a superstable theory (see Proposition 6.3.3) does not generalize completely to the class of all models. In this section we restrict to a t.t. theory and develop a similar dimension theory for a subclass of regular types (the strongly regular types) and the class of all models of the theory. While all of the results here hold in an arbitrary t.t. theory we will simplify the notation by restricting to countable theories; i.e., **we assume throughout the section that every theory is $\omega-$stable**. The relevant types are the following:

**Definition 6.4.1.** *Let $p \in S(A)$ be a stationary nonalgebraic type and $\varphi \in p$. The pair $(p, \varphi)$ is called* strongly regular *if for all sets $B \supset A$ and $q \in S(B)$, $\varphi \in q$ and $q \not\perp p$ implies that $q = p|B$. $p$ is* strongly regular *if $(p, \psi)$ is strongly regular for some $\psi \in q$.*

It is common to write SR instead of strongly regular. It is easy to see that a SR type is regular. (Let $q \in S(B)$ be a forking extension of $p \in S(A)$. Then $\varphi \in q$ and $q \neq p|B$, hence $q \perp p$.) If $(p, \varphi)$ is SR then $(p, \psi)$ is also SR for any formula $\psi \in p$ which implies $\varphi$. Thus, when $p$ is SR there is a $\varphi \in p$ such that $(p, \varphi)$ is SR, $MR(\varphi) = MR(p)$ and $\deg(\varphi) = 1$. For $p \in S(A)$ stationary and $\varphi \in p$ let $A_0$ be a finite set such that $p$ is based on $A_0$ and $\varphi$ is over $A_0$. Then, $(p, \varphi)$ is SR if and only if $(p \restriction A_0, \varphi)$ is SR. Thus, when checking to see if a pair is strongly regular we can always assume the type to be over a finite set.

Compare the following equivalents with Corollary 6.3.3.

**Lemma 6.4.1.** *For $p \in S(A)$ a stationary type and $\varphi \in p$ the following are equivalent.*

(1) *$(p, \varphi)$ is SR.*

(2) *For any model $M \supset A$ there is a model $N \supset M$ realizing $p|M$ such that any $a \in \varphi(N) \setminus M$ realizes $p|M$.*
(3) *There is a model $M \supset A$ and a model $N \supset M$ such that $(*)$ $p|M$ is realized in $N$ and any $a \in \varphi(N) \setminus M$ realizes $p|M$.*

*Proof.* (2) $\Longrightarrow$ (3) holds trivially.

(1) $\Longrightarrow$ (2). Let $b$ realize $p|M$ and $N$ be the prime model over $M \cup \{b\}$. By Corollary 3.3.4, any $a \in N \setminus M$ depends on $b$ over $M$. Thus, $a \in \varphi(N) \setminus M \implies tp(a/M) \not\perp p|M \implies tp(a/M) = p|M$ (by the strong regularity of $(p, \varphi)$).

Before turning to the remaining nontrivial implication, (3) $\Longrightarrow$ (1), we prove

*Claim.* If $(p, \varphi)$ is not SR then for any $a$–model $M \supset A$ there is $q \in S(M)$ containing $\varphi$ such that $q \not\perp p$ and $q \neq p|M$.

Let $r$ be a stationary type witnessing that $(p, \varphi)$ is not SR. Let $A_0$ be a finite subset of $A$ containing the parameters in $\varphi$ and on which $p$ is based. We can assume $r$ to be over a finite set $B \supset A_0$. Let $B' \subset M$ realize $tp(B/A_0)$ and $r'$ be a conjugate of $r$ over $B'$. Since $p$ is based on $A_0$, the nonorthogonality of $r$ and $p$ implies the nonorthogonality of $r'$ and $p$ (see Exercise 5.6.2). Again, using that $p$ is based on $A_0$, $p|\mathfrak{C}$ does not split over $A_0$, hence the specified automorphism taking $B$ to $B'$ over $A_0$ maps $p|B$ to $p|B'$. Thus, $r'$ is not $p|B'$. Since $r'$ contains $\varphi$ it witnesses that $(p, \varphi)$ is not strongly regular.

(3) $\Longrightarrow$ (1) As we said above, we can assume $A$ to be a finite set, which we take to be $\emptyset$ (without loss of generality). Let $M_0 \subset N_0$ be models satisfying $(*)$ in (3) and $a \in N$ a realization of $p|M_0$. We will use a theory in an expanded language to show

*Claim.* There are $a$–models $M$ and $N$ satisfying $(*)$.

Let $L$ be the language of the relevant theory $T$, $P$ a new unary predicate and $L' = L \cup \{P\} \cup \{a\}$. Let $\Psi$ be the collection of formulas $\psi(x, y)$ over $\emptyset$ such that $\models \psi(a, b) \implies a \not\downarrow b$. Let $T' \supset T$ be the theory in $L'$ expressing the following properties of a model $M'$ (the detailed formulation is left to the reader):

- $P(M')$ is an elementary submodel of $M'$ with respect to the language $L$;
- $a$ realizes $p \restriction \emptyset$ and $a \notin P(M')$;
- for all $b \in \varphi(M') \setminus P(M')$, $b$ realizes $p \restriction \emptyset$;
- for all $b \in \varphi(M') \setminus P(M')$ and for all formulas $\psi(x, y) \in \Psi$, $\models \neg\psi(b, c)$ for all $c \in P(M')$.

The model obtained by interpreting $P$ by $M$ on $N$ gives a model of $T'$, proving its consistency. By a now standard elementary chain argument there is a model $M'$ of $T'$ such that $M'$ and $P(M')$ are $\aleph_0$–saturated (i.e., $a$–saturated) as models of $T$. This proves the claim.

Assuming $M$ and $N$ to be $a$–models it suffices to show (by the first claim) that $p|M$ is the only type in $S(M)$ which contains $\varphi$ and is nonorthogonal to $p$. Suppose, to the contrary, that $q \neq p|M$ in $S(M)$ contains $\varphi$ and is nonorthogonal to $p$. Let $B \subset M$ be a finite set on which $q$ is based such that $p|B \overset{a}{\not\perp} q|B$. Since $N$ is an $a-$model there is a $b \in N$ realizing $q|B$ depending on $a$ over $B$. Since $a \smile M$, $b$ is not in $M$. Since $b$ satisfies $\varphi$ the hypothesis implies that $b$ realizes $p|M$. This contradicts that $b$ realizes $q|B \neq p|B$.

**Lemma 6.4.2 (Existence).** *For any distinct models $M \subset N$ there is an $a \in N$ such that $tp(a/M)$ is strongly regular. We may choose $a$ to satisfy any formula over $M$ which is satisfied in $N \setminus M$. Moreover, if $q \in S(M)$ is an SR type nonorthogonal to $tp(a/M)$ then $q$ is realized in $N$.*

*Proof.* Given a formula $\theta$ over $M$ satisfied in $N \setminus M$, let $a \in N \setminus M$ be such that $MR(a/M)$ is minimal in $\{ MR(b/M) : b \in N \setminus M \text{ and } \models \theta(b) \}$. Choose $\varphi \in p = tp(a/M)$ such that $MR(\varphi) = MR(p)$ and $\deg(\varphi) = 1$. Without loss of generality, $\varphi$ implies $\theta$. The minimal rank condition guarantees that every element of $N \setminus M$ satisfying $\varphi$ has the same Morley rank as $p$. Since $\varphi$ has degree 1 every such element realizes $p|M$. By Lemma 6.4.1, $(p, \varphi)$ is SR.

Turning to the moreover part of the lemma suppose that $q \in S(M)$ is strongly regular and nonorthogonal to $p$. Choose $\psi \in q$ such that $(q, \psi)$ is a strongly regular pair. Without loss of generality, $N$ is prime over $M \cup \{a\}$, hence every type over $M$ realized in $N$ is nonorthogonal to $p$. Let $b$ be an element independent from $a$ over $M$ such that there is $c$ realizing $q|(M \cup \{b\})$ with $a$ and $c$ dependent over $M \cup \{b\}$. Let $d$ be an element of $M$ over which $tp(abc/M)$ is based. Then $a$ depends on $cb$ over $d$ and we can assume $\psi$ to be a formula over $d$. Let $\theta(x, y, a)$ be a formula in $tp(cb/ad)$ with the properties:

(1) $\exists y \theta(x, y, a)$ implies $\psi$, and
(2) $\models \theta(c', b', a)$ implies $a \underset{d}{\not\perp} c'b'$.

Since $b$ is independent from $a$ over $M$ and $tp(a/M)$ is definable over $d$ there is a $b'$ in $M$ such that $\models \exists x \theta(x, b', a)$. Let $c' \in N$ satisfy $\theta(x, b', a)$. Item (1) and the dependence guaranteed by (2) force $c'$ to be in $\psi(N) \setminus M$. Thus, $tp(c'/M)$ is nonorthogonal to $tp(a/M)$, hence nonorthogonal to $q$ (since these are regular types). Since $(q, \psi)$ is SR $tp(c'/M)$ must be $q$ as required.

**Corollary 6.4.1.** *Every regular type is nonorthogonal to a strongly regular type. In fact, a stationary type of least Morley rank nonorthogonal to a given regular type is strongly regular.*

*Proof.* See Exercise 6.4.2.

As with Lemma 6.4.1 the scheme throughout the section is to generalize properties about regular types and $a-$models in superstable theories to strongly regular types and models in t.t. theories. The difficult part of this extension is

**Proposition 6.4.1.** *If a type $p$ is nonorthogonal to a model $M$, then there is a strongly regular type over $M$ nonorthogonal to $p$.*

*Proof.* Again the idea is to use a theory in an expanded language to reduce our attention to $a$-models, then use properties proved earlier for regular types.

Without loss of generality $p$ is a regular type over a $|M|^+$-saturated model $N \supset M$. Let $a$ realize $p$ and $N'$ be an $a$-prime model over $N \cup \{a\}$. Let $q \in S(N)$ be a nonalgebraic type realized in $N'$ such that $MR(q \restriction M)$ is minimal among all such. Require, furthermore, that $MR(q)$ is minimal in $\{MR(r) : r \in S(N) \text{ and } MR(r \restriction M) = MR(q \restriction M)\}$. It follows that $q$ is strongly regular. (Take $\psi \in q$ the conjunction of two formulas, one which determines $MR(q \restriction M)$ and another which determines $MR(q)$. By Corollary 6.4.2 there is a SR type over $N$ containing $\psi$ and realized in $N'$. The minimal rank assumptions on $q$ guarantee that this SR type is $q$.) The rest of the proof is needed to show that $q$ does not fork over $M$.

Fix a formula $\psi \in q$ so that $(q, \psi)$ is SR, $MR(\psi) = MR(q)$ and $\deg(\psi) = 1$. Let $d \in N$ contain the parameters in $\psi$ (in which case $q$ is based on $d$). Since $p$ is nonorthogonal to $M$ and $p \sqsupseteq q$ (these are nonorthogonal regular types) $q$ is nonorthogonal to $M$. Since $N$ is $|M|^+$-saturated there is a $d' \in N$ realizing $tp(d/M)$ and independent from $d$ over $M$. By Proposition 5.6.2 the type $q'$ over $d'$ conjugate to $q|d$ over $M$ is nonorthogonal to $q$, hence domination equivalent. This yields (using Proposition 5.6.1):

- finite Morley sequences $I \subset N$ in $q|d$ and $I' \subset N'$ in $q'$, and
- elements $b$ realizing $q$ and $b'$ realizing $q'|N$ in $N'$ such that $b$ depends on $b'$ over $I \cup I' \cup \{d, d'\}$.

Since $I \cup \{d\}$ is $M$-independent from $I' \cup \{d'\}$ and conjugate over $M$ we may as well absorb these Morley sequences into the original domains and assume that $b$ and $b'$ are dependent over $dd'$. Let $e \in M$ be an element on which $tp(dd'/M)$ is based. Without loss of generality there is a formula $\psi_0 \in q_0 = q \restriction e$ with Morley rank $MR(q \restriction M)$. Note: the original assumptions about $q$ imply that any element of $N' \setminus N$ satisfying $\psi_0$ also satisfies $q_0|M = q \restriction M$.

To use previously proved facts about regular types we need to work within the class of $a$-models.

*Claim.* There are $a$-models $M_0$, $N_0$ and $N_0'$ such that

(1) $M_0 \subset N_0 \subset N_0'$;
(2) $e \in M_0$, $d$, $d' \in N_0$ and $b$, $b' \in N_0'$;
(3) $d \underset{M_0}{\perp} d'$;
(4) $b$ realizes $q|N_0$ and $b'$ realizes $q'|N_0$;
(5) for any $b^* \in N_0' \setminus N_0$ satisfying $\psi_0$, $tp(b^*/M_0) = q_0|M_0$.

As in the proof of the second claim in Lemma 6.4.1 we proceed by expressing the desired properties with a first-order theory in an expanded language with predicates $P$ and $Q$ representing the two models $M_0$ and $N_0$. In that earlier proof we showed how to express, e.g., $b$ realizes $q|Q(N^*)$ (where $N^*$

is an arbitrary model of the expanded theory). Notice that the condition "$d$ and $d'$ are $P(N^*)$−independent" can be obtained by requiring $dd'$ to realize $stp(dd'/e)|P(N^*)$. The reader is asked to fill in the details from these hints in the exercises.

Now fix $a$−models $M_0 \subset N_0 \subset N_0'$ as in the claim. The elements $b$ and $b'$ witness the nonorthogonality of $q|N_0$ and $q'|N_0$. These types are based on $d$ and $d'$, respectively, and $\{d, d'\}$ is independent over $e$. Since $M_0$ is an $a$−model and $tp(d/M_0)$ is based on $e$, an element $d'' \in M_0$ realizing $tp(d'/e)$ also realizes $stp(d'/ed)$. Thus, the conjugate $q''$ of $q|d$ over $d''$ is a strongly regular type nonorthogonal to $q$. Since $q$ and $q''$ are domination equivalent, there is $b'' \in N_0'$ realizing $q''|N_0'$. Certainly, $b''$ satisfies $\psi_0$. We assumed that $q$ forks over $M$; i.e., has Morley rank $< MR(\psi_0)$. Thus, $MR(b''/M_0) \leq MR(b''/d''e) < MR(\psi_0)$, contradicting the last condition listed in the claim. This contradiction proves that $q$ does not fork over $M$, and finally completes the proof of the proposition.

Most matters involving orthogonality of types in superstable theories reduce to properties of regular types through:

> for all distinct $a$−models $M \subset N$ with $wt(N/M)$ finite there is $I$, an $M$−independent set of realizations of regular types over $M$ such that $N$ is $a$−prime over $M \cup I$.

The literal generalization of this result to models and SR types is:

> for all distinct models $M \subset N$ with $wt(N/M)$ finite there is $I$, an $M$−independent set of realizations of strongly regular types over $M$ such that $N$ is prime over $M \cup I$.

However, the result in the $a$−model context was proved by first showing $N$ is dominated by $I$ over $M$, then noting that domination and $a$−atomicity (with respect to a finite set) are equivalent over $a$−models. The next lemma states the only general connection between domination and ordinary atomicity, so we must be content with the subsequent proposition.

**Lemma 6.4.3.** *If $A$ is atomic over $B \cup M$, where $M$ is a model, then $A \cup B$ is dominated by $B$ over $M$.*

*Proof.* This is a mild generalization of Lemma 3.4.7. Without loss of generality, $A = a$ and $B = b$ are finite. Suppose, to the contrary, there is a $c$ independent from $b$ over $M$ which depends on $ab$ over $M$; i.e., $c$ depends on $a$ over $M \cup \{b\}$. Let $d \in M$ be an element over which $tp(abc/M)$ is based. Let $\psi(x, y)$ be a formula in $tp(ac/bd)$ such that

- $\exists y \psi(x, y)$ isolates $tp(a/M \cup \{b\})$ and
- whenever $a'$ realizes $tp(a/bd)$ and $\models \psi(a', c')$, $a'$ and $c'$ are dependent over $bd$.

Since $b$ and $c$ are $M$-independent and $tp(b/M)$ is based on $d$ there is a $c' \in M$ such that $\exists x\psi(x, c')$. Let $a'$ satisfy $\psi(x, c')$. Then $a'$ realizes $tp(a/M \cup \{b\})$, hence, $tp(a'/M \cup \{b\})$ does not fork over $bd$, contradicting the dependence of $a'$ and $c'$ over $bd$.

**Proposition 6.4.2.** *For models $M \subset N$, $M \neq N$, there is $I$, an $M$-independent set of realizations of strongly regular types over $M$ such that $N$ is dominated by $M \cup I$ over $M$.*

*Proof.* Let $I$ be a maximal $M$-independent subset of $N$ consisting of realizations of SR types over $M$. Let $M' \subset N$ be a maximal set dominated by $I$ over $M$. Note that $M'$ is a model. (The prime model $M'' \subset N$ over $M'$ is dominated by $M'$ over $M$ by Lemma 6.4.3. By the transitivity of domination, $M''$ is dominated by $I$ over $M$, hence $M'' = M'$.) Supposing, towards a contradiction, that $M' \neq N$ let $a \in N$ realize a strongly regular type over $M'$. Assuming first that $tp(a/M')$ is orthogonal to $M$, $M' \cup \{a\}$ is dominated by $M'$ over $M$, contradicting the maximality of $M'$. Thus, $tp(a/M')$ is nonorthogonal to $M$. By Proposition 6.4.1 there is a strongly regular $p \in S(M)$ nonorthogonal to $tp(a/M')$. By Corollary 6.4.2 there is a $b \in N$ realizing $p|M'$. This element $b$ contradicts the maximality of $I$, completing the proof.

Throughout Section 5.6 we studied orthogonality and domination relative to the class of $a$-models. Using the proven facts about SR types we can extend some of these results to the class of all models of a t.t. theory. As a first installment:

**Lemma 6.4.4.** *If $M$ is a model and $p$, $q \in S(M)$ are nonorthogonal types, then $p \overset{a}{\not\perp} q$.*

*Proof.* Let $a$ realize $p$ and $N$ be a prime model over $M \cup \{a\}$. By Proposition 6.4.2 there is an $M$-independent sequence $(b_0, \ldots, b_n) = b$ in $N$ such that $p_i = tp(b_i/M)$ is SR and $N$ is dominated by $b$ over $M$. Let $c$ be a realization of $q$, $N'$ a prime model over $M \cup \{c\}$ and $(d_0, \ldots, d_m) = d \in N'$ an $M$-independent sequence dominating $N'$ over $M$ with $q_i = tp(d_i/M)$ strongly regular. Since $p$ and $q$ are nonorthogonal some $p_i$ must be nonorthogonal to some $q_j$, say $p_0 \not\perp q_0$. By Corollary 6.4.2 $q_0$ is realized by some $d_0'$ in $N$. Replacing $N'$ by some conjugate over $M$ we can assume that $d_0 \in N$. Since $N$ is dominated by $a$ over $M$ and $N'$ is dominated by $c$ over $M$, $a$ and $c$ are dependent over $M$. We conclude that $p \overset{a}{\not\perp} q$ as desired.

The domination relation on types was motivated by the question:

For $M$ an $a$-model, when does realizing $p \in S(M)$ in some $a$-model $N \subset M$ force $q \in S(M)$ to be realized in $N$.

In this generalization from $a$-models to models the natural definition is:

**Definition 6.4.2.** *For $M$ a model and $p, q \in S(M)$ we write $p <_{RK} q$ if $p$ is realized in any model $N \supset M$ which realizes $q$. If $p <_{RK} q$ and $q <_{RK} p$ we write $p \sqsubseteq_{RK} q$ and says that $p$ and $q$ are $RK$*-equivalent.

The relation $<_{RK}$ abbreviates the term *Rudin-Keisler ordering* which is an ordering on ultraproducts. (We will not go into the rationale behind this definition here. See [Las86].) Parsing the definition, in a t.t. theory

$$
\begin{aligned}
p <_{RK} q \quad & \iff \quad p \text{ is realized in the prime model over } M \cup \{b\}, \\
& \qquad\quad \text{where } b \text{ realizes } q \\
& \iff \quad \text{there are } a \text{ realizing } p \text{ and } b \text{ realizing } q \\
& \qquad\quad \text{with } tp(a/M \cup \{b\}) \text{ isolated.}
\end{aligned}
$$

While we did not include the model $M$ in the notation, $<_{RK}$ is only defined on $S(M)$ for a fixed model $M$. For instance, $<_{RK}$ is not invariant under parallelism. (When $p, q \in S(M)$ and $N \supset M$ it is possible that $p$ is not $<_{RK} q$ while $p|N <_{RK} q|N$.) The reason $\triangleleft$ is better behaved than $<_{RK}$ is because domination is equivalent to $a$-isolation over an $a$-model, but not equivalent to isolation (in general).

Certainly, $p <_{RK} q$ implies $p \not\perp q$. In fact

*Remark 6.4.1.* Given a model $M$ and $p, q \in S(M)$, $p <_{RK} q$ implies $p \triangleleft q$. (See Exercise 6.4.5.)

For example, $<_{RK}$ is not a parallelism invariant; the relation is firmly attached to the model $M$. An exception is found when dealing with SR types.

**Lemma 6.4.5.** *Let $M$ be a model.*

*(i) If $p \in S(M)$ is an SR type nonorthogonal to $q \in S(M)$, then $p <_{RK} q$.*

*(ii) Nonorthogonal SR types are RK-equivalent.*

*(iii) An element of $S(M)$ is minimal in the $<_{RK}$ order if and only if it is RK-equivalent to a strongly regular type.*

*Proof.* To prove the first part of the lemma choose $b$ realizing $q$ and $N$ a prime model over $M \cup \{b\}$. The nonorthogonality of $p$ and $q$ forces $p$ to be nonorthogonal to some regular type $r$ nonorthogonal to $q$. In fact, we can take $r$ to be an SR type over $M$ realized in $N$. By Corollary 6.4.2, $p$ is realized in $N$.

(ii) This follows immediately from (i).

Part (iii) of the lemma follows quickly from

*Claim.* An SR type $p \in S(M)$ is $<_{RK}$ –minimal.

Let $a$ realize $p$ (an arbitrary SR type) and $N$ be prime over $M \cup \{a\}$. Let $r \in S(M)$ be a $<_{RK}$ –minimal type realized in $N$. Then, $r$ is nonorthogonal to $p$, hence $p <_{RK} r$ (by the first part of the lemma). By the minimality of $r$, $p$ is $RK$–equivalent to $r$, so $p$ is also $<_{RK}$ –minimal.

This leads to an interesting minimality condition, whose proof is left to Exercise 6.4.4:

*Remark 6.4.2.* Suppose that $M$ is a model, $tp(a/M)$ is SR and $N$ is prime over $M \cup \{a\}$. Then any model $N' \neq M$, $M \subset N' \subset N$, is isomorphic to $N$ over $M$.

As the following example shows $RK-$equivalence and domination equivalence can differ even in rather simple $\omega-$stable theories.

*Example 6.4.1.* The language consists of a binary relation symbol $E$ and a unary function symbol $S$. The axioms for the theory say that $E$ is an equivalence relation with infinitely many infinite classes and no finite classes, $S$ defines a bijection on the universe with no cycles and $\forall v E(v, s(v))$. This theory is complete, quantifier-eliminable and $\omega-$stable. Let $M$ be any model, $a$ an element not $E-$equivalent to any element of $M$ and $b$ such that for all $n$, $b \neq S^n(a)$ and $a \neq S^n(b)$. Let $p = tp(a/M)$ and $q = tp(ab/M)$. It is easy to see that $p \sqsubseteq q$ and $p$ and $q$ are not $RK-$equivalent.

Turning our attention to the dimensions of SR types in models, the following are proved like the corresponding results about regular types and $a-$models. The details are left to the reader.

**Proposition 6.4.3 (Additivity of Dimension).** *Let $M \subset N$ be models, $A \subset M$ a finite set and $p \in S(A)$ a strongly regular type. Then,*

$$\dim(p, N) = \dim(p, M) + \dim(p|M, N).$$

(The property of SR types which corresponds to Lemma 5.4.2 is Lemma 5.1.9. When $(p, \varphi)$ is SR (as in the statement of the proposition) and $I$ is a basis for $p$ in $M$ the strong regularity of the pair implies that $tp(\varphi(M)/A \cup I) \perp p$. The proposition is proved by inserting these changes in the earlier proof.)

**Corollary 6.4.2.** *Let $M$ be a model, $A \subset M$ finite, $p \in S(A)$ a strongly regular type and $I$ a basis for $p$ in $M$. Given $J$ a Morley sequence in $p$ depending on $M$ over $A$, $J$ depends on $I$ over $A$.*

Combining these results, the proof of Proposition 6.3.3 and the fact that nonorthogonal SR types are $RK-$equivalent yields

**Proposition 6.4.4.** *Let $p \in S(A)$ be a strongly regular type where $A$ is finite.*

*(i) If $M \supset A$ is prime over a finite set, $p$ has dimension $\leq \aleph_0$ in $M$. Furthermore, if $I$ is an infinite Morley sequence in $p$ and $M$ is prime over $A \cup I$, $I$ is a basis for $p$ in $M$.*

*(ii) For any model $M \supset A$, if $B \subset M$ is finite and $q \in S(B)$ is a SR type nonorthogonal to $p$, then* $\dim(p, M) + \aleph_0 = \dim(q, M) + \aleph_0$.

*(iii) For any $\kappa \geq |T|$ there is a model $M \supset A$ of cardinality $\kappa$ such that* $\dim(p, M) = \kappa$ *and for any strongly regular type $q$ over a finite subset of $M$,* $q \perp p \implies \dim(q, M) \leq \aleph_0$.

*(iv) Let $M_0$ be a model of cardinality $\lambda$ and $X$ a set of strongly regular types over $M_0$. Then, there is a model $M$ of cardinality $\lambda$ containing $M_0$ such that* $\dim(q, M) = 0$, *for all $q \in X$, and* $\dim(r, M) = \lambda$ *for all strongly regular types $r$ over a finite subset of $M$ such that $r \perp q$ for all $q \in X$.*

(The first part of (i) is obvious since the prime model over a finite set in an $\omega-$stable theory is countable. The result is, however, true more generally in any t.t. theory.)

We close with another lemma which expresses the intuition that the SR types in an $\omega-$stable theory forms a basis for all of the complete types. The proof is assigned as Exercise 6.4.7.

**Lemma 6.4.6.** *Let $T$ be $\omega-$stable and $M$ a countable model of $T$. Suppose that for any SR type $p$ over a finite subset of $M$,* $\dim(p, M)$ *is infinite. Prove that $M$ is saturated.*

**Historical Notes.** Strongly regular types were defined by Shelah in Definition 3.6 of [She90, V], although he does not require the type to be stationary. Lemma 6.4.1 is found in Theorem 3.18 of that chapter, as is Corollary 6.4.2. The results 6.4.1 through 6.4.5 are by Lascar [Las82], as is the Rudin-Keisler order. Proposition 6.4.3 is stated by Bouscaren and Lascar explicitly in [BL83, 4.2]. Analogues of Proposition 6.4.4 can be found in [Mak84], where Makkai attributes them to handwritten notes by Shelah on his proof of Vaught's conjecture for $\omega-$stable theories.

**Exercise 6.4.1.** Prove that a stationary type of least Morley rank which is nonorthogonal to a given type $p$ is strongly regular.

**Exercise 6.4.2.** Prove Corollary 6.4.1.

**Exercise 6.4.3.** Prove the claim in the proof of Proposition 6.4.1.

**Exercise 6.4.4.** Prove: Suppose that $M$ is a model, $tp(a/M)$ is SR and $N$ is prime over $M \cup \{a\}$. Then any model $N' \neq M$, $M \subset N' \subset N$, is isomorphic to $N$ over $M$.

**Exercise 6.4.5.** Prove Remark 6.4.1.

**Exercise 6.4.6.** Suppose that $p = tp(a/M)$ and $N$ is a prime model over $M \cup \{a\}$. Let $C = \{c_0, \ldots, c_n\} \subset N$ be a maximal $M-$independent set of realizations of SR types over $M$ and let $q_i = tp(c_i/M)$, for $i \leq n$. Show that $p \sqsubseteq q_0 \otimes \ldots \otimes q_n$. (Hence, the $q_i$'s are a regular decomposition of $p$.)

**Exercise 6.4.7.** Prove Lemma 6.4.6.