VI. Preservation of Additional Properties, and Applications

This chapter contains results from three levels of generality: some are specific consistency results; some are preservation theorems for properties like "properness + ${}^{\omega}\omega$-bounding", and some are general preservation theorems, with the intention that the reader will be able to plug in suitable parameters to get the preservation theorem he needs. We do not deal here with "not adding reals" - we shall return to it later (in VIII §4 and XVIII §1,§2).

Results of the first kind appear in 3.23, §4, §5, §6, §7, §8. In §4 we prove the consistency of "there is no P-point (a kind of ultrafilter on ω)". We do this by CS iteration, each time destroying one P-point; but why can't the filter be completed later to a P-point? (If we add enough Cohen reals it will be possible.) For this we use the preservation of a property stronger than ${}^{\omega}\omega$-bounding, enjoyed by each iterand.

More delicate is the result of §5 "there is a Ramsey ultrafilter (on ω) but it is unique, moreover any P-point is above it" (continued in XVIII §4). Here we need in addition to preserve "D continues to generate an ultrafilter in each V^{P_α}".

In 3.23 we prove the consistency of $\mathfrak{s} > \mathfrak{b} = \aleph_1$; i.e. for every subalgebra $\mathbb{B}$ of $\mathcal{P}(\omega)$/finite of cardinality $\aleph_1$, there is $A \subseteq \omega$ which induce on $\mathbb{B}$ an ultrafilter $\{B/\text{finite}: B \in \mathbb{B} \text{ and } A \subseteq^* B\}$; but there is $F \subseteq {}^{\omega}\omega, |F| = \aleph_1$ with no $g \in {}^{\omega}\omega$ dominating every $f \in F$. We use a forcing Q providing a "witness" A for $\mathbb{B} = (\mathcal{P}(\omega)/\text{finite})^V$; not adding g dominating $({}^{\omega}\omega)^V$; we iterate it (CS). After ω_2 steps the first property is O.K., but we need a preservation lemma to show the second is preserved. The definition of this Q and the proof of its relevant properties are delayed to §6. In §7 (i.e. 7.1) we prove the consistency of $\mathfrak{a} > \mathfrak{b}$. Lastly in §8 (i.e. in 8.2) we prove the consistency of $\mathfrak{h} < \mathfrak{b} = \mathfrak{a}$. On history concerning §6, §7, §8 see introduction to §6. See relevant references in the section.

* * *

We now review most of the preservation theorems appearing here for countable support iteration of proper forcing; actually this is done for more general iterations (including RCS, a pure finite/pure countable, FS-finite support), see 0.1 and we can weaken "proper". You can read it being interested only in CS iteration of proper forcing, ignoring all adjectives "pure" and the properties "has pure (θ_1, θ_2)-decidability" (or feeble pure (θ_1, θ_2)-decidability), so letting $\leq_{\text{pr}} = \leq$.

0.A Theorem. For any CS iteration $\langle P_i, \underset{\sim}{Q}_j : i \leq \delta, j < \delta \rangle$ if for each $i < \alpha$ we have $\Vdash_{P_i}$ "Q_i satisfies X" *then* P_δ satisfies X; for each of the following cases:

1) $X =$ "Q is proper and ${}^\omega\omega$-bounding" [Why? By 2.8D, i.e. by 2.3 + 2.8B + 2.8C].
2) Let $f, g : \omega \to \omega + 1 \setminus \{0, 1\}$ be functions diverging to infinity [i.e. $(\forall n < \omega)(\exists k < \omega)(\forall m)(k < m < \omega \Rightarrow f(m) > n \,\&\, g(m) > n)$] and: $X =$ "Q is proper and *for every* $\ell < \omega$ and $\eta \in (\prod_n f(n)^{[g(n)^\ell]})^{V^Q}$ there is a sequence $\langle u_n : n < \omega \rangle \in V$ such that $\bigwedge_n \eta(n) \in u_n$ and $|u_n| > 1 \Rightarrow |u_n| \leq g(n)^{1/\ell}$. [Why? By 2.11F.]
3) $X =$ "Q proper and every dense open $A \subseteq {}^{\omega>}\omega$ includes an old such set". [Why? See 2.15D; or see 2.15B(2) for an equivalent formulation, then by 2.15C, 2.3(5) we can apply 2.3(2)].

Remark. Particular cases of 0.A(2) are the Sacks property (f constantly ω, all g's), and the Laver property (f, g vary on all legal members of ${}^\omega\omega$), the names were chosen for the most natural forcing notions with these properties. Other pairs $f, g \in {}^\omega\omega$ were introduced in and important for [Sh:326 §2]. Concerning the PP-property and the strong PP-property see 2.12, 3.25-6.

For some other properties we can prove that in limit stages, violation does not arise; but leave to the specific iteration the burden for the successor stages. We say "X is preserved in limit".

0.B Theorem. For CS iteration of proper forcing, $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leq \delta, \beta < \delta \rangle$, δ a limit ordinal.

1) If for $\alpha < \delta$, in V^{P_α} there is no new $f \in {}^\omega\omega$ dominating all $h \in ({}^\omega\omega)^V$ *then* this holds for V^{P_δ} [see 3.17(1)],
2) If for $\alpha < \delta$, in V^{P_α} there is no new $f \in {}^\omega\omega$ dominating all $h \in ({}^\omega\omega)^V$ and no real which is Cohen over V *then* this holds for V^{P_δ} [see 2.13D(2); more on Cohen see 2.17].
3) If for $\alpha < \delta$ in V^{P_α} there is no random real over V *then* this holds for V^{P_δ} [see 3.18].

* * *

We now turn to the third kind of results.

In §1 we present a general context suitable for something like: for every $\eta \in ({}^\omega\omega)^{V^Q}$ there is a "small" tree $T \subseteq {}^{\omega>}\omega$ from V such that $\eta \in \lim(T)$; so we assume that the family of small trees has some closure properties. In 2.1 - 2.7 we more specify our context, so that we can get preservation in successor stages too. In 1.16, 1.17 we deal with a generalization where we have several kinds of $\eta \in {}^\omega\omega$ (but for simplifying the presentation, we restrict generality in other directions). A reader who feels our level of generality is too high (or goes over to this view while reading 2.1-2.8) can prefer a simplified version (which is [Sh:326, A2 pp 387-399]), so read only 1.16, 1.17 for the case $k^* = 1$ and then look at any of 2.9 - 2.17 (each dedicated to a specific property being preserved) ignoring the undefined notions.

In 3.1 - 3.13 we give another context (tailored for "there is no dominating reals"). Here for successor stages we use a stronger property (like almost ${}^\omega\omega$-bounding). In XVIII §3 we give another such general theorem.

The reader is tuned now to countable support iteration of proper forcing but we shall later consider other contexts (semiproperness in Chapter X; forcing with additional "partial order $\leq_{\rm pr}$" (pr for pure) plus some substitute of properness in Chapter XIV, XV). To save repetition, in 0.1 below we describe the various contexts. The subscript θ has a role only when $\leq_{\mathrm{pr}}$ is present (cases D-F below) and its meaning is described in 0.1(3). Note that also FS iteration of c.c.c. forcing is a particular case: $\leq_{\mathrm{pr}}$ is equality and $\theta = \aleph_1$ (the relevant results will be presented in §3). Let θ missing mean $\theta \equiv 1$. We may write e.g. $0.1_{\theta=\aleph_0}$ rather than $0.1_{\aleph_0}$ to stress this.

0.1_θ *Iteration Context*:

1) We shall use iteration $\bar{Q} = \langle P_j, \underset{\sim}{Q}_i : j \leq \alpha, i < \alpha \rangle$ of one of the following forms:

   (A) Countable support iteration of proper forcing (see III). In this case $\leq_{\mathrm{pr}}$ is the usual order, 1.11 is just III 1.7; "purely" can be omitted; similarly for (B) (C).

   (B) Like (1) but for $\delta < \alpha$ limit we weaken "$\underset{\sim}{Q}_\delta$ is proper" to "for arbitrarily large $i < \delta, P_{\delta+1}/P_{i+1}$ is proper or even just E-proper" where $E \subseteq \mathcal{S}_{\leq\aleph_0}(\mu)$ is a fixed stationary set (we can use similar variants of the other cases).

   (C) RCS iteration which is a semiproper iteration (see Chapter X).

   (D) Each forcing notion $\underset{\sim}{Q}_i$ has also a partial order $\leq_{\mathrm{pr}}$, $[p \leq_{\mathrm{pr}} q \Rightarrow p \leq q]$; a minimal element $\emptyset_Q$ and is purely proper (i.e. *if* $p \in Q \cap N, Q \in N$, N countable and $N \prec (H(\chi), \in, <^*_\chi)$, *then* there is a (N,Q)-generic $q, p \leq_{\mathrm{pr}} q \in P$). The iteration is defined as $P_i = \{p : p$ a function with domain a countable subset of i, for $j \in \mathrm{Dom}(p)$ we have: $\Vdash_{P_j}$ "$p(j) \in Q_j$" and $\{j :$ not $\Vdash_{P_j}$ "$\emptyset_{Q_j} \leq_{\mathrm{pr}} p(j)$" $\}$ is finite$\}$.

   A particular case is FS iteration of c.c.c forcing. This (i.e. clause (D)) is a particular case of Chapter XIV.

   (E) The iterations $\overline{Q}$ which are GRCS as in XV §1 (and see 0.3), such that: for each $\alpha < \ell g(\overline{Q})$ for some n we have $\Vdash_{P_{\alpha+n}}$ "$(2^{\aleph_1}) + |P_\alpha|$ is collapsed to $\aleph_1$" and each $\underset{\sim}{Q}_\alpha$ is purely semiproper.

   (F) The GRCS iterations as in XV §3 (so each Q_i satisfies $UP(\mathbb{I}, \mathbf{W})$, where $\mathbf{W} \subseteq \omega_1$ is stationary.

(G) The GRCS iteration as in XV §4.

2) We say "P purely adds no f such that $(\forall x \in V)\varphi(x, f)$" *if* for every $p \in P$ and P-name $\underset{\sim}{f}$, for some $q \in P$ and $x \in V$: $p \le_{\text{pr}} q$ and $q \Vdash$ "$\underset{\sim}{f}$ does not satisfy $\varphi(x, \underset{\sim}{f})$".

3) $\theta \in \{1, 2, \aleph_0, \aleph_1\}$ and: $\theta = 1$ means no demand, $\theta \geq \aleph_0$ means each Q_α (or each $P_\alpha, P_\alpha/P_{\beta+1}$) has pure $(\theta, 2)$-decidability (see Definition 1.9) and $\theta = 2$ means they have pure $(2, 2)$-decidability (see Definition 1.9).

Remark. We shall concentrate on case F in 0.1(1) as it is the hardest.

0.2 Definition.

1) We say W is absolute if it is a *definition* (possibly with parameters) of a set so that if $V^1 \subseteq V^2$ are extensions of V (but still models of ZFC with the same ordinals) and $x \in V^1$ then: $V^2 \models$ "$x \in W$" iff $V^1 \models$ "$x \in W$". Note that a relation is a particular case of a set. It is well known that Π^1_2 relations on reals and generally κ-Souslin relations are absolute.

2) We say that a player absolutely wins a game if the definition of legal move, the outcomes and the strategy (which need not be a function with a unique outcome) are absolute and its being a winning strategy is preserved by extensions of V.

3) We can relativize absoluteness to a family of extensions, e.g. for a given universe V and family K of forcing notions we can look only at $\{V^Q : Q \in K\}$; so for V^{Q_0} we consider only the extensions $\{V^Q : Q_0 \lessdot Q \in K\}$, or even demand Q/Q_0 has a specified property. We do not care to state this all the time.

Though Case D is covered by Chapter XIV, (and XV) we may note:

0.3 Theorem. 1) The iteration in case (D) preserves "purely proper".

2) X§2 is generalized to "purely semiproper is preserved" by GRCS iterations.

§1. A General Preservation Theorem

An important part of many independence proofs using iterated forcing, is to show that some property X is preserved (if satisfied by each iterand). We have dealt with such problems in Chapter V (preserving e.g. "ω-properness + the ${}^{\omega}\omega$-bounding property"), [Sh:b] Chapter VI (general context and many examples), [Sh:207], [Sh:177] (replacing the weak form of ω-proper by proper), Blass and Shelah [BsSh:242] (preserving ultrafilters which are P-points), [Sh:326]; in [Sh:b] Chapter X §7 we have dealt with semiproperness. Here we redo [Sh:b] Chapter VI §1, giving a general context which serves for many examples replacing proper by the weaker condition semiproper and even UP and "CS iteration" by "GRCS iterations" i.e. revised countable/finite support with purity (and correcting it). You may read this section replacing everywhere: UP by proper, RCS iteration by countable support iteration, $\leq_{\rm pr}$ by the usual order, $\mathbf{S}$ by the class of regular cardinals, $\mathbf{W} = \omega_1$, semi-generic by generic, omit $\mathbb{I}$-suitable, then 1.9, 1.10 are not necessary.

In fact there is more in common between the examples discussed later even than expressed by the stricter context suggested here (fine covering model) (i.e., the use of trees $T, T\cap {}^{n}\omega$ finite and absoluteness in the definitions of covering models) but the saving will not be so large; we shall return to this in §2.

Unfortunately "adding no reals" will require special treatment (as is the case even if we assume properness). We have dealt with it separately in Chapter V and will return to it in VIII §4, XVIII §1, §2.

For applications it suffices to read Definitions 1.1 - 1.5 (the fine covering models and preservation of them); also 1.9 and Theorem 1.12 (on more general preservation theorems). Another general way to get such preservation theorems is presented in XVIII §3. A simpler version of the theorem is presented in 1.16, 1.17 here (and see 1.3(10); earlier see [Sh:326, Appendix A2 pp. 387-399] (but also for a finite sequence of covering models)).

1.1 Definition. We call (D, R) a *weak covering model* (in V) if:

a) D a set, R a two place relation on D, xRT implies that T is a closed subtree of ${}^{\omega>}\omega$ (i.e., $\langle\rangle \in T$, T is closed under initial segments, and above any $\eta \in T$ there are arbitrarily long members of T),

b) (D, R) covers, i.e. for every $\eta \in {}^{\omega}\omega$ and $x \in \mathrm{Dom}(R)(= \{x : (\exists T)xRT\})$ there is $T \in D$ such that xRT and $\eta \in \lim T$, where

$$\lim T = \{\eta \in {}^{\omega}\omega : \eta \restriction k \in T \text{ for every } k < \omega\}$$

1.1A Remark. The intuitive meaning is: xRT means T is a closed tree of "size" at most x. In Definition 1.2, which exploits more of our intuition, we have an order on the set of possible x's, $x \leq y$, with the intuitive meaning "x is a smaller size than y". So it would be natural to demand:

$$xRT, x < y \Rightarrow yRT \text{ and } xRT, T^{\dagger} \subseteq T \Rightarrow xRT^{\dagger}$$

However, no need arises. Note also that sometimes x appears trivially (e.g. see the ${}^{\omega}\omega$-bounding model in 2.8).

1.2 Definition. (1) A *fine covering model* is $(D, R, <)$ such that:

(α) (D, R) is a weak covering model

(β) $<$ is a partial order on $\mathrm{Dom}(R)$, such that

(i) $(\forall y \in \mathrm{Dom}(R))(\exists x \in \mathrm{Dom}(R))(x < y)$

(ii) $(\forall y, x \in \mathrm{Dom}(R))(\exists z \in \mathrm{Dom}(R))(x < y \rightarrow x < z < y)$

(iii) if $y < x, yRT$ then for some $T^* \in D$, $T \subseteq T^*$ and xRT^*

(iv) if $y < x$ and for $l = 1, 2$ yRT_l *then* there is $T \in D$ such that: xRT, $T_1 \subseteq T$ and for some n, $[\nu \in T_2 \;\&\; \nu \restriction n \in T_1 \Rightarrow \nu \in T]$

(γ) (a) If $x > x^{\dagger} > y_{n+1} > y_n$ for $n < \omega$ and $T_n \in D, y_n RT_n$ (for $n < \omega$) *then* there is $T^* \in D, xRT^*$ and an infinite set $w \subseteq \omega$ such that:

$$\lim T^* \supseteq \{\eta : \eta \text{ is in } {}^{\omega}\omega \text{ and for every } i \in w, \eta \restriction \min(w \setminus (i+1)) \in \bigcup_{\substack{j<i \\ j \in w}} T_j \cup T_0\}$$

(b) if $\eta, \eta_n \in {}^\omega\omega$, $\eta{\restriction}n = \eta_n{\restriction}n$ for each $n < \omega$ and $x \in \mathrm{Dom}(R)$ *then* for some $T \in D$, xRT, $\eta \in \lim T$ and $\eta_n \in \lim T$ for infinitely many n.

(δ) condition (γ) continues to hold in any generic extension in which (α) holds.

(2) For a property X of forcing notions, $(D, R, <)$ is a fine covering model for X-forcing if Definition 1.2(1) holds when we restrict ourselves in (δ) to X-forcing notions only.

(3) We say $(D, R, <)$ is a temporarily fine covering model if it satisfies $(\alpha), (\beta), (\gamma)$ i.e. is a fine covering model for trivial forcing.

1.3 Remark. 1) In an abuse of notation we do not always distinguish between $(D, R, <)$ and (D, R).

2) Look carefully at (δ), it is in a sense, meta-mathematical.

3) So if $(D, R, <)$ is a fine covering model and P is a (D, R)-preserving forcing notion (see Definition 1.5 below) *then* in V^P the model $(D, R, <)$ is still a fine covering model. [Why? In Definition 1.2(1) clause (α) holds as P is (D, R)-preserving, clause (β) holds as it is absolute, clause (γ) holds as in V, $(D, R, <)$ is a fine covering model by clause (δ) of Definition 1.2(1) and clause (δ) by its transitive nature.]

4) In $(\gamma)(a)$ of 1.2(1), we can replace "$y_n RT_n$" by $x^\dagger RT_n$ (by (β) (ii) (iii)).

5) We write in 1.2(1)(β) (iv)$^+$ if $n = 0$.

6) If we assume 1.2(1)(β)(iv)$^+$, then in 1.2(1)(γ)(a) w.l.o.g. $T_n \subseteq T_{n+1}$ hence the conclusion in $(\gamma)(a)$ is:

$$\lim T^* \supseteq \{\eta \in {}^\omega\omega : \text{ for every } i \in w,\ \eta \restriction i \in T_{\max[(w \cap i) \cup \{0\}]}\}.$$

7) We can in (γ) add "and $0 \in w$".

8) A condition stronger than $(\gamma) = (\gamma)_0$ of 1.2(1) is:

$(\gamma)_1 = (\gamma)^+$ if $x > x^\dagger > y_{n+1} > y_n$ for $n < \omega$ and $T_n \in D, y_n RT_n$ (for $n < \omega$) *then* there is $T^* \in D, xRT^*$ and an infinite set $w \subseteq \omega$ such that:

$$\lim T^* \supseteq \{\eta : \eta \text{ is in } {}^\omega\omega \text{ and for every } i \in w, \eta \restriction i \in \bigcup_{\substack{j \le i \\ j \in w}} T_j\}$$

(I.e. it implies both (a) and (b) of $1.2(\gamma)$ (when (D, R) covers, of course).)
If we assume $(\beta)(\mathrm{iv})^+$, then in $1.2(1)(\gamma)$ w.l.o.g. $T_n \subseteq T_{n+1}$ hence the demand in $(\gamma)^+$ is $\lim T^* \supseteq \{\eta \in {}^\omega\omega : \text{for every } i \in w,\ \eta \restriction i \in T_i\}$.
Why? Let y'_n be: $y'_0 = y_0$, $y'_{n+1} = y_{n+2}$. We choose by induction on n, T'_n such that $y_n R T'_n$ and $T'_0 = T_0$, and $T'_n \subseteq T'_{n+1}$ and for some k_n we have $\eta \restriction k_n \in T'_n$ & $\eta \in \bigcup_{m \le n+1} T_m \Rightarrow \eta \in T'_{n+1}$. Now by clause $(\gamma)^+$ there are an infinite $w' \subseteq w$ and T^* such that xRT^* and $\lim(T^*) \supseteq \{\eta \in {}^\omega\omega :$ for every $i \in w'$ we have $\eta \restriction i \in \bigcup_{\substack{j \le i \\ j \in w}} T'_j\}$. Let $w' = \{n_i : i < \omega\}$ with $n_i < n_{i+1}$. Let $j(\ell)$ $(\ell < \omega)$ be increasing fast enough, i.e. $n_{j(\ell+1)} > k_{n_{j(\ell)-1}}$, $w \stackrel{\text{def}}{=} \{n_{j(\ell)} : \ell < w\}$. It is enough to prove that w and T^* are as required in clause (γ). So assume $\eta \in {}^\omega\omega$ belongs to the set on the right hand side of the inclusion in clause (γ), and we shall prove $\eta \in \lim T^*$. So we are assuming that for every $\ell < \omega$ we have $\eta \restriction n_{i(\ell+1)} \in \bigcup_{j < \ell} T_{n_j} \cup T_0$. So it is enough to prove that η appears in the right side of the inclusion in (γ) for w', $\langle T'_i : i < \omega\rangle$. So let $i < \omega$ and we should prove that $\eta \restriction n_i \in \bigcup_{\ell \le i} T'_{n_\ell}$ (as $w' = \{n_i : i < \omega\}$, n_i increasing with i). Let ℓ be such that $j(\ell) \le i < j(\ell+1)$, so by the assumption on η we have $\eta \restriction n_i \lhd \eta \restriction n_{j(\ell+1)} \in \bigcup_{m < \ell} T_{j(m)} \cup T_0$. We prove this by induction on i.

Case 1: $\eta \restriction n_{j(\ell+1)} \in T_0$

So $\eta \restriction n_i \in T_0$, but $T - 0 = T'_0 \subseteq T'_{n(i)}$ hence $\eta \restriction n_i \in T_{n(i)} \subseteq \bigcup_{\ell \le i} T'_{n(\ell)}$ as required.

Case 2: There is $m < \ell$ such that $\eta \restriction n_{j(\ell+1)} \in T_{n_{j(m)}}$

Necessarily $i \ge j(\ell) > j(m)$ so by the induction hypothesis on i we have $\eta \restriction n_{j(\ell)-1} \in \bigcup_{k \le j(\ell)-1} T'_{n_k}$ but $T'_n \subseteq T'_{n+1}$ so $\eta \restriction n_{j(\ell)-1} \in T'_{n_{j(\ell)-1}}$ as by assumption $\eta \restriction n_{j(\ell+1)} \in T_{n_{j(m)}}$, $m < \ell$, by the choice of $T'_{n_{j(\ell)}}$ as $j(\ell+1) > k_{n_{j(\ell)-1}}$ necessarily $\eta \restriction n_{j(\ell+1)} \in T'_{n_{j(\ell)}}$ but $T'_{n_{j(\ell)}} \subseteq T'_{n_i}$ and $n_i \le n_{j(\ell+1)}$ hence $\eta \restriction n_i \in T'_{n_i} \le \bigcup_{\ell \le i} T'_{n_\ell}$ as required.

8A) In clause (γ) w.l.o.g. $T_n \subseteq T_{n+1}$ (i.e. this weaker version implies the original version using (α), (β) of course).
[Why? By 2.4D (note 2.4A, 2.4B, 2.4C, 2.4D do not depend on the intermediate material).]

9) Note in 1.2(1)(γ)(a), that any infinite $w' \subseteq w$ is o.k.

10) In some circumstances clause (b) of 1.2(1)(γ) is a too strong demand, e.g. preservation of P-points. We can overcome this by letting $R = \bigvee_{\ell<k} R_\ell$ (k is finite) and demanding 1.2(γ)(a) for each R_ℓ whereas instead 1.2(γ)(b) we demand

(b)$'$ if $\eta_n, \eta \in {}^\omega\omega$, $\eta_n \restriction n = \eta \restriction n$ for $n < \omega$, and for some $m < k$ we have

$$(\forall x \in \mathrm{Dom}(R_m))(\exists T)(x R_m T \ \& \ \eta \in \lim T) \quad \text{and}$$

$$(\forall x \in \mathrm{Dom}(R_m)) \bigwedge_n (\exists T)(x R_m T \ \& \ \eta_n \in \lim T)$$

then for every $x \in \mathrm{Dom}(R_m)$ for some T we have: $x R_m T$ and for infinitely many $n < \omega$, $\eta_n \in \lim(T)$.

See more on this in 1.16, 1.17 and §5.

Proof. E.g.

9) Assume $\eta \in {}^\omega\omega$ and

$(*)_0$ $i \in w' \Rightarrow \eta \restriction \min(w' \setminus (i+1)) \in \bigcup_{j \in w', j<i} T_j \cup T_0$;

we have to prove $\eta \in \lim(T^*)$. For this it suffices to prove:

$(*)_1$ $i \in w \Rightarrow \eta \restriction (\min(w \setminus i+1)) \in \bigcup_{j \in w, j<i} T_j \cup T_0$.

Let $i \in w$, define $i_1 = i$, $j_1 = \min(w' \setminus i)$, $i_2 = \min(w \setminus (i_1+1))$, $j_2 = \min(w' \setminus (j_1+1))$; so in particular $i_1 \leq j_1 \in w'$, $i_1 < i_2 \leq j_2$, $j_1 < j_2$. As $(*)_0$ holds apply it to j_1 and get $\eta \restriction j_2 \in \bigcup_{j \in w', j<j_1} T_j \cup T_0$, hence for some j_0, $j_0 = 0 \ \vee \ (j_0 < j_1 \ \& \ j_0 \in w')$ and we have $\eta \restriction j_2 \in T_{j_0}$. As $i_2 \leq j_2$ clearly $\eta \restriction i_2 \in T_{j_0}$. As $j_1 = \min(w' \setminus i_1)$ we know that $i_1 \leq j_1$ and $[i_1, j_1) \cap w' = \emptyset$ and thus $j_0 = 0 \vee \ (j_0 < i_1 \ \& \ j_0 \in w')$. Hence $j_0 = 0 \ \vee \ (j_0 < i_1 \ \& \ j_0 \in w)$. So $\eta \restriction i_2 \in T_{j_0} \subseteq \bigcup_{j \in w, j<i} T_j \cup T_0$, as required. (See more 2.4D.) $\square_{1.3}$

1.4 Convention. If the order $<$ is not specified then $< = <_{\mathrm{dis}}$ (see below). Let $<_0$ be such that:

$$x <_0 y \ \text{ iff } \ x, y \in {}^\omega\omega \ \& \ x(0) <'_0 y(0)$$

where $(\omega, <'_0)$ is isomorphic to $(\mathbb{Q}, <)$ (i.e. the rationals). Let $<_{\rm dis}$ be:

$$x <_{\rm dis} y \textit{ iff } x, y \in {}^\omega\omega, 1 \le x(n) \le y(n) \text{ for every } n$$
$$\text{and } y(n)/x(n), x(n) \text{ diverge to } \infty.$$

Let $<^*_{\rm dis}$ be: $x <^*_{\rm dis} y$ iff $y <_{\rm dis} x$. (Note: in $\mathrm{DP}({}^\omega\omega) \stackrel{\rm def}{=} \{x \in {}^\omega\omega : x(n) \ge 1$, $\langle x(n) : n < \omega\rangle$ diverges to infinity$\}$, $<_0$, $<_{\rm dis}$ and $<^*_{\rm dis}$ satisfy clauses (β)(i), (ii), (iii) of Definition 1.2(1)).

1.5 Definition. Let (D, R) be a weak covering model. We say that a forcing notion P *preserves* (D, R) or is (D, R)*-preserving* if $\Vdash_P$ "(D, R) is a weak covering model". We add "purely" if: for every $p \in P$ and $\underset{\sim}{f}$ such that $p \Vdash$ "$\underset{\sim}{f} \in {}^\omega\omega$" and $x \in \mathrm{Dom}(R)$, for some q, T we have $p \le_{\rm pr} q \in P, xRT$ and $q \Vdash$ "$\underset{\sim}{f} \in \lim T$ ".

1.6 Definition. 1) For a weak covering model (D, R) and $y \in \mathrm{Dom}(R)$, $(D, R) \models$ "$\varphi_{\rm dis}(y)$" if:

for every $\eta^* \in {}^\omega\omega$ and function F from $D \times \omega$ to $\mathrm{Rang}(R) = \{T \colon (\exists x \in D) xRT\}$ such that $(\forall n)(\forall z \in \mathrm{Dom}(R))[zRF(z, n)$ and $\eta^* {\restriction} n \in F(z, n)]$.
there are T^*, yRT^* and an infinite set w of natural numbers, and $z_\ell \in \mathrm{Dom}(R)$ for $\ell \in w$ such that:

$$T^* \supseteq \{\eta \in {}^{\omega>}\omega : \text{there is } \ell \in w \text{ such that } \eta \restriction \ell \lhd \eta^*, \text{ and } \eta \in F(z_\ell, \ell)\}.$$

Note that the truth value of $(D, R) \models$ "$\varphi_{\rm dis}(y)$" depends on V (remember $\lhd$ means initial segment).

1A) For a weak covering model (D, R) and $y \in \mathrm{Dom}(R)$ we write $(D, R) \models$ "$\varphi^*_{\rm dis}(y)$" if:
for every $\eta^* \in {}^\omega\omega$, and a function $F : D \times \omega \to \mathrm{Rang}(R)\}$ such that $(\forall n)(\forall z \in \mathrm{Dom}(R)) zRF(z, n)$ and a function H from $D \times \omega$ into ${}^\omega\omega$ such that $\eta^* \restriction n \lhd H(z, n) \in \lim F(z, n)$ (so $\eta^* \restriction n \in F(z, n)$)
there are T^*, yRT^* and an infinite $w \subseteq \omega$ and $n_\ell < \omega$, and $z_\ell \in \mathrm{Dom}(R)$ for

$\ell \in w$ such that:

$$T^* \supseteq \{\eta \in {}^{\omega>}\omega : \text{there is } \ell \in w \text{ such that } \eta \restriction \ell \lhd \eta^*, \\ \text{and } \eta \restriction n_\ell \lhd H(z_\ell, \ell) \text{ and } \eta \in F(z_\ell, \ell)\}.$$

(If $n_\ell \geq \ell$, then "$\eta \restriction \ell \lhd \eta^*$" is not necessary and if $n_\ell \leq \ell$, then $\eta \restriction n_\ell \lhd H(z_\ell, \ell)$ is not necessary). Note that the truth value of $(D, R) \models \varphi^*_{\text{dis}}(y)$ depends on V and $\varphi_{\text{dis}}(y) \Rightarrow \varphi^*_{\text{dis}}(y)$ (as in φ^*_{dis} the set of $\eta \in {}^{\omega>}\omega$ which we demand to be in T^* is smaller than for φ_{dis}).

2) We call (D, R) a *covering model* if it is a weak covering model and

(c) for every $y \in \mathrm{Dom}(R)$, $(D, R) \models \varphi_{\text{dis}}(y)$ or at least $(D, R) \models \varphi^*_{\text{dis}}(y)$.

3) For a weak covering model (D, R) and $\bar{x} = \langle x_n : n < \omega \rangle$ and z, where $\{x_n : n < \omega\} \cup \{z\} \subseteq \mathrm{Dom}(R)$ we say that $(D, R) \models \psi_{\text{dis}}(\bar{x}, z)$ if:

$(*)$ *for every* $\eta^* \in {}^{\omega}\omega$ and a set $\{T_{n,j} : n, j < \omega\}$ such that $x_n R T_{n,j}$ for $n, j < \omega$ *there are* $\langle T^\alpha : \alpha \leq \omega \rangle$ such that:

(i) $T^n \subseteq T^{n+1}$ and $T^0 \subseteq T^\omega$

(ii) zRT^ω (so $T^\omega \in D$)

(iii) $\eta^* \in \lim T^0$

(iv) if $n, j < \omega$ and $\nu \in (\lim T_{n,j}) \cap (\lim T^n) \cap (\lim T^\omega)$, *then* for some k:

$$(\forall \rho)[\nu \restriction k \trianglelefteq \rho \in T_{n,j} \Rightarrow \rho \in T^{n+1} \cap T^\omega]$$

4) (D, R) is a strong covering model if it is a covering model and

(d) For every $z \in \mathrm{Dom}(R)$ there are $x_n (n < \omega)$ such that:

$$(D, R) \models \psi_{\text{dis}}(\langle x_0, x_1, \ldots \rangle, z)$$

1.7 Definition. 1) Let K be a property of weak covering models. We say that a forcing notion P is K-preserving if:

for any $(D, R) \in V$ satisfying K, P preserves (D, R). We add "purely" if for any (D, R) satisfying K, P purely preserves (D, R).

2) We call a covering model (D, R) *smooth* if:

for any (D, R)-preserving forcing notion P, $\Vdash_P$ "(D, R) is a covering model".

3) We call a strong covering model (D, R) *strongly smooth* if:
for any (D, R)-preserving forcing notion P we have $\Vdash_P$ "(D, R) is a strong covering model".

1.8 Claim. 1) If $(D, R, <)$ is a fine covering model *then* (D, R) is a strongly smooth strong covering model.
2) The following is a sufficient condition for $(D, R) \models \psi_{\text{dis}}(\langle x_n : n < \omega\rangle, z)$:
$(*)$ for some $y_n \in \text{Dom}(R)$ (for $n < \omega$) :
$(a)_n$ *if* $n < \omega$, $x_n R T_j$ for $j < \omega$ and $y_n R T$ *then* for some $\langle n_j : j < \omega\rangle$ and T^*:

(i) $y_{n+1} R T^*$

(ii) $T \subseteq T^*$

(iii) $\eta \in T_j$ & $\eta \restriction n_j \in T \Rightarrow \eta \in T^*$

(b) if $y_n R T^n$ for $n < \omega, T^n \subseteq T^{n+1}$ *then* for some T^*, zRT^* and $\bigwedge_n T^n \subseteq T^*$.
3) For a weak covering model (D, R) we have: if (D, R) is a strong covering model then it is a covering model, and strongly smooth implies smooth.

Proof. 1) By 1.2(1)(α) we have: (D, R) is a weak covering model. Now we show that it is a strong covering model. So by 1.6(2), (4) we have to check conditions (c), (d) of Definition 1.6.
Proof of (c) We are going to prove that for $y \in \text{Dom}(R)$ we have $(D, R) \models \varphi^*_{\text{dis}}(y)$.
So suppose $\eta^* \in {}^\omega\omega$, F is a function from $D \times \omega$ to $\text{Rang}(R)$, and H is a function from $D \times \omega$ to ${}^\omega\omega$ such that:

$$(\forall n)(\forall x \in \text{Dom}(R))[xRF(x, n) \ \& \ \eta^* \restriction n \lhd H(x, n) \in \lim F(x, n)]$$

First we use a stronger assumption.

Proof of (c) assuming $(\gamma)^+$ *of 1.3(8)*: So there exist, by (β)(i), (ii) of Definition 1.2, $y^\dagger$, x_n (for $n < \omega$) such that $y > y^\dagger > x_{n+1} > x_n > \ldots > x_0$ (choose $y^\dagger$ and then, inductively on n, x_n). Let $z_\ell \stackrel{\text{def}}{=} x_\ell$ and $n_\ell \stackrel{\text{def}}{=} \ell$ and let $T_n \stackrel{\text{def}}{=} F(x_n, n)$. Apply condition $(\gamma)^+$ of Definition 1.2(1) (i.e. 1.3(8)) to get T^* and an infinite

$w \subseteq \omega$ such that $\lim T^* \supseteq \{\eta \in {}^\omega\omega : \text{for every } i \in w, \eta \restriction i \in \bigcup_{\substack{j \in w \\ j \le i}} T_j\}$ and yRT^*. Remember $n_\ell \ge \ell$.

We shall show that T^* and w and $\langle n_\ell : \ell < \omega \rangle$ are as required in 1.6(1A). We have to prove that (for each $\ell \in w$ and $\eta \in {}^{\omega >}\omega$):

$(*)$ $\eta \restriction n_\ell \trianglelefteq H(z_\ell, \ell)$ & $\eta \in F(z_\ell, \ell) \Rightarrow \eta \in T^*$.

(Note: $\eta \restriction \ell \triangleleft \eta^*$ follows from $\eta \restriction \ell = \eta \restriction n_\ell \trianglelefteq H(z_\ell, \ell)$ because $\eta^* \restriction \ell \triangleleft H(z_\ell, \ell)$ by the assumptions on H in 1.6(1A).)

So assume $\eta \restriction n_\ell \trianglelefteq H(z_\ell, \ell)$ and $\eta \in F(z_\ell, \ell)$ (so $\eta \in T_\ell$), and we have to prove $\eta \in T^*$. We can choose ν, $\eta \triangleleft \nu \in \lim F(z_\ell, \ell)$, so it suffices to show that for any $i \in w$ we have $\nu \restriction i \in \bigcup_{\substack{j \le i \\ j \in w}} T_j$. If $i \ge \ell$, then: $\nu \restriction i \in F(z_\ell, \ell) = T_\ell \subseteq \bigcup_{\substack{j \le i \\ j \in w}} T_j$ and if $i < \ell$ then: $\nu \restriction i = \eta \restriction i = \eta^* \restriction i \triangleleft H(z_i, i) \in \lim F(z_i, i)$, hence $\nu \restriction i \in F(z_i, i) = T_i \subseteq \bigcup_{\substack{j \le i \\ j \in w}} T_j$ (remember $i \in w$). So by the conclusion of $(\gamma)^+$ (in 1.3(8) which we have applied) $\nu \in \lim T^*$ hence $\eta \in T^*$ is as required; so we have proved condition (c).

The full proof of (c): Let $y^\dagger, x_n$ be as above. Now we prove (c) using (γ) of 1.2(1) only. So we are given η^*, F and H as in the assumptions of 1.6(1A). Apply condition (γ)(b) of Definition 1.2(1) with $x_0, \eta^*, H(x_n, n)$ (for $n < \omega$) here standing for x, η, η_n (for $n < \omega$) there, and get an infinite $w_0 \subseteq \omega$ and $T_0 \in \mathrm{Rang}(R)$ such that $x_0 R T_0$ and $\bigwedge_{n \in w_0} H(x_n, n) \in \lim T_0$, hence $\eta^* \in \lim T_0$.

Let $w_0 = \{k_\ell : \ell < \omega\}$, k_ℓ increasing with ℓ, of course w.l.o.g. $k_\ell + 1 < k_\ell$ (hence $\ell + 1 < k_\ell$). Applying (β)(iv) (of 1.2(1)) choose $T_{\ell+1}$ such that $x_{k_\ell+1} R T_{\ell+1}$, $T_0 \subseteq T_{\ell+1}$, even $T_\ell \subseteq T_{\ell+1}$, and for some $m_\ell < \omega$ we have: $[\rho \restriction m_\ell \in T_0 \;\&\; \rho \in F(x_{k_\ell}, k_\ell) \Rightarrow \rho \in T_{\ell+1}]$. So by (γ)(a) (of 1.2(1)) for some T^* and infinite $w_1 \subseteq \omega$ we have: yRT^* and for every $\eta \in {}^\omega\omega$ we have $[\bigwedge_{i \in w_1} \eta \restriction \min(w_1 \setminus (i+1)) \in \bigcup_{\substack{j < i \\ j \in w_1}} T_j \cup T_0] \Rightarrow \eta \in T^*$. We define $w = \{k_\ell : \ell + 1 \in w_1\}$ and for $\ell + 1 \in w_1$ let us define n_{k_ℓ} as the first natural number $n = n_{k_\ell}$ such that $n_{k_\ell} > \ell$, in the interval (ℓ, n_{k_ℓ}) there are at least two members of w_1, and $n_{k_\ell} > m_\ell$.

We are going to prove that w, $\langle n_j : j \in w \rangle$ are as required (in 1.6(1A)). Remembering that the general members of w have the form k_ℓ with $\ell + 1 \in w_1$, it suffices to prove that (the replacement of $\eta \in {}^{\omega>}\omega$ by $\nu \in {}^{\omega}\omega$ is as in the proof above of clause (c) from $(\gamma)^+$ of 1.3(8)):

$(*)$ for any ℓ and ν we have $\otimes \Rightarrow \oplus$ where

$\otimes$ (A) $k_\ell \in w$ (i.e. $\ell+1 \in w_1$)

(B) $\nu \in {}^{\omega}\omega$

(C) $\nu \restriction k_\ell \lhd \eta^*$

(D) $\nu \restriction n_{k_\ell} \lhd H(z_{k_\ell}, k_\ell)$

(E) $\nu \in \lim(F(z_{k_\ell}, k_\ell))$

$\oplus$ $\nu \in \lim(T^*)$

By the choice of T^*, for getting $\oplus$ it suffices to prove

$\otimes_1$ if $i = i_1 \in w_1$, and $i_2 = \min(w_1 \setminus (i_1+1))$ then $\nu \restriction i_2 \in \bigcup_{j \in w_1, j < i_1} T_j \cup T_0$.

Note that $k_\ell \in w_0$ (see above before choice of the T_ℓ's). We split the proof of $\otimes_1$ accordingly to how large i is.

Case 1: $\neg(\exists j)[\ell < j \in w_1 \cap i_1]$

By the choice of n_{k_ℓ} we know that in interval (ℓ, n_{k_ℓ}) there are at least two members of w_1, but $i_1 \leq \min(w_1 \setminus (\ell+1))$ and $i_2 = \min(w_1 \setminus (i_1+1))$ so necessarily $i_2 < n_{k_\ell}$. Hence (by the previous sentence, by $\otimes$(D), by the choice of T_0, and trivially respectively) we have

$$\nu \restriction i_2 \lhd \nu \restriction n_{k_\ell} \lhd H(z_{k_\ell}, k_\ell) \in \lim(T_0) \text{ and thus } \nu \restriction i_2 \in \bigcup_{j \in w_1, j < i_1} T_j \cup T_0$$

as required (in $\otimes_1$).

Case 2: $(\exists j)[\ell < j \in w_1 \cap i_1]$

Let $i_0 = \max(w_1 \cap i_1)$, so by the assumption of the case not only i_0 is well defined but also it is $> \ell$. Looking at the desired conclusion of $\otimes_1$ and the definition of i_0 it suffices to prove that $\nu \restriction i_2 \in T_{i_0}$. But we know that $[n < \omega \Rightarrow T_n \subseteq T_{n+1}]$ and (by the previous sentence) $\ell < i_0$, hence $T_{\ell+1} \subseteq T_{i_0}$, so it suffices to prove $\nu \restriction i_2 \in T_{\ell+1}$. For this by the choice of $T_{\ell+1}$ it suffices to show the following:

$\otimes_2$ (A) $\nu \restriction m_\ell \in T_0$

(B) $\nu \restriction i_2 \in F(x_{k_\ell}, k_\ell)$

As for clause $\otimes_2$(B), by the assumption $\otimes$(E) it holds. As for clause $\otimes_2$(A), we know

$$\nu{\restriction}m_\ell \lhd \nu{\restriction}n_{k_\ell} \lhd H(z_{k_\ell}, k_\ell) \in \lim(T_0).$$

[Why? first $\lhd$ holds as $m_\ell < n_{k_\ell}$ by the choice of n_{k_ℓ}, second $\lhd$ holds by assumption $\otimes$(D), and the last "$\in$" holds as $k_\ell \in w_0$ and the choice of T_0, w_0.]

So both clauses of $\otimes_2$ hold hence $\otimes_1$ holds in case 2 hence in general, hence we have proved $(*)$. Thus we have finished proving clause (c) in the general case. Having proved condition (c) we shall now prove condition (d).

Proof of (d). Choose $x^\dagger$ and then by induction on $n < \omega, x_n$ such that $x_n < x_{n+1} < \ldots < x^\dagger < z$ (they exist by (β) of Definition 1.2(1)).

So it suffices to prove that $(D, R) \models \psi_{\text{dis}}(\langle x_0, x_1, \ldots \rangle, z)$. Let $\eta^* \in {}^\omega\omega$ and $\langle T_{n,j} : n, j < \omega \rangle$ be as in $(*)$ of Definition 1.6(3).

For each $n < \omega$, by applying ω times Def 1.2(1)(β)(ii), we can find $x_{n,j}$ (for $j \leq \omega$) such that $x_n < x_{n,0} < x_{n,1} < \ldots < x_{n,\omega} < x_{n+1}$ (first choose $x_{n,\omega}$ and then $x_{n,0}, x_{n,1}, \ldots$). We now define by induction on n, T^*_n such that $x_n R T^*_n$ and $T^*_n \subseteq T^*_{n+1}$. First let T^*_0 be such that $x_0 R T^*_0, \eta^* \in \lim T^*_0$ (possible by 1.1(b) and 1.2(1)(α)). Second, assuming T^*_n was defined, we can choose by induction on j trees $T'_{n,j}$ satisfying: $T'_{n,j} \subseteq T'_{n,j+1}$, $T'_{n,0} = T^*_n$, $x_{n,j} R T'_{n,j}, \eta^* \in \lim T'_{n,j}$ and such that for some $m = m(n,j)$ we have

$$(\forall \rho)[\rho \in T_{n,j} \& \rho \restriction m \in T'_{n,j} \Rightarrow \rho \in T'_{n,j+1}]$$

(possible by 1.2(1)(β)(iv)). Now by (γ)(a) of Def 1.2(1) we can find $w(n) \subseteq \omega$ infinite and T^*_{n+1} such that $x_{n+1} R T^*_{n+1}$ and

$$T^*_{n+1} \supseteq \{\eta : \text{ for every } i \in w(n),\ \eta \restriction \min(w(n) \setminus (i+1)) \in \bigcup_{\substack{j<i \\ j \in w(n)}} T'_{n,j} \cup T^*_n\}.$$

Necessarily $T^*_n \subseteq T^*_{n+1}$.

Then applying (γ)(a) of Definition 1.2(1) we can find $w \subseteq \omega$ infinite and T^*_ω such that zRT^*_ω and:

$$T^*_\omega \supseteq \{\eta : \text{ for every } i \in w,\ \eta \restriction \min(w \setminus (i+1)) \in \bigcup_{\substack{j<i \\ j \in w}} T^*_j \cup T^*_0\}.$$

As said above $T^*_n \subseteq T^*_{n+1}$ for each n, clearly from the condition above zRT^*_ω and $T^*_0 \subseteq T^*_\omega$ and in particular $\eta^* \in \lim T^*_0 \subseteq \lim T^*_\omega$. So in 1.6(3) $(*)$, (with T^*_α here for T^α there) conditions (i), (ii), (iii) are satisfied. As for condition (iv), let $\nu \in (\lim T_{n,j}) \cap (\lim T^*_n) \cap (\lim T^*_\omega)$. Then any $k < \omega$ such that: $k > \min\{i \in w : |i \cap w \setminus (n+1)| > 1\}$ and $k > \min\{i : |i \cap w(n) \setminus (j+1)| > 1\}$ and $k > m(n,j)$ is as required. So T^*_ω is as required in 1.6(3), i.e. we have proved (d) from 1.6(4).

Now why is (D,R) strongly smooth? By remark 1.3(3). Suppose P is (D,R)-preserving then in V^P still (D,R) is a weak covering model as P is (D,R)-preserving, hence (α) of Definition 1.2(1) holds in V^P, (β) is trivial, and $(\gamma),(\delta)$ hold by (δ). So $(D,R,<)$ is a fine covering model in V^P hence, by what we already proved it is temporarily a strong covering model. As this holds for every P we finish.

2) Similar proof.

3) Read the definitions. $\square_{1.8}$

1.9 Definition. A forcing notion Q has pure (θ_1,θ_2)-decidability if: for every $p \in Q$ and Q-name $\underset{\sim}{t} < \theta_1$, there are $a \subseteq \theta_1, |a| < \theta_2$ (but $|a| > 0$) and $r \in Q$ such that $p \leq_{\rm pr} r$, and $r \Vdash_Q$ "$\underset{\sim}{t} \in a$" (for $\theta_1 = 2$, alternatively, $\underset{\sim}{t}$ is a truth value), [if $\theta = \theta_1 = \theta_2$ we write just θ].

1.9A Remark. 1) If $\aleph_0 > \theta_2 > 2$, pure $(\theta_2, 2)$-decidability is equivalent to pure $(2,2)$-decidability.

2) Q purely semiproper implies Q has $(\aleph_1, \aleph_1)$-decidability.

3) If Q is purely proper *then* Q has $(\lambda, \aleph_1)$-decidability for every λ.

4) If $\leq_{\rm pr} = \leq$ and Q is proper or Q has the c.c.c. (and we let $\leq_{\rm pr}$ be equality if not defined) *then* Q is purely proper (see 0.1 case D).

1.10 Lemma. For $(\theta_1, \theta_2) \in \{(2,2), (\aleph_0, 2)\}$ the property "Q has pure (θ_1, θ_2)-decidability" is preserved by GRCS iteration as 0.1.

Proof: In quoting we refer to case F. We prove it by induction on the length of the iteration (for all $q, \underset{\sim}{t}$ and generic extension of V). By the distributivity of the iteration (in case F claim XV 1.7) it suffices to deal with the following five cases:

Case 1. $\underline{\alpha \leq 1}$ Trivial.

Case 2. $\underline{\alpha = 2}$ Easy.

case 3. $\underline{\alpha = \omega_1}$ If there is q_1, $q \leq_{\text{pr}} q_1 \in P_\alpha$ such that for some $\beta < \alpha$, and P_β-name $\underset{\sim}{t}_1$,: $q_1 \in P_\beta$ and $q_1 \Vdash_{P_\alpha}$ "$\underset{\sim}{t} = \underset{\sim}{t}_1$", then we can use the induction hypothesis. By XV 3.3 this holds

Case 4. $\underline{\alpha \text{ strongly inaccessible, } \alpha > |P_i| \text{ for } i < \alpha}$: Even easier than the case $\alpha = \omega_1$.

case 5. $\underline{\alpha = \omega}$: So $\theta_2 = 2$, and w.l.o.g. $p = \{p_n : n < \omega\}, p_n$ a P_n-name of a member of $\underset{\sim}{Q}_n$. We define q_n such that:

(i) q_n a P_n-name of a member of $\underset{\sim}{Q}_n$

(ii) $\Vdash_{P_n}$ "$p_n \leq_{\text{pr}} q_n$"

(iii) in V^{P_n}, q_n decides $\underset{\sim}{s}_n$, where:

for $G_{n+1} \subseteq P_{n+1}$ generic over V, $\underset{\sim}{s}_n[G_{n+1}]$ is $k+1$ *iff* there is $r \in P_\omega/G_{n+1}$ such that $\text{Dom}(r) = [n+1, \omega)$, $P_\omega/G_{n+1} \models p \restriction [n+1, \omega) \leq_{\text{pr}} r$ and $r \Vdash_{P_\omega/G_{n+1}}$ "$\underset{\sim}{t} = k$", with k minimal under those conditions; otherwise (i.e. if there is no such k) $\underset{\sim}{s}_n = 0$. (Actually q_n is a P_n-name of a member of $\underset{\sim}{Q}_n[\underset{\sim}{G}_n]$.) (If $\theta_1 = \aleph_0$ - clear, if $\theta_1 = 2$ - use Definition 1.9 twice, see 1.9A(1)).

Now $q = \{q_n : n < \omega\} \in P_\omega, p \leq_{\text{pr}} q$; clearly there is r, $q \leq r \in P_\omega$ and $\ell < \theta_1$ such that $r \Vdash$ "$\underset{\sim}{t} = \ell$". Also w.l.o.g. for some $n(*)$, $[n(*) \leq n < \omega \Rightarrow r \restriction \{n\}$ is pure]; hence $r \restriction n(*) \Vdash_{P_{n(*)}}$ "$P_\omega/P_{n(*)} \models p \restriction [n(*), \omega) \leq_{\text{pr}} r \restriction [n(*), \omega)$".

We can prove by downward induction on $m \leq n(*)$ that for some $\ell > 0$ we have $(r \restriction m) \cup \{q_m\} \Vdash$ "$\underset{\sim}{s}_m = \ell$".

For $m = 0$ we easily finish (by the definition of $\underset{\sim}{s}_m$). $\square_{1.10}$

1.10A Claim. Assume that Q is (D,R)-preserving, (D,R) a is weak covering model, Q has pure (θ_1,θ_2)-decidability and for some λ and stationary $S \subseteq \mathcal{S}_{\leq \aleph_0}(\lambda)$, the forcing notion Q is purely S-proper (or the parallel for semiproper and $|D| = \aleph_1$, follows from 0.1(1) in all cases there).

(a) If $(\theta_1,\theta_2) = (\aleph_0, 2)$ *then* Q is purely (D,R)-preserving.

(b) If $(\theta_1,\theta_2) = (\aleph_0,\aleph_0)$ and for every $x \in \mathrm{Dom}(R)$ there is $y \in \mathrm{Dom}(R)$ such that for each $n < \omega$:

$$(\forall T_1,\dots,T_n \in \mathrm{Rang}(R))(\exists T \in \mathrm{Rang}(R))[\bigwedge_{\ell=1}^{n} yRT_\ell \rightarrow xRT \ \&\ \bigwedge_{\ell=1}^{n} T_\ell \subseteq T]$$

then Q is purely (D,R)-preserving.

Proof. Straight. $\square_{1.10A}$

1.11 Claim. 1) Assume $\bar{Q} = \langle P_n, \underset{\sim}{Q}_n : n < \omega\rangle$ a GRCS iteration with $\underset{\sim}{Q}_n$ having pure $(\aleph_0,2)$ decidability, as XV 3.1. Then for every $p \in P_\omega$, $p \Vdash$ "$\underset{\sim}{f} \in {}^\omega\omega$" there is $q, p \leq_{\mathrm{pr}} q \in P_\omega$, such that $q \Vdash$ "$\underset{\sim}{f}(n) = \underset{\sim}{k}_n$" where $\underset{\sim}{k}_n$ is a P_n-name.

2) If we assume in addition: $p \Vdash_{P_\omega}$ "$\underset{\sim}{f} \leq g$", $g \in {}^\omega\omega$" (and $g \in V$) *then* we can replace "having pure $(\aleph_0,2)$-decidability" by "having pure $(2,2)$-decidability".

Proof. Straightforward. $\square_{1.11}$

1.12 Theorem. Suppose (D,R) is a smooth strong covering model, $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \delta\rangle$ a GRCS iteration as in 0.1, e.g. satisfying $\langle \mathbb{I}_{i,j}, \lambda_{i,j}, \mu_{i,j}, \mathbf{S}_{i,j}, \mathbf{W} : \langle i,j\rangle \in W\rangle$ (as in XV 3.1), $\mathbb{I} = \cup\{\mathbb{I}_{i,j} : \langle i,j\rangle \in W\}$, and $S \subseteq \mathbf{S}_{i,j}$ for $\langle i,j\rangle \in W$, each $\underset{\sim}{Q}_i$ with pure (θ_1,θ_2)-decidability and

$(*)$ $(\theta_1, \theta_2) \in \{(\aleph_0, 2), (2,2)\}$ and† if $(\theta_1,\theta_2) = (2,2)$ then for each T, x, k there is $F \in {}^\omega\omega$ such that

$$\forall \eta[xRT \ \&\ \eta \in {}^{\omega>}\omega \ \&\ (\exists k)(\eta(k) \geq F(k)) \ \&\ \eta \restriction k \in T) \Rightarrow \eta \in T]$$

† The meaning of this is like 1.11(2), i.e. we are not interested in all ${}^\omega\omega$ just in $\{\eta : \eta \in {}^\omega\omega$ and $\eta \leq g$ (i.e. $(\forall n)(\eta(n) \leq g(n))\}$.

and

(**) $|D| \leq \aleph_1$ or at least every regular uncountable $\kappa \leq |D|$ belongs to **S**.

If P_i is purely (D, R)-preserving for each $i < \delta$ (δ a limit ordinal) *then* P_δ is purely (D, R)-preserving.

1.12A Remark. If $\leq^{Q_i}_{\text{pr}} = \leq^{Q_i}$, the pure (θ_1, θ_2)-decidability is always trivially true for $(\aleph_0, 2)$ and even $(\infty, 2)$.

Proof: By XV 1.7 it is enough to consider only the cases $\delta = \omega, \delta = \omega_1, \delta$ strongly inaccessible $\bigwedge_{i<\delta} \delta > |P_i|$. In the last two cases $\mathbb{R}^{V^{P_\delta}} = \bigcup_{i<\delta} \mathbb{R}^{V^{P_i}}$ so w.l.o.g. $\delta = \omega$.

Suppose $p \in P_\omega$, $\underset{\sim}{f}$ a P_ω-name, $p \Vdash_{P_\omega}$ "$\underset{\sim}{f} \in {}^\omega\omega$" and $z \in \text{Dom}(R)$.

By 1.11 above (using part (1) if $(\theta_1, \theta_2) = (\aleph_0, 2)$ and using part (2) and (*) if $(\theta_1, \theta_2) = (2, 2)$) w.l.o.g $\underset{\sim}{f}(k)$ is a P_k-name of a natural number.

For notational simplicity we shall write the members of P_n as $\langle \underset{\sim}{q}_\ell : \ell < n \rangle, \Vdash_{P_\ell}$ "$\underset{\sim}{q}_\ell \in \underset{\sim}{Q}_\ell$" and similarly for P_ω. Let $p = \langle \underset{\sim}{q}_\ell : \ell < \omega \rangle$, and let for $m < n$ (in V^{P_m}) $P_n/P_m = \underset{\sim}{Q}_m * \underset{\sim}{Q}_{m+1} * \cdots * \underset{\sim}{Q}_{n-1}$.

Now we define by induction on $n < \omega$, a condition $p^n \in P_n$ such that $p^n = \langle \underset{\sim}{q}^n_0, \ldots, \underset{\sim}{q}^n_{n-1} \rangle$, and for each $m \leq n$ a P_m-name $\underset{\sim}{t}_{n,m}$ such that:

α) $p \restriction n \leq_{\text{pr}} p_n$, and $p_n \leq_{\text{pr}} p_{n+1} \restriction n$, moreover

$$\Vdash_{P_\ell} \text{“}\underset{\sim}{q}_\ell \leq_{\text{pr}} \underset{\sim}{q}^n_\ell \leq_{\text{pr}} \underset{\sim}{q}^{n+1}_\ell\text{”}.$$

β) If $G_m \subseteq P_m$ is generic (over V) $m \leq n$, then in $V[G_m]$ we have

$$\langle \underset{\sim}{q}^n_m, \ldots, \underset{\sim}{q}^n_{n-1} \rangle \Vdash_{P_n/P_m} \text{“}\underset{\sim}{f}(n) = \underset{\sim}{t}_{n,m}[G_m]\text{”};$$

so $\underset{\sim}{t}_{n,n} = \underset{\sim}{f}(n)$. Equivalently, $\langle \emptyset, \emptyset, \ldots, \emptyset, \underset{\sim}{q}^n_m, \ldots, \underset{\sim}{q}^n_{n-1} \rangle \Vdash_{P_n}$ "$\underset{\sim}{f}(n) = \underset{\sim}{t}_{n,m}$".

γ) $\Vdash_{P_m} \left[\underset{\sim}{q}^n_m \Vdash_{\underset{\sim}{Q}_m} \text{“}\underset{\sim}{t}_{n,m+1} = \underset{\sim}{t}_{n,m}\text{”} \right]$

This is easily done: define $\langle \underset{\sim}{q}^n_\ell : \ell < n < \omega \rangle$ by induction on n, for each n let $\underset{\sim}{t}_{n,n} = \underset{\sim}{f}(n)$ and define $\underset{\sim}{q}^n_\ell, \underset{\sim}{t}_{\ell,n}$ by downward induction on $\ell < n$.

Let $\underset{\sim}{f}_m$ be the P_m-name of a function from ω to ω defined by: for $n \geq m$ we have: $\underset{\sim}{f}_m(n) = \underset{\sim}{t}_{m,n}$ and for $n < m$ we have: $\underset{\sim}{f}_m(n) = \underset{\sim}{f}(n)$. So clearly we have:

δ) $\langle \emptyset, \ldots, \emptyset, q^n_m \rangle \Vdash_{P_{m+1}}$ "$\underset{\sim}{f}_m \restriction n = \underset{\sim}{f}_{m+1} \restriction n$".

ε) $\Vdash_{P_n}$ "$\underset{\sim}{f}_n \restriction n = \underset{\sim}{f} \restriction n$".

By Definition 1.6(4)(d) there are $x_n (n < \omega)$ such that

$$(D, R) \models \psi_{dis}(\langle x_0, \ldots \rangle, z).$$

Now let χ be large enough, and we split our requirement according to the kind of iteration. (The cases are from 0.1, cases A,B of 0.1 are covered by the later cases).

Let N be countable (the cases listed cover all possibilities):

Case D or C: $N \prec (H(\chi), \in, <^*_\chi)$, N is countable such that $(*)$ below holds.

Case E,F,G: Let $\langle N_\eta : \eta \in (T, \mathbf{I}) \rangle$ be an $(\mathbb{I}, \mathbf{W})$-suitable tree of models, $N = N_{<>}$ such that

$(*)$ $\langle P_\ell, \underset{\sim}{Q}_\ell : \ell < \omega \rangle$, $P \in N$, and also $(D, R), \underset{\sim}{f}, \langle \underset{\sim}{q}^n_\ell : \ell < n < \omega \rangle$, $\langle \underset{\sim}{t}_{n,m} : m \leq n < \omega \rangle$ belong to N and $N \cap \omega_1 \in \mathbf{W}$ in cases E, F, G.

Let $\langle T_{n,j} : j < \omega \rangle$ enumerate $\{T \in D \cap N : x_n R T\}$ and η^* be f_0 (which is a P_0-name, i.e. a function in V). Now let $\langle T^\alpha : \alpha \leq \omega \rangle$ be as guaranteed in $(*)$ of 1.6(3).

We now define (in V!) by induction on n conditions $r^n = \langle r_0, \ldots, r_{n-1} \rangle \in P_n$ (so trivially $r^n = r^{n+1} \restriction n$) such that:

a) $p \restriction n \leq_{\mathrm{pr}} r^n$,

b) $r^n \Vdash$ "$\underset{\sim}{f}_n \in (\lim T^n) \cap (\lim T^\omega)$",

c) *Case D*: r^n is generic for (N, P_n).

Case C: r^n is semi-generic for (N, P_n).

Case E,F,G: for some P_n-name $\underset{\sim}{\eta}_n$, letting $N_{\underset{\sim}{\eta}_n} = \bigcup_{k<\omega} N_{\underset{\sim}{\eta}_n \restriction k}$ we have: r^n is semi-generic for $(N_{\underset{\sim}{\eta}_n}, P_n)$ and $r^n \Vdash_{P_n}$ "$N_{\underset{\sim}{\eta}_n}[G_{P_n}]$ is $(\bigcup_{l \geq n} \mathbb{I}_\ell)$-suitable model for χ"

If we succeed then we easily finish; clearly $r^\dagger = \langle r_0, r_1, \dots, r_n, \dots \rangle$ satisfies $p \le_{\mathrm{pr}} r^\dagger$; also for $n < \omega$:

$$(r^\dagger \restriction n) \Vdash_{P_n} \text{“}\underset{\sim}{f} \restriction n = \underset{\sim}{f}_n \restriction n\text{”}.$$

Hence $(r^\dagger \restriction n) \Vdash_{P_n}$ “$\underset{\sim}{f} \restriction n \in T^n \cap T^\omega$” and therefore $r^\dagger \Vdash_{P_\omega}$ “$\underset{\sim}{f} \in \lim T^\omega$ ”. As zRT^ω (by 1.6(3)(*)(ii)) clearly $r^\dagger, T^\omega$ are as required.

So we have just to carry out the induction. There is no problem for $n = 0$ (by the choice of η^*). So we have to do the induction step. Assume r^n is defined, and we shall define r^{n+1}.

Note. as P_{n+1} purely preserves (D, R) we can deduce:

$\otimes$ $\quad Q_n$ (in V^{P_n}) purely preserves (D, R).

Let $G_n \subseteq P_n$ be generic over $V, r^n \in G_n$, so $\underset{\sim}{f}_{n+1}$ becomes a $\underset{\sim}{Q}_n[G_n]$-name $\underset{\sim}{f}_{n+1}/G_n$ of a member of ${}^\omega\omega$. But $(D, R, <)$ is purely preserved by P_{n+1}, hence for every $q \in \underset{\sim}{Q}_n[G_n]$, and $y \in \mathrm{Dom}(R)$ there is a condition $q^\dagger$, $q \le_{\mathrm{pr}} q^\dagger$ (where $q^\dagger \in \underset{\sim}{Q}_n[G_n]$), such that $q^\dagger \Vdash_{\underset{\sim}{Q}_n[G_n]}$ “$\underset{\sim}{f}_{n+1}/G_n \in \lim T^\dagger$ ” for some $T^\dagger \in D$ satisfying $yRT^\dagger$. Also there are $\langle q^\dagger_\ell : \ell < \omega \rangle$, and ν such that:

$\nu \in \lim T^\dagger$, in $\underset{\sim}{Q}_n[G_n]$ we have $q^\dagger \le_{\mathrm{pr}} q^\dagger_0 \le_{\mathrm{pr}} q^\dagger_1 \le_{\mathrm{pr}} \dots$ and $q^\dagger_\ell \Vdash_{\underset{\sim}{Q}_n[G_n]}$ “$\underset{\sim}{f}_{n+1} \restriction \ell = \nu \restriction \ell$”. [Why? $\nu = \underset{\sim}{f}[G_n]$ can serve.]

We can use choice functions, so let $\nu = F_1(q, z)$ and $q^\dagger_\ell = F_{2,\ell}(q, z)$ and $T^\dagger = F_0(q, z)$, and $q^\dagger = F_2(q, z)$. By our hypotheses (smoothness) in $V[G_n]$ we know that (D, R) is still a covering model. Note also that w.l.o.g. $F_0, F_1, F_{2,\ell}$ belong to $N[G_n]$. Remember (by (a)) that in Case F $N[G_n]$ is an $(\bigcup_{\ell \ge r} \mathbb{I}_\ell)$-suitable model for χ.

So now we apply condition c) of Definition 1.6(2) (the definition of a covering model) and get that in $V[G_n]$ the statement $\varphi^*_{\mathrm{dis}}(x_{n+1})$ holds. Look at the definition of φ^*_{dis} (1.6(1A)) and apply it to $\eta^* \stackrel{\mathrm{def}}{=} \underset{\sim}{f}_n[G_n]$ (which is an actual member of ${}^\omega\omega$ in $N[G_n]$), the function H with domain $D \times \omega$, $H(z, m) \stackrel{\mathrm{def}}{=} F_1(q^m_n, z)$ and the function $F : \mathrm{Dom}(R) \times \omega \to D$ defined by $F(z, m) \stackrel{\mathrm{def}}{=} F_0(q^m_n, z)$. So we get a tree $T^*_n \in D$, and an infinite set w_n as described there.

However note: $D \subseteq V$, so though T^*_n is defined in $V[G_n]$ it is an element of V. Working in V we have P_n-names $\underset{\sim}{T}^*_n, \underset{\sim}{w}_n$.

In fact without loss of generality $\underset{\sim}{T}^*_n \in N$, hence (by assumption (**) of 1.12 and condition (c) on r^n) we have $\langle r_0, \dots, r_{n-1}\rangle \Vdash_{P_n}$ "$\underset{\sim}{T}^*_n \in D \cap N$" so for some P_n-name $\underset{\sim}{j} = \underset{\sim}{j}(n)$ (of a natural number) we have $\langle r_0, \dots, r_{n-1}\rangle \Vdash_{P_n}$ "$\underset{\sim}{T}^*_n = T_{n,\underset{\sim}{j}}$". Now $\langle r_0, \dots, r_{n-1}\rangle$ forces $\underset{\sim}{f}_n \in (\lim T^n) \cap (\lim T^\omega) \cap (\lim \underset{\sim}{T}^*_n) = (\lim T^n) \cap (\lim T^\omega) \cap (\lim T_{n,\underset{\sim}{j}(n)})$.

Hence, working in $V[G_n]$, by the choice of $\langle T^\alpha : \alpha \leq \omega\rangle$ (see 1.6(3)(iv)) there is $k < \omega$ (which depends on $\underset{\sim}{f}_n[G_n]$) such that:

(A) $\underset{\sim}{f}_n[G_n] \restriction k \lhd \rho \in T_{n,\underset{\sim}{j}(n)[G_n]} \cap T^\omega \Rightarrow \rho \in T^{n+1} \cap T^\omega$.

Now w.l.o.g. we can increase k, so w.l.o.g. $k \in \underset{\sim}{w}_n[G_n]$ (and $k > n$); (k was defined in $V[G_n]$). By the choice of q^k_n and the $\underset{\sim}{f}_\ell$'s:

(B) $q^k_n \Vdash_{\underset{\sim}{Q}_n[G_n]}$ "$\underset{\sim}{f}_n[G_n] \restriction k \lhd \underset{\sim}{f}_{n+1}$",

also by the choice of $F_0, F_1, F_2, F_{2,\ell}$:

(C) $F_{2,\ell}(q^k_n, x_{n+1}) \Vdash_{\underset{\sim}{Q}_n[G_n]}$ "$\underset{\sim}{f}_{n+1} \restriction \ell \lhd F_1(q^k_n, x_{n+1})$" and

(D) $q^k_n \leq F_{2,\ell}(q^k_n, x_{n+1}) \in \underset{\sim}{Q}_n[G_n] \cap N[G_n]$, and

(E) $H(x_{n+1}, k) = F_1(q^k_n, x_{n+1})$ and $F(x_n, n) = F_0(q^k_n, x_n)$.

Now by the choice of $T^*_n = T^*_{n,\underset{\sim}{j}[G_n]}$ for some ℓ

(F) $H(x_{n+1}, k) \restriction l \trianglelefteq \rho \in F(x_n, n) \Rightarrow \rho \in T_{n,\underset{\sim}{j}[G_n]}$

So together.

(G) $F_{2,l}(q^k_n, x_{n+1})$ is a member of $\underset{\sim}{Q}_n[G_n] \cap N[G_n]$, it is a pure extension of p_n and it forces $\underset{\sim}{f}_{n+1}$ (really $\underset{\sim}{f}_{n+1}[G_n]$) to belong to $\lim T^{n+1} \cap \lim T^\omega$.

Now we can choose $r_n, F_{2,l}(q^k_n, x_{n+1}) \leq_{\mathrm{pr}} r_n \in Q_n[G_n]$ to satisfy (c) thus finishing the induction and the proof. $\square_{1.12}$

So e.g.

1.13 Corollary. Suppose

(α) $\langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha\rangle$ is a GRCS iteration as in XV 3.1. (i.e. 0.1F)

(β) (D, R) is a fine covering model,

(γ) $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ is purely (D,R)-preserving"

(δ) D has cardinality $\aleph_1$ (or just (**) of 1.12)

(ε) each $\underset{\sim}{Q}_i$ has pure $(\aleph_0, 2)$-decidability.

Then P_α is purely (D,R)-preserving.

Proof: We prove by induction on α that $\Vdash_{P_\alpha}$ "(D,R) is a smooth strong covering model" and P_α is purely (D,R)-preserving.

Case 1. $\alpha = 0$ By 1.8(1) we know (D,R) is a smooth strong covering model.

Case 2. $\alpha = \beta + 1$ By the induction hypothesis $V^{P_\beta} \models$ "(D,R) is a smooth strong covering model", as $\underset{\sim}{Q}_\beta$ is (D,R)-preserving $V^{P_\beta} \models [\Vdash_{Q_\beta}$ "(D,R) is a weak covering model"], hence (see Definition 1.7(3)):

$V^{P_\beta} \models [\Vdash_{Q_\beta}$ "(D,R) is a strong covering model"].

Let $\underset{\sim}{R}$ be a P_α-name of a (D,R)-preserving forcing notion; easily $\Vdash_{P_\beta}$ "$\underset{\sim}{Q} * \underset{\sim}{R}$ is (D,R)-preserving" so as above $\Vdash_{P_\beta}$ "[$\Vdash$ "$_{\underset{\sim}{Q}_\beta * \underset{\sim}{R}}$ (D,R) is a strong covering model"]" .

So in $V^{P_\alpha} = (V^{P_\beta})^{\underset{\sim}{Q}_\beta}$, for every (D,R)-preserving forcing notion $\underset{\sim}{R}$,

$\Vdash_{\underset{\sim}{R}}$ "(D,R) is a strong covering model".

So in V^{P_α}, (D,R) is a smooth strong covering model. As for "P_α is purely (D,R)-preserving", by 1.10A it follows by the previous sentence and clause (α).

Case 3. α limit The real case, done in 1.12. $\square_{1.13}$

1.13A Corollary. Suppose:

(α) $\bar{Q}$ is a countable support iteration of proper forcing

(β) (D,R) is a fine covering model

(γ) $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ is (D,R)-preserving "

Then P_α is (D,R)-preserving.

1.13B Remark. 1) We have parallel conclusions to 1.13 weakening (ε) to

(ε)$'$ "$\underset{\sim}{Q}_i$ has (2,2)-decidability "

if we add the requirement from 1.12(*) for $(\theta_1, \theta_2) = (2,2)$.

2) We can have parallel conclusions to 1.13 weakening (ε) to

$(\varepsilon)''$ "$\underset{\sim}{Q}_i$ has $(\aleph_0, \aleph_0)$-decidability"

if we add

(ζ) each $\underset{\sim}{Q}_i$ is purely ${}^\omega\omega$-bounding.

1.14 Definition. 1) A class ($\equiv$ property) K of objects $(D, R, <)$ is a fine class of covering models if:

(i) each member satisfies $(\alpha), (\beta), (\gamma)$ of Definition 1.2.

(ii) *if* Q is a forcing notion, K-preserving (i.e. each $(D, R, <) \in K^V$ is a weak covering model even in V^Q) *then* in V^Q: each $(D, R, <) \in K^V$ is in K^{V^Q} and satisfies (γ) of Definition 1.2(1); note that clauses (α), (β) of 1.2(1) follows.

2) "K is a (smooth) (strong) class of covering models" are defined similarly.

1.15 Theorem. In 1.12, 1.13 (and 1.13B) we can replace the covering model by a class of covering models.

* * *

1.16 Definition.

1) $(\overline{D}, \overline{R})$ is a weak covering k^*-model if: $\overline{D} = \langle D_k : k < k^* \rangle$, $\overline{R} = \langle R_k : k < k^* \rangle$, $k^* < \omega$ and

(a) for each $k < k^*$, D_k is a set, R_k is a two place relation on D_k, xR_kT implies T is a closed subtree of ${}^{\omega>}\omega$.

(b) $(\overline{D}, \overline{R})$ covers, i.e. for every $\eta \in {}^\omega\omega$, for some $k < k^*$, η is of the k-th kind which means: for every $x \in \mathrm{Dom}(R_k) = \{x : (\exists T) xR_kT\}$ there is $T \in D_k$ such that xR_kT and $\eta \in \lim(T)$.

2) $(\overline{D}, \overline{R}, \overline{<})$ is a fine covering k^*-model if

(α) $(\overline{D}, \overline{R})$ is a weak covering k^*-model

(β) $\overline{<} = \langle <_k : k < k^* \rangle$, $<_k$ is a partial order on $\mathrm{Dom}(R_k)$ such that

(i) $(\forall y \in \mathrm{Dom}(R_k))(\exists x \in \mathrm{Dom}(R_k))(x <_k y)$

(ii) $(\forall y, x \in \mathrm{Dom}(R_k))(\exists z \in \mathrm{Dom}(R_k))(x <_k y \rightarrow x <_k z <_k y)$

(iii) if $y <_k x, yR_kT$ then for some $T^* \in D_k$, $T \subseteq T^*$ and xR_kT^*

(iv) if $y <_k x$ and for $\ell = 1, 2$ we have yR_kT_ℓ *then* there is $T \in D_k$ such that:

xR_kT, $T_1 \subseteq T$ and for some n, $[\nu \in T_2 \ \& \ \nu \restriction n \in T_1 \Rightarrow \nu \in T]$

(γ) (a) for each $k < k^*$ the following holds. If $x >_k x^\dagger >_k y_{n+1} >_k y_n$ for $n < \omega$ and $T_n \in D_k, y_nR_kT_n$ (for $n < \omega$) *then* there is $T^* \in D_k, xR_kT^*$ and an infinite set $w \subseteq \omega$ such that:

$$\lim T^* \supseteq \{\eta \in {}^\omega\omega : \text{ for every } i \in w, \eta \restriction \min(w \setminus (i+1)) \in \bigcup_{\substack{j<i \\ j \in w}} T_j \cup T_0\}$$

(b) if $k < k^*$, $\{\eta\} \cup \{\eta_n : n < \omega\} \subseteq {}^\omega\omega$, $\eta \restriction n = \eta_n \restriction n$ and $x <_k y$, and η, η_n are of the k-kind (see below), *then* for some $T \in D_k$ we have $yR_kT \ \& \ \eta \in \lim(T)$ and for infinitely many n, $\eta_n \in \lim(T)$.

(δ) condition (γ) continues to hold in any generic extension in which (α) holds.

3) For a property X of forcing notions, $(\overline{D}, \overline{R}, \overline{<})$ is a fine covering k^*-model for X-forcing if Definition 1.16(2) holds when in (δ) we restrict ourselves to X-forcing notions only.

4) We say $(\overline{D}, \overline{R}, \overline{<})$ is a temporarily fine covering k^*-model if it satisfies $(\alpha), (\beta), (\gamma)$ i.e. it is a fine covering k^*-model for trivial forcing.

5) We say $\eta \in {}^\omega\omega$ is of (k, x)-kind (or just the x-th kind when $\langle \mathrm{Dom}(R_k) : k < k^* \rangle$ are pairwise disjoint) if there is T such that $\eta \in \lim(T)$ and xR_kT (note: $(\bar{D}, \bar{R})$ covers iff for any $\eta \in {}^\omega\omega$ and $\bar{x} = \langle x_k : k < k^* \rangle \in \prod_{k<k^*} \mathrm{Dom}(R_k)$ for some k, the sequence η is of the (k, x_k)-kind). We say η is of the k-th kind if it is of the (k, x)-kind for every $x \in \mathrm{Dom}(R_k)$.

For simplicity we restrict ourselves to the fine case (and not the parallel of smooth strong covering).

1.17 Theorem. Assume $(\overline{D}, \overline{R}, \overline{<})$ is a fine covering k^*-model.

1) If $\overline{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \delta, j < \delta \rangle$ is a CS iteration, each $\underset{\sim}{Q}_j$ preserves $(\overline{D}, \overline{R}, \overline{<})$ then so does P_δ

2) Similarly for other iterations as in 0.1 (with pure preserving).

Proof. For simplicity $\mathrm{Dom}(R_k)$ are pairwise disjoint so let $<= \bigcup_{k<k^*} <_k$. We concentrate on part 1). By V 4.4, if δ is of uncountable cofinality then there is no problem, as all new reals are added at some earlier point. So we may suppose that $\mathrm{cf}(\delta) = \aleph_0$ hence by associativity of CS iterations of proper forcing (III) without loss of generality $\delta = \omega$.

We claim that $\Vdash_{P_\omega}$ "$(\overline{D}, \overline{R}, \overline{\leq})$ covers." (Note that this suffices for the proof of the theorem.)

So let p^* be a member of P_ω and $\underset{\sim}{f}$ a P_ω-name such that $p^* \Vdash_{P_\omega}$ "$\underset{\sim}{f} \in {}^\omega\omega$", and $x^*_k \in \mathrm{Dom}(R_k)$ for $k < k^*$. It suffices to prove that for some k, T, and p we have: $p^* \leq p \in P_\omega$, $x_k R_k T_k$ (so $T_k \in D_k$) and $p \Vdash_{P_\omega}$ "$\underset{\sim}{f} \in \lim(T_k)$". As we can increase p^* w.l.o.g. above p^*, for every n, $\underset{\sim}{f}(n)$ is a P_n-name. Let χ be large enough and let N be a countable elementary submodel of $(H(\chi), \in, <^*_\chi)$ to which $\{x_0, \dots, x_{k^*-1}, p^*, \underset{\sim}{f}, \overline{Q}\}$ belongs.

For clarity think that our universe V is countable in the true universe or at least $\beth_3(|P_\omega|)^V$ is. We let $K = \{(n,p,G) : n < \omega,\ p \in P_\omega$ is above p^*, $G \subseteq P_n$ is generic over V and $p{\restriction}n \in G\}$. On K there is a natural order: $(n,p,G) \leq (n',p',G')$ if $n \leq n'$, $P_\omega \models p \leq p'$ and $G \subseteq G'$. Also for $(n,p,G) \in K$ and $n' \in (n,\omega)$ there is G' such that $(n,p,G) \leq (n',p,G')$ as $\bar{Q}$ is an iteration of proper forcing notions. Also if $(n,p,G) \in K$ and $p \leq p' \in P_\omega/G$ (i.e. $p' \in P_\omega$ and $p'{\restriction}n \in G$) then $(n,p,G) \leq (n,p',G)$. For $(n,p,G) \in K$ let $L_{(n,p,G)} = \{g : g \in ({}^\omega\omega)^{V[G]}$ and there is an increasing sequence $\langle p_\ell : \ell < \omega\rangle$ in $V[G]$ of conditions in P_ω/G, $p \leq p_0$, such that $p_\ell \Vdash \underset{\sim}{f}{\restriction}\ell = g{\restriction}\ell\}$. So:

$(*)_1$ $K \neq \emptyset$

$(*)_2$ $(n,p,G) \in K \Rightarrow L_{(n,p,G)} \neq \emptyset$

$(*)_3$ $g \in L_{(n,p,G)} \Rightarrow (\underset{\sim}{f}{\restriction}n)[G] = g{\restriction}n$.

Note also

$(*)_4$ $L_{(n,p,\underset{\sim}{G}_n)}$ is a P_n-name.

$(*)_5$ if $(n,p,G) \leq (n',p',G')$ then $L_{(n',p',G')} \cap V \subseteq L_{(n,p,G)}$

1.17A Fact. There are $k < k^*$ and $(n,p,G) \in K$ such that if $(n,p,G) \leq (n',p',G') \in K$ *then* for some $(n'',p'',G'') \in K$, $(n',p',G') \leq (n'',p'',G'')$ there is $g \in L_{(n'',p'',G'')}$ which is of the k'th kind.

[Why? otherwise choose by induction (n^ℓ, p^ℓ, G^ℓ) for $\ell \le k^*$, in K, increasing such that: $L_{(n^{\ell+1}, p^{\ell+1}, G^{\ell+1})}$ has no members of the ℓ'-kind for $\ell' \le \ell$. So $L_{(n^{k^*}, p^{k^*}, G^{k^*})} = \emptyset$, a contradiction.]

So choose k and $(n^\otimes, p^\otimes, G^\otimes) \in K$ as in the fact, w.l.o.g. $n^\otimes = 0$. Remember that $\underset{\sim}{f}(n)$ is a P_n-name for each n.

1.17B Fact. If $(n^\otimes, p^\otimes, G^\otimes) \le (n, p, G) \in K$ and $x \in \mathrm{Dom}(R_k)$ then there is $g \in L_{(n,p,G)}$ which is of the (k, x)-kind.

Proof. By the choice of $(n^\otimes, p^\otimes, G^\otimes)$ there is $(n', p', G') \in K$ such that $(n, p, G) \le (n', p', G')$ and $L_{(n',p',G')}$ has a member g of the k-th kind. So there are $T \in \mathrm{Rang}(R_k)$ and $\langle p'_\ell : \ell < \omega \rangle$ such that $g \in \lim(T)$, $x R_k T$, $p'_0 = p'$, $p'_\ell \le p'_{\ell+1}$, $p'_\ell \in P_\omega/G'$, $p'_\ell \Vdash_{P_\omega/G'}$ "$\underset{\sim}{f} \restriction \ell = g \restriction \ell$". Note that $T \in V$. From the point of view of $V[G]$, all this is just forced by some $q \in G'$, so q forces that $\langle \underset{\sim}{p}'_\ell : \ell < \omega \rangle$, T, $\underset{\sim}{g}$ are as above. So we can find $\langle q_\ell : \ell < \omega \rangle$, $q_\ell \in P_{n'}/G$, q_ℓ increasing, $q \le q_\ell$ and q_ℓ forces a value to $\underset{\sim}{p}'_\ell$, say p''_ℓ and to $\underset{\sim}{g} \restriction \ell$ and is above $p''_\ell \restriction n'$.

And we are done.

1.17C Fact. If $\mathcal{T} \subseteq \mathrm{Rang}(R_k)$ is countable and $x <_k y$, and $(\forall T \in \mathcal{T})(\exists z \le_k x)(z R_k T)$ and $T^0 \in \mathcal{T}$ then for some $T^1 \in \mathrm{Rang}(R_k)$ we have $y R_k T^1$, $T^0 \subseteq T^1$ and for each $T \in \mathcal{T}$ for some m we have:

$$(\forall \nu)(\nu \in T \ \&\ \nu \restriction m \in T^0 \Rightarrow \nu \in T^1).$$

Proof. Let $\langle T_n : n < \omega \rangle$ list $\mathcal{T}$ (possibly with repetitions) such that $T_0 = T^0$. Let $x <_k x' <_k y$, choose inductively x_n, $x <_k x_n <_k x_{n+1} <_k x'$ (possible by clause (β)(ii) of Definition 1.16(2)). Choose inductively $T'_n \in \mathrm{rang}(R_k)$ such that $T'_0 = T_0 = T^0$ and $x_n R_k T'_n$, $T'_n \subseteq T'_{n+1}$ and for some $k_n < \omega$ we have: $\nu \in T_n$, $\nu \restriction k_n \in T'_n \Rightarrow \nu \in T'_{n+1}$ (possible by clause (β)(iv) of Definition 1.16(2)). Choose, for each n, $T''_n \in \mathrm{Rang}(R_k)$ such that $x' R_k T''_n$, $T'_n \subseteq T''_n$ (possible by clause (β)(iii) of Definition 1.16(2)). Next use 1.16(1)(γ)(a) to find an infinite $w \subseteq \omega$ and $T^1 \in \mathrm{Rang}(R_k)$ such that yRT^1, $T_0'' \subseteq T^1$ and

$$i \in w \ \&\ \nu\restriction\min(w \setminus (i+1)) \in \bigcup_{j<i, j\in w} T_j'' \cup T_0'' \Rightarrow \nu \in T^1.$$

Check that T^1 is as required. $\square_{1.17C}$

Continuation of the proof of 1.17.

Choose $x' < x_k^*$ and then inductively on n choose x_n such that $x_n <_k x'$ $x_n <_k x_{n+1}$, and choose a countable $N \prec (H(\chi), \in)$ (with $\chi = \mathrm{cf}(\chi) > \beth_\omega(|P_\omega|)$) such that all the elements $\langle x_n : n < \omega\rangle$, $\langle P_n, \underset{\sim}{Q}_n : n < \omega\rangle$, $\underset{\sim}{f}$, $p^\otimes$ belong to N. Now, working in V, we choose by induction on n sequences $\langle p_\eta : \eta \in {}^n\omega\rangle$, $\langle \underset{\sim}{f}_\eta : \eta \in {}^n\omega\rangle$, $\langle q_\eta : \eta \in {}^n\omega\rangle$, and T_n such that

(A) p_η is a $P_{\ell g(\eta)}$-name of a member of $P_\omega \cap N$, $p_{\langle\rangle} = p^\otimes$, $p_\eta \le p_{\eta\char`\^\langle \ell\rangle}$, $p_\eta\restriction n \le q_{\eta\restriction n}$.

(B) q_η is $(N, P_{\ell g(\eta)})$-generic, $q_\eta \in P_{\ell g(\eta)}$ and $[\ell < \ell g(\eta) \Rightarrow q_\eta\restriction\ell = q_{\eta\restriction\ell}]$.

(C) $\underset{\sim}{f}_\eta$ is a $P_{\ell g(\eta)}$-name of a member of ${}^\omega\omega$ and $q_\eta \Vdash_{P_{\ell g(\eta)}}$ "$\underset{\sim}{f}_\eta \in \lim(T_n) \cap N[G_{P_{\ell g(\eta)}}]$ is of the (k, x_{3n})-kind and belongs to $L_{(n, p_\eta[\underset{\sim}{G}_{P_{\ell g(\eta)}}], \underset{\sim}{G}_{P_{\ell g(\eta)}})}$" when $\eta \in {}^n\omega$.

(D) $x_{3n} R_k T_n$ and $T_n \subseteq T_{n+1}$.

(E) $p_{\eta\char`\^\langle\ell\rangle} \Vdash_{P_\omega}$ "$\underset{\sim}{f}_\eta\restriction\ell = \underset{\sim}{f}_{\eta\char`\^\langle\ell\rangle}\restriction\ell = \underset{\sim}{f}\restriction\ell$".

Suppose we succeed in this endeavour. By (β)(iii) of 1.16(1) we can find T_n' such that $T_n \subseteq T_n'$, $x' R_k T_n'$ (as $x_{3n} <_k x'$). Let w and T^* be as guaranteed by clause (γ)(a) of Definition 1.16(1) (for $\langle T_n' : n < \omega\rangle$, x', x) and let $\langle n_i : i < \omega\rangle$ be the increasing enumeration of w. So xR_kT^* and: if $\eta\restriction n_{i+1} \subseteq \bigcup_{j<i} T_{n_j}' \cup T_0'$ for each $i < \omega$ then $\eta \in T^*$.

Let $g(i) \overset{\mathrm{def}}{=} n_i$. Let $\nu = \langle n_{2j+1} : j < \omega\rangle$.

So it is enough to prove that for some $q \in P_\omega$ which is above $p^\otimes$, we have $q \Vdash_{P_\omega}$ "$\underset{\sim}{f} \in \lim(T^*)$". We choose $q \in P_\omega$ by $q\restriction i = q_{\nu\restriction i}$, by clause (B) we have: $q \in P_\omega$ is well defined and above each $q_{\nu\restriction i}$ and above each $p^\otimes\restriction i$ hence above $p^\otimes$.

We just have to prove: $q \Vdash$ "$\underset{\sim}{f}\restriction n_{i+1} \in \bigcup_{j<i} T_{n_j}' \cup T_0'$". As $q\restriction(i+1) = q_{\nu\restriction(i+1)}$, by clause (A) we have $p_{\nu\restriction(i+1)} \le q\restriction(i+1)$; by clause (E) letting $\eta = \nu\restriction i$, $\ell = \nu(i)$

we have $q \Vdash_{P_\omega}$ "$\underset{\sim}{f}_\eta \restriction \ell = \underset{\sim}{f}_{\eta \frown \langle \ell \rangle} \restriction \ell = \underset{\sim}{f} \restriction \ell$", but $\ell = \nu(i) = n_{2i+1}$, so we have $q \Vdash$ "$\underset{\sim}{f} \restriction n_{2i+1} = \underset{\sim}{f}_{\nu \restriction i} \restriction n_{2i+1}$"; now by clause (C) applied to $\eta = \nu \restriction i$ remembering $T_n \subseteq T'_n$ we have $q \Vdash$ "$\underset{\sim}{f}_{\nu \restriction i} \in \lim(T'_i)$" hence by the last two statements $q \Vdash$ "$(\underset{\sim}{f} \restriction n_{2i+1}) \in T'_i$". So, as $n_j < n_{j+1}$, for $i = 0$ we have $q \Vdash$ "$\underset{\sim}{f} \restriction n_i \in T'_0$", and for $i > 0$ we have $q \Vdash$ "$\underset{\sim}{f} \restriction n_{2i+1} \in T'_i \subseteq T'_{2i-1}$" and for $i \geq 0$ we have $q \Vdash$ "$\underset{\sim}{f} \restriction n_{2i+2} \trianglelefteq \underset{\sim}{f} \restriction n_{2i+3} \in T'_{i+1} \subseteq T'_{2i}$" so $q \Vdash$ "$\underset{\sim}{f} \restriction n_{i+1} \in \bigcup_{j<i} T'_{n_j} \cup T'_0$" holds (check by cases).

Hence we have finished proving $\Vdash_{P_\omega}$ "$(\overline{D}, \overline{R})$ covers ${}^\omega\omega$". So it suffices to carry out the induction.

There is no problem for $n = 0$.

Let us deal with $n+1$. By fact 1.17C (above) there are $T_{n,i} \in \mathrm{Rang}(R_k)$ for $i < \omega$ such that

(*) (i) $T_{n,0} = T_n$

(ii) $T_{n,i} \subseteq T_{n,i+1}$

(iii) $x_{3n+i} R_k T_{n,i}$

(iv) if $T \in (\mathrm{Rang}(R_k)) \cap N$ and $(\exists z)(z \leq_k x_{3n+i}$ & $z R_k T)$ then for some $m = m_T < \omega$ we have $\nu \in T$ & $\nu \restriction m \in T_{n,i} \Rightarrow \nu \in T_{n,i+1}$.

Let $T_{n+1} = T_{n,3}$.

Next we define $p_{\eta \frown \langle \ell \rangle}$, $\underset{\sim}{f}_{\eta \frown \langle \ell \rangle}$, $q_{\eta \frown \langle \ell \rangle}$ for $\eta \in {}^n\omega$, $\ell < \omega$. It is enough to define then in $N[G_{P_n}]$ where G_{P_n} is any generic subset of P_n to which q_η belongs (note that e.g. $p_{\eta \frown \langle \ell \rangle}$ is a P_{n+1}-name, and if $q_\eta \notin G_{P_n}$ the requirements on it are trivial to satisfy).

Let $\eta \in {}^n\omega$, and let G_{P_n} be a subset of P_n generic over V such that $q_\eta \in G_{P_n}$. So now p_η is in $(P_\omega / G_{P_n}) \cap N[G_{P_n}]$, and $f_\eta \stackrel{\mathrm{def}}{=} \underset{\sim}{f}_\eta[G_{P_n}]$ is a member of ${}^\omega\omega$ of the (k, x_{3n})-kind which belongs to $L_{(n, p_\eta, G_{P_n})}$, moreover $f_\eta \in N[G_{P_n}]$. So in $N[G_{P_n}]$ there is an increasing sequence $\langle p^0_{\eta \frown \langle \ell \rangle} : \ell < k \rangle$ of members of P_ω / G_{P_n}, $p_\eta = p^0_{\eta \frown \langle 0 \rangle}$, $p^0_{\eta \frown \langle \ell \rangle} \Vdash_{P_\omega / G_{P_n}}$ "$\underset{\sim}{f} \restriction \ell = \underset{\sim}{f}_\eta \restriction \ell$" w.l.o.g. $p^0_{\eta \frown \langle \ell \rangle} \restriction n = p_\eta \restriction n$. If $G_{P_{n+1}} \subseteq P_{n+1}$ is generic over V extending G_{P_n} and $p^0_{\eta \frown \langle \ell \rangle} \restriction (n+1) \in G_{P_{n+1}}$ then $(n+1, p^0_{\eta \frown \langle \ell \rangle}, G_{P_{n+1}}) \geq (n^\otimes, p^\otimes, G^\otimes)$ is from K, so by Fact 1.17B there are $f_{\eta,\ell} \in ({}^\omega\omega)^{V[G_{P_{n+1}}]}$ and an increasing sequence $\langle p^1_{\eta \frown \langle \ell \rangle, j} : j < \omega \rangle$ of conditions from $P_\omega/G_{P_{n+1}}$ starting with $p^0_{\eta\char94\langle\ell\rangle}$ such that $p^1_{\eta\char94\langle\ell\rangle,j} \Vdash$ "$\underset{\sim}{f}\restriction j = f_{\eta,\ell}\restriction j$" and $f_{\eta,\ell}$ is of the (k, x_{3n})-kind, say $f_{\eta,\ell} \in \lim(T^0_{\eta,\ell})$, $x_{3n}R_kT^0_{\eta,\ell}$.

Letting $Q_n = \underset{\sim}{Q}_n[G_{P_n}]$ we have Q_n-names for these objects so $\langle \underset{\sim}{f}_{\eta,\ell} : \ell < \omega\rangle$ is a Q_n-name of an ω-sequence of members of ${}^\omega\omega$ of the (k, x_{3n})-kind and $\underset{\sim}{T}^0_\eta$ and $\langle \underset{\sim}{p}^1_{\eta\char94\langle\ell\rangle,j} : j < \omega\rangle$ are $\underset{\sim}{Q}_n$-names as above.

W.l.o.g. $\langle(\underset{\sim}{f}_{\eta,\ell}, \underset{\sim}{T}^0_{\eta,\ell}, \langle \underset{\sim}{p}^1_{\eta\char94\langle\ell\rangle,j} : j < \omega\rangle) : \ell < \omega\rangle \in N[G_{P_n}]$.

So we can find $\langle(p^1_{\eta,\ell}, T^1_{\eta,\ell}) : \ell < \omega\rangle$ such that:

$$Q_n \models \text{"}p^0_{\eta,\ell} \le p^1_{\eta,\ell}(n)\text{"},$$

$$p^1_{\eta,\ell} \Vdash_{Q_n} \text{"}\underset{\sim}{T}^0_\eta = T^1_{\eta,\ell} \text{ hence } \underset{\sim}{f}_{\eta,\ell} \in \lim(T^1_{\eta,\ell})\text{"}$$

$$\text{and } x_{3n}R_kT^1_{\eta,\ell}.$$

Also we can find $g^1_{\eta,\ell}$, $p^1_{\eta,\ell,j}$ ($\ell < \omega$, $j < \omega$) such that $p^1_{\eta,\ell,0} = p^1_{\eta,\ell}$, $p^1_{\eta,\ell,j} \le p^1_{\eta,\ell,j+1}$ and $p^1_{\eta,\ell,j} \Vdash_{P_\omega/G_{P_n}}$ "$\underset{\sim}{f}_{\eta,\ell}\restriction j = g_{\eta,\ell}\restriction j$" where $g_{\eta,\ell} \in {}^\omega\omega$ necessarily $g_{\eta,\ell}\restriction j \in T^1_{\eta,\ell}$ hence $g_{\eta,\ell} \in \lim(T^1_{\eta,\ell})$. W.l.o.g. $\langle p^1_{\eta,\ell,j} : \ell < \omega, j < \omega\rangle$, $\langle g_{\eta,\ell} : \ell < \omega\rangle$ belongs to $N[G_{P_n}]$. So $g_{\eta,\ell}$, $f_\eta \in {}^\omega\omega$ are of the (k, x_{3n})-kind and $g_{\eta,\ell}\restriction\ell = f_\eta\restriction\ell$, so by clause (γ)(b) of Definition 1.16(2), there is $T^2_\eta \in \operatorname{Rang}(R_k)$ such that $x_{3n}R_kT_\eta$ and $B^\eta_0 = \{\ell < \omega : g_{\eta,\ell} \in \lim(T^2_\eta)\}$ is infinite.

Now as $f_\eta \in \lim(T_n)$, $x_{3n}R_kT^2_\eta$ and $T^2_\eta \in N[G_{P_n}] \cap \operatorname{Rang}(R_k)$ clearly, by $(*)$ above, for some $m_\eta < \omega$, $f_\eta\restriction m_\eta \trianglelefteq \nu \in T^2_\eta \Rightarrow \nu \in T_{n+1}$. Hence

$$\ell \in B^\eta_0 \;\&\; \ell \ge m_\eta \;\Rightarrow\; g_{\eta,\ell} \in \lim(T^2_\eta) \;\&\; g_{\eta,\ell}\restriction m_\eta = f_\eta\restriction m_\eta \;\Rightarrow\; g_{\eta,\ell} \in \lim(T_{n,1}).$$

As $x_{3n}R_kT^1_{\eta,\ell}$, $T^1_{\eta,\ell} \in N[G_{P_n}] \cap \operatorname{Rang}(R_k)$ clearly for some $m_{\eta,\ell} \in (m_\eta, \omega)$ we have $g_{\eta,\ell}\restriction m_{\eta,\ell} \triangleleft \nu \in T^1_{\eta,\ell} \Rightarrow \nu \in T_{\eta,2}$ and hence

$$p_{\eta,\ell,m_{\eta,\ell}} \Vdash_{Q_n} \text{"}\underset{\sim}{f}_{\eta,\ell}\restriction m_{\eta,\ell} = g_{\eta,\ell}\restriction m_{\eta,\ell} \text{ and } \underset{\sim}{f}_{\eta,\ell} \in \lim(T^1_{\eta,\ell})\text{"}.$$

Thus $p_{\eta,\ell,m_{\eta,\ell}} \Vdash_{Q_n}$ "$\underset{\sim}{f}_{\eta,\ell} \in \lim(T_{n,2})$". (Note that $\underset{\sim}{B}^\eta_0$, $\underset{\sim}{T}^1_{\eta,\ell}$, $\underset{\sim}{m}_n$, $\underset{\sim}{m}_{\eta,\ell}$ are P_n-names.)

Now, at last, we define $p_{\eta \char`\^ \langle i \rangle}$ for $i < \omega$. So $p_{\eta \char`\^ \langle i \rangle} \restriction n = p_\eta$, and we define $p_{\eta \char`\^ \langle i \rangle}(n)$ in $V[G_{P_n}]$ where $q_\eta \in G_{P_n}$ (justified above). Let $\ell(i)$ be the $\ell$-th member of $B^\eta_o \setminus m_\eta$, and $p_{\eta \char`\^ \langle i \rangle}(n) = p_{\eta, \ell(i), m_{\eta, \ell(i)}}$ and $\underset{\sim}{f}_{\eta \char`\^ \langle i \rangle}$ be $\underset{\sim}{f}_{\eta, \ell(i)})$.

Lastly let $q_{\eta \char`\^ \langle i \rangle} \in P_{n+1}$ be such that $q_{\eta \char`\^ \langle i \rangle} \restriction = q_\eta$, $q_{\eta \char`\^ \langle i \rangle}$ above $p_{\eta \char`\^ \langle i \rangle}$ and is (N, P_{n+1})-generic (possible as in the proof of preservation of properness by iteration. $\square_{1.17}$

§2. Examples

In this section we use the machinery from the previous section. First (2.1–2.7) we try to restate the results in a way easier to apply by putting more of the common part of the examples in the general results, but you can deal directly with the examples i.e. you can essentially ignore 2.4-2.5, start with 2.7, and use 1.15 (instead 2.1 - 2.5) but have to check somewhat more. Then we deal with several properties which we call: ${}^\omega\omega$-bounding property, Sacks property, Laver property, (f, g)-bounding and more. Several have been used (explicitly or implicitly) and we show that their preservation by countable support iteration follows from 1.13A (so actually from 1.12; really we use 1.15). We usually present the "classical" examples of such forcing.

Names (Sacks, Laver) come from the forcing which seems to be "the example" of a forcing with this property. However as Judah comments, maybe "Sacks property" is confusing as Sacks's forcing satisfies a stronger condition. For simplicity:

2.0 Convention. Forcing notions are from the first case of 0.1 (e.g. proper) and $V^\dagger$ subuniverse of V means, if not said otherwise, $V = (V^\dagger)^Q$, Q as above.

2.1 General Discussion and Scheme.

For usual notions we have two variants of the preservation theorem. We first define a family K of candidates for covering models, usually they have all the same definition, φ but applied in *some* subuniverse $V^\dagger$ (with the same $\aleph_1$) and we get $\varphi(V^\dagger)$, and demand that it is a weak covering model (or a family of covering models; this restricts the family of $V^\dagger$; we can further restrict ourselves to the case $V = V^\dagger[G]$ where G is a subset of some forcing notion $P \in V^\dagger$ generic over V). We then write $K = K_\varphi$ (φ - the definition, possibly with parameters). Then we prove:

(A) any model from K_φ is actually a temporary fine covering model.

So

(B) if $(D, R, <) \in (K_\varphi)^V$ still covers in V^P *then* it is (in V^P) still a temporary fine covering model.

This implies that

(C) if $\bar{Q} = \langle P_j, Q_i : j \le \alpha, i < \alpha \rangle$ is an iteration as in 0.1, α a limit ordinal, $(D, R, <) \in K_\varphi$ in V and for every $\beta < \alpha$ we have $\Vdash_{P_\beta}$ "$(D, R, <)$ still covers, so it is a weak covering model" *then* $(D, R, <)$ covers in V^{P_α}.

But we may want a nicer preservation theorem in particular dealing with the composition of two.

2.1A Definition. 1) For a formula $\varphi = \varphi_x$ (possibly with a free parameter x) defining for any universe $V^\dagger$ which satisfies $x \in V^\dagger$ a weak covering model $\varphi_x[V^\dagger]$ (the definition in $V^\dagger$) and a property Pr of forcing notions, we do the following. Let

$$K_\varphi^{Pr} = \varphi^{Pr}(V) = \{\varphi_x[V^\dagger] : V^\dagger \text{ a subuniverse of } V,\ V = (V^\dagger)^Q \text{ for some forcing notion } Q \text{ satisfying } \mathrm{Pr}, x \in V^\dagger, \varphi_x[V^\dagger] \text{ covers in } V, \text{ so } Q \text{ is } \varphi_x[V^\dagger]\text{-preserving}\},$$

so $\varphi_x[V]$ is a member of $\varphi^{Pr}(V)$. We omit Pr if Q fits into the appropriate case of 0.1 (see 2.0); for simplicity we concentrate on this case[†].

2) A forcing notion P is K_φ^{Pr}-preserving or φ-preserving *if* it preserves each $(D, R, <) \in K_\varphi^{Pr}$. We may add "purely" to all of them.

3) Writing $D^\varphi, R^\varphi, <^\varphi$ we mean $\varphi[V] = (D^\varphi, R^\varphi, <^\varphi)$; if φ has a free parameter x and a fixed parameter t we write $\varphi_t[V; x]$, or $\varphi_{t,x}[V]$.

[†] it is reasonable to deal only with Pr preserved by the relevant iterations, and everything is similar.

2.2 Restatement of Definition.

1) φ is a temporarily definition of weak covering models if (each instance satisfies):

(α) (a) 1.1(a)

(b) 1.1(b) (i.e. $\varphi[V^\dagger]$ covers in the $V^\dagger$ in which we define)

1A) φ is a temporary fine definition of covering models if (α) (above) and in addition:

(β) 1.2(1)(β)

(γ) 1.2(1)(γ) i.e. $\varphi[V^\dagger]$ satisfies it in $V^\dagger$

2) φ is a fine definition of covering models if in addition:

(δ) if $Q \in V$ is $\varphi(V)$-preserving (i.e. each member of $\varphi(V)$ covers in V^Q i.e. $(\alpha)(b)$ holds also in V^Q) *then* in V^Q still each member of $\varphi(V^Q)$ satisfies 1.2(1)(γ).

3) φ is a finer definition of covering models (for simplicity with no free parameter) *if* in addition:

$(\alpha)(c)$ $<^\varphi$ is absolute for φ-preserving extensions i.e. if V^1 is a class of $V^2, \varphi(V^1)$ covers in V^2 (remember 2.1A(3) and 2.0), $x, y \in V^1$ then: $V^1 \models x <^\varphi y$ iff $V^2 \models x <^\varphi y$. Similarly for D^φ, R^φ.

(ε) if Q is $\varphi(V)$-preserving, $\varphi[V] \models$ "$y < x$", and $T^* \in V^Q$ and $\varphi[V^Q] \models yRT^*$ *then* for some $T^{**} \in V$ we have: $T^* \subseteq T^{**}$ and $\varphi[V] \models xRT^{**}$ moreover

$(\varepsilon)^+$ like (ε) above but Q is demanded only to be $\varphi[V]$-preserving.

4) φ is a finest definition of covering models if in addition:

(ζ) if Q is $\varphi(V)$-preserving, and $x \in \mathrm{Dom}(R^\varphi[V^Q])$ *then* there is a $y \in \mathrm{Dom}(R^\varphi[V])$, such that $\varphi[V^Q] \models y < x$.

5) the φ-covering model is $\varphi[V]$; a φ-covering model is a $\varphi[V^\dagger]$ for an appropriate subuniverse $V^\dagger$ so it belongs to $\varphi(V)$.

6) $(D, R, <)$ is 2-directed when: if $y < x, yRT_1, yRT_2$ (so $x, y, T_1, T_2 \in D$) *then* for some T, xRT and $T_1 \cup T_2 \subseteq T$. We say φ is 2-directed if every $\varphi[V]$ is (see 1.2(1)(β)(iv) and 1.3(5)).

2.3 Restatement of Theorems.

1) If φ is a fine definition of covering models, $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha \rangle$ is an iteration as in $0.1_{\theta=\aleph_0}$ and $\underset{\sim}{Q}_j$ is purely φ-preserving for $j < \alpha$ *then* P_α is purely φ-preserving, hence: $\varphi[V]$ covers and φ is a fine definition of covering models, even in V^{P_α}.

2) If φ is a finer definition of covering models, $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha \rangle$ is as in $0.1_{\theta=\aleph_0}$ and

   (∗) each $\underset{\sim}{Q}_j$ is purely $\varphi[V^{P_j}]$-preserving (and (see 0.1) $\underset{\sim}{Q}_j$ has pure $(\aleph_0, 2)$-decidability)

   then P_α is purely $\varphi[V]$-preserving.

3) In (2) we can weaken (∗) to

   $(*)^-$ for $i \leq j < \alpha, i$ non limit we have P_{j+1}/P_i is purely $\varphi[V^{P_i}]$-preserving

4) fine $\Leftarrow$ finer.

5) finer $\Leftarrow$ finest.

6) If φ is a finer definition of covering models, and Q is $\varphi[V]$-preserving *then* Q is $\varphi(V)$-preserving.

7) We can replace pure $(\aleph_0, 2)$-decidability by "pure (2,2)-decidability" if each $\varphi(V')$ is as in 1.12(∗).

Proof: Straightforward. E.g.

6) Suppose $\varphi[V'] = (D', R', <') \in \varphi(V)$, so $V = (V')^{Q'}$, Q' as in 0.1 and $(D', R', <')$ covers in V too. Suppose further that $p \in Q$ and $p \Vdash$ "$\underset{\sim}{f} \in {}^\omega\omega$" and $x \in \mathrm{Dom}(D')$; choose $y \in \mathrm{Dom}(R')$, $y <' x$.

By clause (α)(c) (see 2.2(3)) $\varphi[V] \models$ "$y < x$" (and $x, y \in \mathrm{Dom}(R^{\varphi[V]})$. As Q is purely $\varphi[V]$-preserving there are q and T_1 such that: $p \leq_{\mathrm{pr}} q \in Q$, $T_1 \in \mathrm{Dom}(R^{\varphi[V]})$, $\varphi[V] \models$ "yRT_1" and $q \Vdash$ "$\underset{\sim}{f} \in \lim(T_1)$". By clause $(\varepsilon)^+$ (see 2.2(3)) there is $T_0 \in \mathrm{Rang}(R')$ such that $xR'T_0$, $T_1 \subseteq T_0$. So q, T_0 are as required. $\square_{2.3}$

We can save somewhat using: (we shall usually use 2.4(2))

2.4 Claim.

1) Suppose (i) $(D, R, <)$ is a temporarily fine covering model in V, and:

(ii) V is a subuniverse of $V^\dagger$ and $(D, R, <)$ covers in $V^\dagger$, or just $V^\dagger = V^Q, Q$ is $(D, R, <)$-preserving,

(iii) every countable $a \subseteq D$ from $V^\dagger$ is a subset of some countable $b \in V$ (e.g., Q is proper or: Q preserves $\aleph_1, V \models$ "$|D| \leq \aleph_1$"),

(iv)* there are one-to-one functions $h_n : \omega \to \omega$ such that $h_n \restriction n = h_{n+1} \restriction n$, and $(\mathrm{Rang}(h_n)) \cap (\mathrm{Rang}(h_m)) \subseteq \mathrm{Rang}(h_n \restriction \mathrm{Min}\{n, m\})$ and: for every $x \in D$ for some $y \in D$, for every T_1 such that yRT_1 there is T_0, xRT_0 such that: $\eta \in \lim T_1$ implies $\langle \eta(h_n(\ell)) : \ell < \omega \rangle \in \lim T_0$ for infinitely many n. In fact $\langle h_n : n < \omega \rangle$ may depend on x.

Then $(D, R, <)$ *is a temporarily fine covering model in* $V^\dagger$.

2) We can replace (iv)* by

(iv)** there are an infinite $w \subseteq \omega$ and functions $h_n : \omega \to \omega$ and a sequence $\langle (g_k, \bar{f}^k) : k < \omega \rangle$ such that

(α) $h_n \restriction n = h_{n+1} \restriction n$

(β) for $k < \omega$ the set $v_k \stackrel{\text{def}}{=} \{(n, \ell) : \ell \geq n - 1, n < \omega \text{ and } h_n(\ell) = k\}$ is finite, g_k is a function from ${}^{(v_k)}\omega$ to ω, $\bar{f}^k = \langle f^k_{(n,\ell)} : (n, \ell) \in v_k \rangle$, $f^k_{(n,\ell)} : \omega \to \omega$ such that $f^k_{(n_0,\ell_0)}(g_k(\ldots, m_{(n,\ell)}, \ldots)_{(n,\ell) \in v_k}) = m_{(n_0,\ell_0)}$

(γ) for every $x \in \mathrm{Dom}R$ for some $y \in \mathrm{Dom}R$ we have: if yRT_1 then there is T_0 satisfying xRT_0 and

$$(\forall \eta \in \lim T_1)(\exists^\infty n)[\langle f^{h_n(\ell)}_{(n,\ell)}(\eta(h_n(\ell))) : \ell < \omega \rangle \in \lim T_0]$$

3) We can replace (iv)* by

(iv)*** for every $x \in \mathrm{Dom}R$ for some Borel function $\mathbf{B}$ from $\{\langle \eta_\alpha : \alpha \leq \omega \rangle : \eta_\alpha \in {}^\omega\omega \text{ and } \eta_\omega \restriction n = \eta_n \restriction n\}$ into ${}^\omega\omega$, *there is* $y \in \mathrm{Dom}R$ such that: for every T_1 satisfying yRT_1 there is T_0 satisfying xRT_0 such that

$$\langle \eta_\alpha : \alpha \leq \omega \rangle \in \mathrm{Dom}(\mathbf{B}) \;\&\; \mathbf{B}(\langle \eta_\alpha : \alpha \leq \omega \rangle) \in \lim T_1$$
$$\Rightarrow (\exists^\infty n)(\eta_\alpha \in \lim T_0).$$

Remark: Applying 2.4 we may wonder if (iii) is a burden. At first glance, if $V^\dagger = V^Q$, Q not proper, this may be so. But actually we need it only for the limit cases, and there in the cases of iteration of non-proper forcing notions, we usually assume that in some earlier stage the cardinality of D becomes $\aleph_1$.

Before proving 2.4, we make some observations of some interest, among them a proof.

2.4A Observation. If $\bigwedge_n T_n \subseteq T_{n+1}$ and T_n, T are perfect subtrees of ${}^{\omega>}\omega$ and w is a witness for $(*)^1_{\langle T_n:n<\omega\rangle,T}$ *then* u is a witness for $(*)^1_{\langle T_n:n<\omega\rangle,T}$ if $\otimes$ holds, where

$(*)^1_{\langle T_n:n<\omega\rangle,T}$ T_n, T perfect trees $\subseteq {}^{\omega>}\omega$, $T_0 \subseteq T$ and for some $w = \{n_0, n_1, \ldots\}$ (strictly increasing called a witness): $\eta \in {}^{\omega>}\omega$, $\bigwedge_i [\eta{\restriction}n_{i+1} \in \bigcup_{j<i} T_{n_j} \cup T_0] \Rightarrow \eta \in T$,

$\otimes$ $u \subseteq \omega$ is infinite and: if $i_0 < i_1 < i_2$ are successive members of w *then* $|u \cap (i_0, i_2)| \leq 1$ and the second member of w is smaller than the second member of u.

Proof. Let $w = \{n_i : i < \omega\}$, and $u = \{m_i : i < \omega\}$, both in increasing order. Assume $\eta \in {}^{\omega>}\omega$ and $\bigwedge_i \eta{\restriction}m_{i+1} \in \bigcup_{j<i} T_{m_j} \cup T_0$ and it suffices to prove that $\bigwedge_i \eta{\restriction}n_{i+1} \in \bigcup_{j<i} T_{n_j} \cup T_0$. As each T_j is perfect without loss of generality $\ell g(\eta) = n_{i(*)}$ for some $i(*) > 0$, and we shall prove by induction on $i < i(*)$ that $\eta{\restriction}n_{i+1} \in \bigcup_{j<i} T_{n_j} \cup T_0$. For $i = i(*) - 1$ we will get the desired conclusion by the choice of w. For $i = 0$ we have $\bigcup_{j<i} T_{m_j} \cup T_0 = T_0 = \bigcup_{j<i} T_{n_j} \cup T_0$ so as $m_1 \geq n_1$ the conclusion should be clear.

For $i + 1 > 1$, as $|u \cap (n_{i-1}, n_{i+1})| \leq 1$ (holds by $\otimes$), if $u \cap [0, n_{i-1}] = \emptyset$ then by $u \cap [0, n_{i+1})$ has at most one member hence $m_1 \geq n_{i+1}$ and we do as above. So there is $j < \omega$ such that $m_{j+1} \geq n_{i+1}$, $m_{j-1} \leq n_{i-1}$. Now we know $\eta{\restriction}m_{j+1} \in \bigcup_{\varepsilon<j} T_{m_\varepsilon} \cup T_0$, so if $\eta{\restriction}m_{j+1} \in T_0$ then $\eta{\restriction}n_{i+1} \trianglelefteq \eta{\restriction}m_{j+1} \in T_0 \subseteq \bigcup_{\varepsilon<i} T_{n_\varepsilon} \cup T_0$ and we are done. So for some $\varepsilon < j$, $\eta{\restriction}m_{j+1} \in T_{m_\varepsilon}$ hence $\eta{\restriction}n_{i+1} \trianglelefteq \eta{\restriction}m_{j+1} \in T_{m_\varepsilon} \subseteq T_{m_{j-1}} \subseteq T_{n_{i-1}} \subseteq \bigcup_{\zeta<i} T_{n_\zeta} \cup T_0$ as required. $\square_{2.4A}$

2.4B Observation. Suppose $h : \omega \to \omega$ is one to one (or just finite to one), T_n, S_n, T are perfect subtrees of ${}^{\omega>}\omega$, $\bigwedge_n S_n \subseteq S_{n+1}, T_0 \subseteq S_0$ and for each n for some $m \in [n, \omega)$ we have $(*)^2_{T_{h(n)}, S_n, S_{m+1}}$ holds (see below).

Then $(*)^1_{\langle S_n:n<\omega\rangle,T} \Rightarrow (*)^1_{\langle T_n:n<\omega\rangle,T}$ where:

$(*)^1_{\langle T_n:n<\omega\rangle,T}$ T_n, T perfect trees $\subseteq {}^{\omega>}\omega$, $T_0 \subseteq T$ and for some $w = \{n_0, n_1, \dots\}$ (strictly increasing): $\eta \in {}^{\omega>}\omega$, $\bigwedge_i [\eta{\restriction}n_{i+1} \in \bigcup_{j<i} T_{n_j} \cup T_0] \Rightarrow \eta \in T$,

$(*)^2_{T_1,T_2,T_3}$ for some $k < \omega$ (the witness) $\rho \in T_1$ & $\rho{\restriction}k \in T_2 \;\Rightarrow\; \rho \in T_3$.

Remark. Note $(*)^2_{T_1^*,T_2^*,T_3^*}$ & $T_1' \subseteq T_1^*$ & $T_2' \subseteq T_2^*$ & $T_3^* \subseteq T_3' \Rightarrow (*)^2_{T_1',T_2',T_3'}$

Proof. We want to prove $(*)^1_{\langle T_n:n<\omega\rangle,T}$, so we have to find an appropriate w. Let $w_1 = \{n_i : i < \omega\}$ (the increasing enumeration) witness $(*)^1_{\langle S_n:n<\omega\rangle,T}$ and for $j < \omega$ let k_j be such that it witnesses $(*)^2_{T_{h(j)},S_j,S_m}$ (for the first possible $m > n$, note that $(*)^2_{T_{h(j)},S_j,S_{m+1}}$ is preserved by increasing m as $S_m \subseteq S_{m+1}$). By 2.4A above without loss of generality

$\oplus$ $\bigwedge_i n_i \in \mathrm{Rang}(h)$, and $k_{n_i} < n_{i+1}$ and $(\forall k)(h(k) \leq n_i \Rightarrow k < n_{i+1})$, hence $n_i < h(n_{i+1}) < n_{i+2}$, and also for some $m \in (n_i, n_{i+1})$ we have $(*)^2_{T_{h(n_i)},S_{n_i},S_m}$ we get $(*)^2_{T_{h(n_i)},S_{n_i},S_{n_{i+1}}}$).

Choose $m_i = h(n_{4i+4})$. Now we shall prove that $w \stackrel{\text{def}}{=} \{m_i : i < \omega\}$ is a witness to $(*)^1_{\langle T_n:n<\omega\rangle,T}$ thus finishing the proof of 2.4B. So we assume $\eta \in {}^{\omega>}\omega$, $\bigwedge_i \eta{\restriction}m_{i+1} \in \bigcup_{j<i} T_{m_j} \cup T_0$ and we have to prove that $\eta \in T$. As $w_1 = \{n_i : i < \omega\}$ witnesses $(*)^1_{\langle S_n:n<\omega\rangle,T}$, it suffices to prove: for each $i < \omega$ we have $\eta{\restriction}n_{i+1} \in \bigcup_{j<i} S_{n_j} \cup S_0$.

We prove it by induction on i. If $n_{i+1} \leq m_1$ then as $\eta{\restriction}m_1 \in T_0$, $T_0 \subseteq S_0$, T_0 is perfect, clearly $\eta{\restriction}n_{i+1} \in S_0 \subseteq \bigcup_{j<i} S_j \cup S_0$. But $n_{i+1} \leq m_1$ holds if $n_{i+1} \leq h(n_8)$ what implies $i < 9$. So we assume $i \geq 9$. Let $4i(*) + 2 \leq i < 4(i(*) + 1) + 2$ (so $i(*) \geq 1$). So by the assumption $\otimes$, we have $\eta{\restriction}n_{i+1} \lhd \eta{\restriction}h(n_{4(i(*)+1)+4}) = \eta{\restriction}m_{i(*)+1} \in \bigcup_{j<i(*)} T_{m_j} \cup T_0$. Stipulating $m_{-1} = 0$, for some $j(*) \in \{-1, 0, \dots, i(*) - 1\}$ we have $\eta{\restriction}m_{i(*)+1} \in T_{m_{j(*)}}$. If $j(*) = -1$, then $\eta{\restriction}n_{i+1} \lhd \eta{\restriction}m_{i(*)+1} \in T_0 \subseteq S_0 \subseteq \bigcup_{j<i} S_{n_j} \cup S_0$ as required. So assume $j(*) \in \{0, \dots, i(*) - 1\}$. But we know that $(*)^2_{T_{m_{j(*)}},S_{n_{4j(*)+4}},S_{n_{4j(*)+5}}}$ [Why? As by the definition $m_{j(*)} = h(n_{4j(*)+4})$, and by $\oplus$ above], and we want to apply it to $\rho \stackrel{\text{def}}{=} \eta{\restriction}m_{i(*)+1}$. The first assumption of $(*)^2_{T_{m_{j(*)}},S_{n_{4j(*)+4}},S_{4j(*)+5}}$ was deduced above: $\rho = \eta{\restriction}m_{i(*)+1} \in T_{m_{j(*)}}$. The second assumption there is

$\rho{\restriction}k_{n_{4j(*)+4}} \in S_{n_{4j(*)+4}}$ (by the choice of $k_{n_{4j(*)+4}}$), now we know $j(*) < i(*)$ and $\rho{\restriction}k_{n_{4j(*)+4}} = \eta{\restriction}k_{n_{4j(*)+4}} \lhd \eta{\restriction}n_{4j(*)+5} \in S_{n_{4j(*)+4}}$

[Why? First, the equality holds as:

(a) $\rho = \eta{\restriction}m_{i(*)+1}$

(b) $k_{n_{4j(*)+4}} \leq m_{i(*)+1}$, because $m_{i(*)+1} = h(n_{4(i(*)+1)+4}) > n_{4(i(*)+1)+3}$ $= n_{4i(*)+7} \geq n_{4(j(*)+1)+7} > n_{4j(*)+6} > k_{n_{4j(*)+4}}$ (why? by definition of $m_{i(*)+1}$, by $\oplus$, arithmetic, as $i(*) \geq j(*)$, arithmetic and $\oplus$ respectively).

Secondly, the $\lhd$ holds as $k_{n_{4j(*)+4}} \leq n_{4j(*)+5}$ by $\oplus$.

Finally, the membership holds - by the induction hypothesis on i, and $\langle S_n : n < \omega\rangle$ being increasing, note the induction hypothesis can be applied as $j(*) < i(*)$ hence $4j(*) + 5 \leq 4(i(*) - 1) + 5 = 4i(*) + 1 < i$].

So we can actually apply $(*)^2_{T_{m_{j(*)}}, S_{n_{4j(*)+4}}, S_{n_{4j(*)+5}}}$ and get $\rho = \eta{\restriction}m_{i(*)+1}$ belongs to $S_{n_{4j(*)+5}}$. As $\eta{\restriction}n_{i+1} \lhd \eta{\restriction}m_{i(*)+1} = \rho$ (see above) and $4j(*) + 5 \leq 4(i(*) - 1) + 5 < 4i(*) + 2 \leq i$, really $\eta{\restriction}n_{i+1} \in \bigcup_{j<i} S_{n_j} \cup S_0$ as required, thus we have finished. $\square_{2.4B}$

2.4C Observation. If $(*)^1_{\langle T_n : n<\omega\rangle, T}$ holds as witnessed by w, and $\bigwedge_n T_n \subseteq T_{n+1}$ and $h : \omega \to \omega$ is such that $\mathrm{Rang}(h)$ is infinite, $h(0) = 0$ and we let $T^1_n \stackrel{\mathrm{def}}{=} T_{h(n)}$ *then* $(*)^1_{\langle T^1_n : n<\omega\rangle, T}$.

Proof. Let $u \subseteq \omega$ be infinite such that $h{\restriction}u$ is one to one (possible as $\mathrm{Rang}(h)$ is infinite), $h{\restriction}u$ is strictly increasing and for $i < j$ in u, $h(i) < j$ & $i < h(j)$, moreover, $|(h(i), j) \cap w| \geq 2$. Now u is as required by 2.4A above. $\square_{2.4C}$

2.4D Observation. In 1.2(1)(γ)(a) we can add the assumption $T_n \subseteq T_{n+1}$ and get an equivalent condition (assuming 1.2(1)(α), (β) of course).

Proof. Of course we only need to assume this apparently weaker version and prove the original version. Let $x_0 < \ldots < x_n < x_{n+1} < \ldots y^+ < y$, $x_n R T^0_n$ be given. We define by induction on n, T^1_n such that: $T^1_n \subseteq T^1_{n+1}$, $T^1_0 = T^0_0$, $x_{n+1} R T^1_n$ and $(*)^2_{T^0_n, T^1_n, T^1_{n+1}}$ (possible by 1.2(1)(β)(iv) which says: if $y < x$, $y R T_\ell$ then $(\exists T)[T_1 \subseteq T$ & $(*)^2_{T_2, T_1, T}]$). So $T^1_n \subseteq T^1_{n+1}$ and (as we are assuming the weaker version of 1.2(1)(γ)(a)) $(*)^1_{\langle T^1_n:n<\omega\rangle,T}$ holds for some T such that yRT. By 2.4B, (with $S_n \overset{\text{def}}{=} T^1_n$, $T_n \overset{\text{def}}{=} T^0_n$ and $h(n)=n$) we get $(*)^1_{\langle T^0_n:n<\omega\rangle,T}$ as required. $\square_{2.4D}$

2.4E Observation. If V, $V^\dagger$, $(D,R,<)$ satisfy conditions (i), (ii), (iii) of claim 2.4(1) *then* $(D,R,<)$ satisfies $(\gamma)(a)$ of 1.2(1) also in $V^\dagger$.

Proof: Let $x > x^\dagger > y_{n+1} > y_n$ for $n<\omega$ and $T_n \in V$ be such that y_nRT_n (but the sequence $\langle T_n : n<\omega\rangle$ may be from $V^\dagger$. Let b be a countable set from V such that $\{T_n : n<\omega\} \subseteq b \subseteq V$. Let $\langle S^0_n : n<\omega\rangle \in V$ enumerate $\{T \in b : (\exists y)(y<x^\dagger \& yRT)\}$, so $\{T_n : n<\omega\} \subseteq \{S^0_n : n<\omega\}$. Without loss of generality $S^0_0 = T_0$ and for each n for infinitely many m we have $S^0_m = S^0_n$. By 1.2(1)(β) we can find in V a sequence $\langle z_n, S^1_n, k_n : n<\omega\rangle, z^\dagger$ such that $x^\dagger < z_n < z_{n+1} < z^\dagger < x$ for $n<\omega$ (of course $S^1_n \in V$), $k_n<\omega$ such that $z_nRS^1_n$, and $S^1_0 = T_0, S^1_n \subseteq S^1_{n+1}$ and $[\rho \in S^0_n \ \& \ \rho{\restriction}k_n \in S^1_n \Rightarrow \rho \in S^1_{n+1}]$ (choose them inductively). So $(*)^2_{S^0_n,S^1_n,S^1_{n+1}}$; now $(*)^2_{-,-,-}$ has obvious monotonicity properties in its variables (see 2.4B), hence $n_0 \le n < n_1 \Rightarrow (*)^2_{S^0_n,S^1_{n_0},S^1_{n_1}}$. Choose by induction on n, $h(n)$ as

$$\min\{m : T_n = S^0_m \text{ and } m>n \text{ and } m > \sup\{h(k) : k<n\}\},$$

well defined by the choice of $\langle S^0_m : m<\omega\rangle$. So we know $(*)^2_{S^0_{h(n)},S^1_n,S^1_{h(n)+1}}$.

We want to apply 2.4B with $\langle S^1_n : n<\omega\rangle$, $\langle T_n : n<\omega\rangle$, h here standing for $\langle S_n : n<\omega\rangle$, $\langle T_n : n<\omega\rangle$, h there; we have here almost all the assumptions (including h is one to one (even strictly increasing) and $\bigwedge\limits_n \bigvee\limits_{m\ge n} (*)^2_{T_{h(n)},S^1_n,S^1_{m+1}}$) but still need to choose T^* and prove that xRT^* and $(*)^1_{\langle S^1_n:n<\omega\rangle,T^*}$.

Apply (γ)(a) of 1.2(1) in V (which holds by (i)) with $\langle S^1_n : n<\omega\rangle, \langle z_n : n<\omega\rangle, z^\dagger, x$ here standing for $\langle T_n : n<\omega\rangle, \langle y_n : n<\omega\rangle, x^\dagger$, x there, and get T^* (in V!) such that $(*)^1_{\langle S^1_n:n<\omega\rangle,T^*}$ holds and xRT^*. So we can really apply 2.4B hence get that $(*)^1_{\langle T_n:n<\omega\rangle,T^*}$ holds, as required. $\square_{2.4E}$.

2.4F Proof of 2.4(1). From definition 2.2(1A), part (α) and (β) should be clear. By 2.4E we know that (γ)(a) of 1.2(1) holds, so it suffices to prove (γ)(b) of 1.2(1).

Given $x \in \mathrm{Dom}(R)$ and η, $\eta_n \in {}^\omega\omega$ such that $\eta \restriction n = \eta_n \restriction n$, let y and h_n $(n < \omega)$ be as in (iv)*. We can find $\nu \in {}^\omega\omega$ such that: for each $n < \omega$ we have $\eta_n(k) = \nu(h_n(k))$ (note that there is such $\nu \in {}^\omega\omega$ because if $\ell = h_{n_1}(k_1) = h_{n_2}(k_2)$ then $\ell \in \mathrm{Rang}(h_{n_1}) \cap \mathrm{Rang}(h_{n_2})$ hence $k_1, k_2 < \min\{n_1, n_2\}$, so $h_{n_2}(k_2) = h_{n_1}(k_1) = h_{n_2}(k_1)$, but h_{n_2} is one to one so $k_1 = k_2$). As $(D, R, <)$ covers we can find T_1 such that yRT_1 and $\nu \in \lim(T_1)$. Now let T_0 be as guaranteed by (iv)* (of 2.4). $\square_{2.4}$

2.4G Proof of 2.4(2), (3). Similar.

2.5 Claim.

(1) We can get the conclusion of 2.4 and even strengthen it by " in $V^\dagger$ the model $(D, R, <)$ still satisfies $(\gamma)_1$ (see 1.3(8))" if we replace (iv)* by:

  (iv) every $f \in ({}^\omega\omega)^{V^\dagger}$ is dominated by some $g \in ({}^\omega\omega)^V$,

  (v) $(D, R, <)$ satisfies $(\gamma)_1$ of 1.3(8),

  (vi)′ $(D, R, <)$ is 2-directed (see 2.2(6)).

(2) In 2.4(1) we can replace (iv)$^+$ by (iv)$^-$, (v)′ below and (vi)′ above, where

  (iv)$^-$ no $f \in ({}^\omega\omega)^{V^\dagger}$ dominates $({}^\omega\omega)^V$

(v)′$(\gamma)_2$ if $y, x, y^\dagger, x_n, T_n \in D_\varphi[V]$, $x_0 < x_1 < \ldots < y^\dagger < y$, $x_n R T_n$, $T_n \subseteq T_{n+1}$ (for each $n < \omega$) *then* for some $k < \omega$ and $\langle T^\ell : \ell < k\rangle$, $\langle B_\ell : \ell < n\rangle$ we have:

  (a) $\omega = \bigcup\limits_{\ell < k} B_\ell$

  (b) if $n \in B_\ell$, $\eta \in T_n$, $\eta \restriction n \in T_0$ then $\eta \in T^\ell$

  (c) yRT^ℓ.

(3) Assuming (α), (β) of 2.11 we have $(\gamma)_3 \Rightarrow (\gamma)$, $(\gamma)_3 \Rightarrow (\gamma_2)$ where

  $(\gamma)_3$ like $(\gamma)_2$ replacing (b) by

   (b)$^+$ if $\ell < k$, and $(\forall n)(\eta \restriction n \in \bigcup\{T_m : m \leq n \text{ and } m \in B_\ell\})$ implies $\eta \in \lim T^\ell$.

Remark. Can we phrase a maximal $(\gamma)_n$? Like $(\gamma)_2$ but without T_n.

Proof. 1) As in the proof in 2.4F, we have (α), (β), $(\gamma)(\mathrm{a})$ of 1.2(1) and it suffices to prove $(\gamma)(\mathrm{b})$ and $(\gamma)_1$ of 1.3(8), but the latter implies the former. So let $V^\dagger$, $\langle x_n : n < \omega\rangle$, $x^\dagger$, x and $\langle T_n : n < \omega\rangle$ be given as there. So $x_n R T_n$, $\{x^\dagger, x\} \subseteq \mathrm{Dom}(R)$, $x_n < x_{n+1} < x^\dagger < x$. As in the proof of 2.4E we can find $\langle (z_\ell, S^0_\ell) : \ell < \omega\rangle \in V$ such that $\{(x_n, T_n) : n < \omega\} \subseteq \{(z_n, S^0_n) : n < \omega\}$, and w.l.o.g. $n < \omega \Rightarrow z_n R S^0_n$ & $z_n < x^\dagger$. By the 2-directness we can find a sequence $\langle (y_n, S^1_n) : n < \omega\rangle \in V$ such that $y_n < y_{n+1} < x^\dagger$ and $S^1_n = T_0$ and $y_n R S^1_n$ and $S^0_n \cup S^1_n \subseteq S^1_{n+1}$ (possible by (vi)$'$ which). Define $h \in ({}^\omega\omega)^{V^\dagger}$ by $h(n) = \min\{m : T_n = S^0_m\}$ and choose a strictly increasing function $g \in ({}^\omega\omega)^V$ such that $[n < \omega \Rightarrow h(n) < g(n)$ & $n < g(n)]$. By $(\gamma)_1$ of 1.3(8) applied to $\langle S^1_{g(i)} : i < \omega\rangle$ in V there are $T^* \in \mathrm{Rang}(R)$ and infinite $w_1 \subseteq \omega$ such that

$(*)_1$ xRT

$(*)_2$ $\lim(T^*) \supseteq \{\eta : \eta \in {}^\omega\omega \text{ and } i \in w_1 \Rightarrow \eta\restriction i \in \bigcup_{j \in w_1, j \le i} S^1_{g(j)}\}$

Let us prove that T^* and w are as required. So we assume

$(*)_3$ $\eta \in {}^\omega\omega$ and

$$i \in w_1 \Rightarrow \eta\restriction i \in \bigcup_{j \in w_1, j \le i} T_j.$$

We should prove that $\eta \in \lim(T^*)$, but by $(*)_2$ it suffice to prove:

$(*)_4$ $i \in w_1 \Rightarrow \eta\restriction i \in \bigcup_{j \in w_1, j \le i} S^1_{g(j)}$

As $T_j \subseteq S^1_{g(j)}$ this is immediate.

2) As in the proof of 2.4(1) we can deal with conditions (α), (β), $(\gamma)(\mathrm{a})$ (the first two: trivially, the last one by 2.4E). For $(\gamma)(\mathrm{b})$ let η, $\eta_n \in ({}^\omega\omega)^{V^\dagger}$, $\eta_n\restriction n = \eta\restriction n$ and $y \in \mathrm{Dom}(R)$ be given and choose $x^\dagger$; $\langle x_n : n < \omega\rangle$, $\langle T_n : n < \omega\rangle$, $(x = y)$ $\langle S^1_n : n < \omega\rangle$, x_n, $y^\dagger$ as in the proof of 2.5(1), so in particular $\eta \in S^1_0$ and $h(n) = \min\{m : m > n \text{ and } \eta_n \in \lim(S^1_n)\}$ are well defined; note $S^1_n \subseteq S^1_{n+1}$. Let $g \in {}^\omega\omega$ be strictly increasing, $g(0) = 0$ such that $A = \{n : h(n) < g(n)\}$ is infinite. We can find such g by clause (iv)$^-$ of the assumption. Now apply $(\gamma)_2$ to y_n $(n < \omega)$, $x^\dagger$, y, $\langle S^1_{g(n)} : n < \omega\rangle$, and get $k < \omega$, $\langle B_\ell : \ell < k\rangle$, $\langle T^\ell : \ell < k\rangle$ as there (in particular yRT^ℓ). Now for each $n \in A$ for some $\ell(n) < k$, we have $(\nu \in {}^{\omega>}\omega)$ & $\nu\restriction n \in S^1_{g(0)}$ & $\nu \in S^1_{g(n)} \Rightarrow \nu \in T^{\ell(n)}$. So, if $n \in A$, $\eta_n\restriction n = \eta\restriction n \in S^1_0 = S^1_{g(0)}$, $\eta_n \in S^1_{h(n)} \subseteq S^1_{g(n)}$ hence $\eta_n \in T^{\ell(n)}$. So for some $\ell < k$, $\{n \in A : \ell(n) = \ell\}$ is infinite and we are done. $\square_{2.5}$

* * *

2.6 Definition. $TTR = \{T \cap {}^{m\geq}\omega : m < \omega,\ T \subseteq {}^{\omega>}\omega$ a closed tree and for every $n < \omega$ we have $T \cap {}^{n\geq}\omega$ finite $\}$, where "T is a closed tree" means, as usual: $T \neq \emptyset$, $[\eta \in T \ \&\ \nu \lhd \eta \Rightarrow \nu \in T]$, $[\eta \in T \Rightarrow \bigvee_{i<\omega} \eta\hat{\ }\langle i\rangle \in T]$. Note that TTR has a natural tree structure: $t < s$ if $t = s \cap {}^{n\geq}\omega$ for some n. For $t \in TTR$ let $\operatorname{ht}(t) = \min\{n : t \subseteq {}^{n\geq}\omega\}$ and $TTR_n = \{t \in TTR : \operatorname{ht}(t) = n\}$.

2.6A Notation. $DP({}^{\omega}\omega) = \{x \in {}^{\omega}\omega : x(n) \geq 1$ for every n and $\langle x(n) : n < \omega\rangle$ diverges to infinity, i.e. for every $m < \omega$ for some $k < \omega$, for every $n \geq k, x(n) \geq m.\}$

2.6B Remark. Usually we can replace x by x', $x'(n) = \operatorname{Min}\{x(m) : n \leq m < \omega\}$, hence without loss of generality x is nondecreasing.

2.7 Fact. Each closed tree $T \subseteq {}^{\omega\geq}\omega$ such that $(\forall n)[|T \cap {}^{n\geq}\omega| < \aleph_0]$ induces a branch $\{T \cap {}^{n\geq}\omega : n < \omega\}$ (in the tree TTR) and is its union. Now TTR is isomorphic to ${}^{\omega>}\omega$.

* * *

Now we deal with some examples: we do not state the aim - the preservation theorems by combining with 2.1-2.7 - for each φ_i separately but usually we mention the case of CS iteration of proper forcing.

2.8A Definition. [${}^{\omega}\omega$-bounding]: 1) We define $\varphi = \varphi_1^{cm}$ (a definition of covering models) by letting $\varphi[V] = (D, R)$ if:

a) $D = H(\aleph_1)^V$

b) xRT iff $x, T \in D$, $x \in DP({}^{\omega}\omega)$, T is a closed tree and $(\forall n)(T \cap {}^{n}\omega$ is finite) (so x has really no role)

c) $<=<_0$ (see 1.4)

2) A forcing notion P is ${}^{\omega}\omega$-bounding (in V) if it is φ_1^{cm}-preserving (see 2.8C-equivalent to the definition from V).

2.8B Claim. $\varphi = \varphi_1^{cm}$ is a finest definition of covering models for proper forcing, and it is 2-directed.

Remark.

1) Instead of "proper" we can use "forcing Q such that every countable subset of $\min\{(2^{\aleph_0})^{V^1} : \varphi^{cm}[V^1] \text{ covers } V\}$ from V^Q is included in one from V".
2) We just "forget" to mention the pure version.

Proof: Let us check the conditions in Definition 2.2, the 2-directed should be clear.

(α) (a) Trivial by a), b) of Definition 2.8A.

(α) (b) Trivial (a tree with one branch).

(α) (c) and (β) are trivial.

(γ), $(\gamma)^+$ Let $x > y, yRT_n$ (remember 1.3(4)).

Put $w \stackrel{\text{def}}{=} \omega$, $T^* \stackrel{\text{def}}{=} \{\eta$: for every $k < \omega, \eta \restriction k \in \bigcup_{j \le k} T_j\}$ and check that T^* is as required.

(δ) By 2.4 for (γ), 2.5(1) for $(\gamma)^+$.

$(\varepsilon)^+$ Now we use Fact 2.7 applied to T^*, (i.e. to the branch of TTR which T^* induced). So there is a closed tree $\mathcal{C} \subseteq TTR, \mathcal{C} \in D, xR\mathcal{C}$ and for every $n, T^* \cap {}^{n>}\omega \in \mathcal{C}$. Let $T^{**} = \{\eta \in {}^{\omega>}\omega$: for some $t \in \mathcal{C}, \eta \in t\}$. Clearly $T^{**} \in D$ (as $\mathcal{C} \in D, D = H(\aleph_1), T^{**}$ is a closed tree $\subseteq {}^{\omega>}\omega$), and $T^* \subseteq T^{**}$. In addition, for every n

$$T^{**} \cap {}^{n}\omega = \bigcup\{t \cap {}^{n}\omega : t \in \mathcal{C} \cap TTR_{n+1}\},$$

so, being a finite union of finite sets, $T^{**} \cap {}^{n}\omega$ is finite.

(ζ) Easy. $\square_{2.8B}$

2.8C Fact. If $V \subseteq V^\dagger$ then $\varphi_1^{cm}[V]$ covers in $V^\dagger$ *if and only if*

$$(\forall f \in ({}^{\omega}\omega)^{V^\dagger})(\exists g)(f <^* g \in ({}^{\omega}\omega)^V)$$

if and only if

$$(\forall y \in \mathrm{DP}\,({}^{\omega}\omega)^{V^\dagger})(\exists x \in \mathrm{DP}\,({}^{\omega}\omega)^V)[y <^*_{\text{dis}} x]$$

if and only if

$$(\forall y \in \mathrm{DP}\,({}^{\omega}\omega)^{V^{\dagger}})(\exists x \in \mathrm{DP}\,({}^{\omega}\omega)^{V})[y <_{\mathrm{dis}} x].$$

2.8D Conclusion. For a CS iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha \rangle$ if $\Vdash_{P_j}$ "$\underset{\sim}{Q}_j$ is proper and ${}^{\omega}\omega$-bounded" for each i *then* P_α is proper and ${}^{\omega}\omega$-bounding.

Proof. By 2.3(2) and 2.8B + 2.8C.

* * *

2.9A Definition. [The Sacks property] Define $\varphi = \varphi_2^{cm}$ (a definition of a covering model) by letting $\varphi[V] = (D, R)$ be

a) $D = H(\aleph_1)$

b) xRT iff $x, T \in D$ and $x \in DP({}^{\omega}\omega), T \subseteq {}^{\omega>}\omega$ and for every $n < \omega$, $T \cap {}^{n}\omega$ has at most $x(n)$ elements.

c) $<=<_{\mathrm{dis}}$ (see 1.4).

2.9B Claim. $\varphi = \varphi_2^{cm}$ is a finest definition of covering models.

Proof. Let us check the conditions in Definition 2.2.

(α) (a) Trivial by a), b) of Definition 2.9A.

(α) (b) Trivial.

(α) (c) Trivial.

(β) Trivial, by the definition of the partial order (1.4).

$(\gamma)^+$ Let $y_n R T_n$ and $y_n <_{\mathrm{dis}} y_{n+1} <_{\mathrm{dis}} x^{\dagger} <_{\mathrm{dis}} x$ (for $n < \omega$). Define n_k inductively as the first $n < \omega$ such that $\ell < k \Rightarrow n_\ell < n$ and for every ℓ, $n \leq \ell < \omega$, we have $(k+2) \cdot x^{+}(\ell) \leq x(\ell)$. Let $w = \{n_k : k < \omega\}$ and

$$T^* = \{\eta : n \in w \Rightarrow \eta \restriction n \in \bigcup_{\substack{m \leq n \\ m \in w}} T_n\}.$$

(δ) Immediate by 2.4(1).

$(\varepsilon)^+$ There is by 2.7 a closed tree $\mathcal{C} \subseteq TTR$ in D, $T^* \cap {}^n\omega \in \mathcal{C}$, $zR\mathcal{C}$ where $z(m) = x(m)/y(m)$. Let $\mathcal{C}_1 = \{t \in \mathcal{C}$: for every $n, |t \cap {}^n\omega| \leq y(n)\}$ and let $\mathcal{C}_2$ be the maximal closed tree $\subseteq \mathcal{C}_1$. Clearly $\mathcal{C}_2 \in D$ and $T^* \cap {}^n\omega \in \mathcal{C}_2$ for every n, now $T^{**} = \bigcup\{t : t \in \mathcal{C}_2\} \in D$ is as required.

(ζ) Easy by 2.8C. $\square_{2.9B}$

2.9C Claim.

1) If $V \subseteq V^\dagger$, $\varphi_2^{cm}[V]$ covers in $V^\dagger$ *iff* for every $\eta \in ({}^\omega\omega)^{V^\dagger}$ and $y \in DP({}^\omega\omega)^V$ there is $\langle a_\ell : \ell < \omega\rangle \in V$, $a_\ell \subseteq \omega$, $|a_\ell| \leq y(\ell)$ and $\bigwedge_\ell \eta(\ell) \in a_\ell$.

2) If $V \subseteq V^\dagger$ and $\varphi_2^{cm}[V]$ covers in $V^\dagger$ *then* $\varphi_1^{cm}[V]$ covers in $V^\dagger$.

Proof. Straight.

2.9D Conclusion. For a CS iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha\rangle$ if $\Vdash_{P_j}$ "$\underset{\sim}{Q}_j$ is proper and has the Sacks property" *then* P_α is proper and has the Sacks property.

Proof. By 2.3(2) and 3.9B + 2.9C.

* * *

2.10A Definition. [The Laver Property] 1) We define $\varphi = \varphi_3^{cm}$ by letting $\varphi[V] = (D, R, <)$ (the Laver model) be

a) $D = H(\aleph_1)^V$.

b) xRT iff $(x, T \in D$ and) $x \in DP({}^\omega\omega)$, $T \subseteq {}^{\omega>}\omega$ a closed tree and: $(\forall n)$ [the set $\{\eta(n) : \eta \in T, \ell g(\eta) = n+1, (\forall i \leq n)\eta(i) < x(2i)\}$ has power $\leq x(2n+1)$].

c) $x < y$ iff $\langle x(2n+1) : n < \omega\rangle <_{\text{dis}} \langle y(2n+1) : n < \omega\rangle$ (see 1.4B) and $\langle x(2n) : n < \omega\rangle = \langle y(2n) : n < \omega\rangle$.

2.10B Claim. $\varphi = \varphi_3^{cm}$ is a finest definition of a covering model.

Proof. It can be proved very similarly to the proof of 2.9B. The proof of (α), (β) is totally trivial and (δ) follows from (γ) by 2.4(1), so we shall prove $(\gamma)^+$.

Let $x > y > y_{n+1} > y_n$, T_n, be given and $y_n R T_n$.

We can choose $n_0 < n_1 < n_2 < \dots$ (by induction) such that: for $k \geq n_\ell$, $(\ell+2) \times y(2k+1) < x(2k+1)$, and let $w = \{n_\ell : \ell < w\}$,

$T^* = T_0 \cup \{\eta : \text{for every } i \in w, \eta \restriction i \in \bigcup_{\substack{j \leq i \\ j \in w}} T_j\}$

$T^0 = \{\eta \in {}^{\omega>}\omega : \text{for every } i < \ell g(\eta), \eta(i) < x(2i)\}$

Clearly $T^* \subseteq {}^{\omega>}\omega$ is a closed tree, and for any k, $|T^0 \cap T^* \cap {}^k\omega| \leq x(2k+1)$, because, letting $n_\ell \leq k < n_{\ell+1}$, $\{\eta(k) : \ell g\eta > k, \eta \in T^0 \cap T^*\}| \leq |\{\eta(k) : \ell g\eta > k, \eta \in T^0 \cap (\bigcup_{j \leq \ell+1} T_{n_j})\}| \leq \sum_{j \leq \ell+1} |\{\eta(k) : \ell g(\eta) > k, \eta \in T^0 \cap T_{n_j}\}| \leq \sum_{j \leq \ell+1} y(2k+1) < x^\dagger(2k+1)$.

So T^* the definition of xRT^* is satisfied and T^* is as required.

$(\varepsilon)^+$, (ζ) left to the reader - similar to the proof of (δ). $\square_{2.10B}$

2.10C Claim. 1) If $V \subseteq V^\dagger$, $\varphi_3^{cm}[V]$ covers in $V^\dagger$ *iff* for every $\eta \in (\prod_{n<\omega}(n+1))^{V^\dagger}$ and $y \in DP({}^\omega\omega)^V$ there is $\langle a_\ell : \ell < \omega \rangle \in V$, $a_\ell \subseteq \omega$, $|a_\ell| \leq y(\ell)$ and $\bigwedge_\ell \eta(\ell) \in a_\ell$ *iff* for every $f \in (DP({}^\omega\omega))^V$ for every $y \in (DP({}^\omega\omega))^{V^+}$ and $\eta \in (\prod\limits_{n<\omega} f(n))^{V^+}$ there is $\langle a_\ell : \ell < \omega \rangle \in V$, $|a_\ell| \leq y(\ell)$ and $\bigwedge\limits_{\ell<\omega} \eta(\ell) \in a_\ell$ *iff* similarly for some f.

2) A forcing notion P has the Sacks property (i.e. is φ_2^{cm}-preserving) iff it has the ${}^\omega\omega$-bounding property (i.e. is φ_1^{cm}-preserving), and the Laver property (i.e. is φ_3^{cm}-preserving).

Proof. Easy.

2.10D Conclusion. For a CS iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha \rangle$, if $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ is proper and has the Laver property" *then* P_α is proper and has the Laver property.

Proof. By 2.3(2) and 2.10B + 2.10C.

* * *

The next example deal with trying to have: every new $\eta \in \prod_{n<\omega} f(n)$ belong to some old $\prod_{n<\omega} a_n$, $\langle |a_n| : n < \omega \rangle$ quite small, where $f(n)$ can be finite. Below we could have used $Y = \{\mathrm{id}\}$, but in applying it is more convenient to have Y. See more on this in [Sh:326] and much more in Roslanowski, Shelah [RoSh:470].

2.11A Definition. Let f denote a one place function, $\mathrm{Dom}(f) = \omega, 1 \leq f(n) \leq \omega$, f diverges to ∞ and g denote a two place function from ω to $\{\alpha : 1 \leq \alpha \leq \omega\}$; both nondecreasing, for clarity. Let $Y \subseteq DP({}^{\omega}\omega)$ have absolute definition and $<=<_Y$ be an absolute dense order on Y with no minimal member, and those properties are absolute (so Y may be countable, if $Y = DP({}^{\omega}\omega)$ we omit it). Finally, let H denote a family of such pairs (f, g). If $H = \{(f, g)\}$ we write f, g.

We define $\varphi = \varphi^{cm}_{4,Y;H}$, but if $Y = DP({}^{\omega}\omega)$ we may omit it, by letting for a universe $V, \varphi[V]$ be $(D, R, <)$ where

a) $D = H(\aleph_1)$,

b) $\mathrm{Dom}(R)$ is the set of triples (z, f, g) for $z \in Y$, $(f, g) \in H$; more formally member $x \in DP({}^{\omega}\omega)$ such that $\langle x(3\ell + i) : \ell < \omega \rangle$ codes z when $i = 0$, f when $i = 1$ and g when $i = 2$; we write $x = (z^x, f^x, g^x)$. We define: xRT *iff* x, $T \in D$, $x \in \mathrm{Dom}(R)$, $T \subseteq {}^{\omega>}\omega$ is a closed tree and for each n the set $\{\eta(n) : \eta \in T,\ \ell g(\eta) = n + 1, (\forall i \leq n)\eta(i) < f^x(i)\}$ has cardinality $< 1 + g^x(n, z^x(n))$. (So for $g^x(n, z^*(n)) = \omega$ this means "finite".)

c) $<_Y$ is the dense order of Y (e.g. $<_0$ or $<_{\mathrm{dis}}$) and $\langle z^1, f^1, g^1 \rangle < \langle z^2, f^2, g^2 \rangle$ iff $f^1 = f^2$, $g^1 = g^2$ and $z^1 < z^2$.

We may use also g with positive real (not integer) values, but still algebraic.

Let us note that φ^{cm}_1 (${}^{\omega}\omega$-bounding), φ^{cm}_2 (Sacks), φ^{cm}_3 (Laver) are particular cases of $\varphi^{cm}_{4,Y;H}$:

2.11B Claim. 1) Let $f = \omega$ (i.e. the function with constant value ω), $g(n, i) = \omega$ and $Y = DP({}^{\omega}\omega)$. Then for universes $V \subseteq V^{\dagger}, \varphi^{cm}_1[V]$ covers in $V^{\dagger}$ iff $\varphi^{cm}_{4,f,g}[V]$ covers in $V^{\dagger}$ (hence a forcing notion Q is φ^{cm}_1-preserving iff it is $\varphi^{cm}_{4,f,g}$-preserving).

2) Let g be $g(n,i) = 1+i$. For universes $V \subseteq V^\dagger, \varphi_3^{cm}[V]$ covers in $V^\dagger$ *iff* for every $f \in DP({}^\omega\omega)^V$, $\varphi^{cm}_{4,f,g}[V]$ covers in $V^\dagger$ (hence a forcing notion Q is φ_3^{cm}-preserving iff it is $\varphi^{cm}_{4,f,g}$-preserving for every $f \in DP({}^\omega\omega)$).

Proof. Check.

2.11C Claim. 1) Assume

(i) H is a family of pairs (f,g) and $Y \subseteq DP({}^\omega\omega)$ (an absolute definition, dense with no minimal element),

(ii) each $(f,g) \in H$ is as in 2.11A and $x <_Y y \Rightarrow \langle g(n,y(n))/g(n,x(n)) : n < \omega\rangle$ diverges to ∞,

(iii) *for every* $(f,g) \in H$ and $y \in Y$ *there are* $x \in Y$ and $(f',g') \in H$ and $h_n : \omega \to \omega$ one to one, $h_n{\restriction}n = h_{n+1}{\restriction}n$, $[n < m \Rightarrow \mathrm{Rang}(h_n) \cap \mathrm{Rang}(h_m) = \mathrm{Rang}(h_n{\restriction}n)]$ and $g'(h_n(\ell), x(h_n(\ell))) \leq g(\ell, y(\ell))$ and $f'(h_n(\ell)) \geq f(\ell)$.

Then $\varphi^{cm}_{4,Y,H}$ is a fine definition of covering models.

2) In part (1) we can replace clause (iii) by

(iii)$^-$ *for every* $(f,g) \in H$ and $y \in Y$ *there are* $x \in Y$ and $(f',g') \in H$ and $h_n : \omega \to \omega$ such that:

(α) $h_n{\restriction}n = h_{n+1}{\restriction}n$

(β) for every $k < \omega$, letting $w_k = \{(n,\ell) : \ell \geq n-1 \text{ and } h_n(\ell) = k\}$ we have $\prod_{(n,\ell)\in w_k} f(\ell) \leq f'(k)$

(γ) $g(n,\ell) \geq g'(n, h_n(\ell))$

3) Assume we replace (iii) by

(iii)* *for* $(f,g) \in H, x_1 < y$ in Y *there are* $x_2 \in DP({}^\omega\omega), (f',g') \in H$ such that: for every n large enough $f'(n) \geq f(n)^{g(n,x_1(n))}$ and $g(n,y(n)) \geq g(n,x_1(n)) \times g'(n, x_2(n))$.

Then $\varphi^{cm}_{4,Y,H}$ is a finest definition of a family of covering models.

Proof. 1) Let us check the conditions in Definition 2.2. Let $(f,g) \in H$ and we deal with each $\varphi^{cm}_{4,Y,f,g}[V]$ separately (this is enough).

(α) (a) Trivial by definition 2.11A.

(α) (b) Trivial.

(α) (c) Trivial.

(β) Check.

$(\gamma)^+$ Let $y < x$, yRT_n for $n < \omega$ (remember 1.3(4) letting $y = x^\dagger$).

Choose n_k by induction on k such that: $\bigwedge_{\ell<k} n_\ell < n_k$ and $n_k < \omega$ and $[n_k \leq \ell < \omega \Rightarrow k \times g(\ell, y(\ell)) < g(\ell, x(\ell))]$ (possible by assumption (ii)). Now $w = \{n_k : k < \omega\}$ and $T^* = \{\eta \in {}^{\omega>}\omega : \text{for every } n \in w, \eta \restriction n \in \bigcup_{\substack{\ell \leq n \\ \ell \in w}} T_\ell\}$ are as required.

(δ) By 2.4(1) (for (iv)* use the assumption (iii) of 2.11C(1)) and 2.7.

2) The proof is similar to the proof of part (1) using 2.4(2) instead 2.4(1) in proving clause (δ).

3) Note that (iii)* $\Rightarrow$ (iii)$^-$ easily, so demands (α), (β), (γ), $(\gamma)^+$, (δ) hold.

$(\varepsilon)^+$ Straightforward (use a tree T, x_2RT, to "catch" the T in a narrow tree $\subseteq TTR$).

(ζ) Check. $\square_{2.11C}$

2.11D Conclusion. For Y, H satisfying (i), (ii), (iii)* of 2.11C, for any CS iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha\rangle$, if $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ is proper and $\varphi^{cm}_{4,Y;H}[V^{P_i}]$-preserving" *then* P_α is proper, $\varphi^{cm}_{4,Y;H}[V]$-preserving.

2.11D Definition. We say that a forcing notion Q is (f, g)-bounding (where $f, g \in {}^\omega(\omega + 1 \setminus \{0, 1\})$) if for every $\eta \in (\prod\limits_{n<\omega} f(n))^{V^Q}$ there is $\langle a_n : n < \omega\rangle \in V$ such that $|a_n| \leq g(n)$ and $\eta \in \prod\limits_{n<\omega} a_n$.

2.11F Conclusion. Assume

$(*)$ $f, g \in {}^\omega(\omega + 1 \setminus \{0, 1\})$ are diverging to infinity.

If $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha\rangle$ is a CS iteration such that Q_i is $(f^{g^\ell}, [g]^{1/\ell})$-bounding in V^{P_i} for every $\ell < \omega$ *then* P_α is proper and (f^{g^ℓ}, g)-bounding for every $\ell < \omega$.

Proof. We use 2.11C(3) (and 2.11A, 2.11B and 2.3). We let $Y = \{x \in {}^\omega\omega : x \text{ constant }\}$, so we can identify x with $x(0)$, let $\{a_n : n < \omega\}$ list the positive rationals, define $x < y \Leftrightarrow a_{x(0)} < a_{y(0)}$ and let $g_\ell(n, x(n)) = [g(n)^{a_{x(n)}/\ell}]$ (=the integer part, note: x is constant so $x(n) = x(0)$) and $f_\ell(n) = f(n)^{[g(n)^\ell]}$.

Lastly let $H = \{(f_\ell, g_\ell) : \ell < \omega\}$, so $\varphi^{cm}_{4,Y;H}$ is well defined.

Now we show that $\varphi^{cm}_{4,Y;H}$ is a finest definition of covering model, to get this we would like to apply 2.11C(3). Among the three assumptions there, clause (i) holds by the choice of Y and H. Also the first phrase in clause (ii) holds, as for the second, if $x < y$ and $(f,g) \in H$, then for some ℓ, $(f,g) = (f_\ell, g_\ell)$, hence

$$g(n, y(n))/g(n, x(n)) = [g(n)^{a_{y(0)}/\ell}]/[g(n)^{a_{x(0)}/\ell}]$$

$$\geq (g(n)^{a_{y(0)}/\ell} - 1)/(g(n)^{a_{x(0)}/\ell}) = g(n)^{(a_{y(0)} - a_{x(0)})/\ell} - g(n)^{-a_{x(0)}/\ell}$$

as $a_{y(0)} > a_{x(0)} > 0$, this clearly diverges to infinity.

Lastly for clause (iii)*, let $(f,g) \in H$ (so for some ℓ, $(f,g) = (f_\ell, g_\ell)$) and let $x_1 < y$ in Y. Now choose $x_2 \in Y$ such that $\varepsilon + x_2(0) < y(0) - x_1(0)$ for some $\varepsilon > 0$ and choose m such that $a_{x_1(0)}/\ell < m$ and let $(f', g') = (f_{\ell+m}, g_{\ell+m})$. Let us check:

$$f'(n) = f_{\ell+m}(n) = f(n)^{g(n)^{\ell+m}} = (f(n)^{g(n)^\ell})^{g(n)^m} = f_\ell(n)^{g(n)^m}$$

$$\geq f_\ell(n)^{g_\ell(n, x_1(n))}$$

(the last inequality because $g_\ell(n, x_1(n)) = [g(n)^{a_{x_1(n)}/\ell}]$ and $a_{x_1(0)}/\ell < m$)

$$g(n, y(n)) = g_\ell(n, y(n)) =$$

$$[g(n)^{a_{y(n)}/\ell}] \geq g(n)^{a_{y(n)}/\ell} - 1 \geq (g(n)^{a_{x_1(n)}/\ell})(g(n)^{a_{x_2(n)}/\ell})(g(n)^{\varepsilon/\ell}) - 1$$

$$\geq g(n, x_1(n))g(n, x_2(n))g(n)^{\varepsilon/\ell} - 1 > g(n, x_1(n))g(n, x_2(n))$$

(the last inequality: for n large enough).

So really (iii)* of 2.11C(3) holds hence 2.11C(3) applies and $\varphi^{cm}_{4,Y;H}$ is a finest definition of covering models, so 2.3(2) applies.

Lastly, we can check that by monotonicity

$\otimes$ Q is $\varphi^{cm}_{4,Y;H}$-preserving iff Q is $(f^{g^\ell}, [g^{1/\ell}])$-bounding for every $\ell < \omega$.

So by the last two sentences we are done. $\square_{2.11E}$

* * *

2.12A Definition. [The PP property] 1) We define $\varphi = \varphi_5^{cm}$ (a definition of a covering model) by letting $\varphi[V] = (D, R, <)$ (the PP model) where:

a) $D = H(\aleph_1)$

b) xRT iff $x, T \in D, x \in {}^\omega\omega$ is strictly increasing, $T \subseteq {}^{\omega >}\omega$ is a closed subtree and $T \cap {}^n\omega$ is finite for every n and:

(*) for arbitrarily large n there are k, and $n < i(0) < j(0) < i(1) < j(1) \leq \ldots < i(k) < j(k) < \omega$ and for each $\ell \leq k$, there are $m(\ell) < \omega$ and $\eta^{\ell,0}, \ldots, \eta^{\ell,m(\ell)} \in T \cap {}^{j(\ell)}\omega$, such that: $j(\ell) > x(i(\ell) + m(\ell))$ and

$$(\forall \eta \in T \cap {}^{j(k)}\omega) \bigvee_{\ell,m} \eta^{\ell,m} \trianglelefteq \eta.$$

c) $<$ is $<^*_{\rm dis}$

Remark: concerning the PP property, there is a strong version ("strong PP-property") proved in 4.4 and 5.6 for the forcing notion there and a weak version ("weak PP-property") derived in 2.12D below and used in 4.7 and 5.8 (though in the statement the "PP-property" appears). See Definition 2.12E.

2.12B Claim. 1) If the forcing notion P is φ_5^{cm}-preserving then it has the ${}^\omega\omega$-bounding property; if P has the Sacks property (i.e. is φ_2^{cm}-preserving) *then* it is φ_5^{cm}-preserving.

2) If $(D, R, <)$ is a Sacks model (i.e. $\varphi_2^{\rm cm}[V]$) *then*
$(\forall \eta \in {}^\omega\omega)(\forall x)(\exists T \in D)$ $[x \in (\mathrm{Dom}(R^{\varphi_5^{cm}})) \cap D \Rightarrow xR^{\varphi_5^{cm}}T \;\&\; \eta \in \lim T]$

3) If $(D, R, <)$ is a PP-model (see 2.2(5)) *then*
$(\forall \eta \in {}^\omega\omega)(\forall x)(\exists T \in D)$ $[x \in (\mathrm{Dom}(R^{\varphi_1^{cm}})) \cap D \Rightarrow xR^{\varphi_1^{cm}}T \;\&\; \eta \in \lim T]$.

Proof. Easy.

2.12C Claim. $\varphi = \varphi_5^{cm}$ is a finest definition of a covering model, 2-directed.

Proof. Let us check the conditions in Definition 2.2.

(α) (a) Trivial by a), b) of Definition 2.12A.

(α) (b) Check.

(α) (c) Check.

(β) Trivial.

(γ)(a) So let $(D, R, <)$ be $\varphi[V]$. Let $x > y$, and yRT_n. Let $h_m : \omega \to \omega$ be such that for any n there are $i(0) < j(0) < \ldots < j(k), \eta_{\ell,i}$ (for $i \le m(\ell), \ell \le k$) witnessing $(*)$ of Definition 2.12A(1)(b) for yRT_m and n (so $n < i(0)$) such that $j(k) < h_m(n)$. Now we define n_i by induction on $i, n_0 = 0$ and n_{i+1} is such that: choose $\ell_0, \ell_1, \ldots, \ell_{i+1}$ as follows: $\ell_0 = n_i, \ell_{j+1} = h_{n_j}(\ell_j) + 1$, and $n_{i+1} = \ell_{i+1}$. Let $T^* = \{\eta : \text{for every } i, \eta \restriction n_{i+1} \in \bigcup_{j<i} T_{n_j}\}$ i.e. we choose $w = \{n_i : i < \omega\}$.

So clearly $xR^{\varphi_5^{cm}}T^*$ is as required.

(γ)(b) Easy.

(δ) We use 2.4(2) with $h_n(\ell) = \ell$. So let $x \in \mathrm{Dom}(R)$, and we choose $y = x$. Now $w_\ell = \{(n, \ell) : \ell \ge n\}$, g_n is any one to one function from ${}^n\omega$ onto ω, $f^n_{(n,\ell)}$ is thus determined. Now check.

$(\varepsilon)^+$ So we know Q is $\varphi_5^{cm}[V]$-preserving, $\varphi_5^{cm}[V] \models y < x$ and $T^* \in V^Q$, and $\varphi[V^Q] \models yRT^*$. We should find $T^{**} \in V$ such that: $T^* \subseteq T^{**}$ and $\varphi[V] \models xRT^{**}$. We work in $V^+ = V^Q$ but $(D, R, <) = \varphi_5^{cm}[V]$. The proof is straight but still we elaborate. Let $h^* : \omega \to \omega$ be defined for T^* as h_m was defined for T_m in the proof of clause (γ)(a). So by 2.12B(3) there is $h^{**} \in (D \cap {}^\omega\omega)^V$ such that h^{**} is strictly increasing and $(\forall n)[h^*(n) \le h^{**}(n)]$.

We now choose z such that for every n, there are $n = m_0^n < m_1^n < \ldots < m_{n+1}^n, m_{\ell+1}^n = h^{**}(m_\ell^n) + m_\ell^n + 1$, and let $z(n) = m_{n+1}^n$. Clearly $z \in D^V$.

So remembering 2.7, we can apply the "covering property" of (D, R) to T^* (i.e., the branch T^* induces in TTR). Apply it for z and we get an appropriate closed subtree $\mathcal{C} \in D = H(\aleph_1)^V$ of TTR, (so $T^* \cap {}^{n\ge}\omega \in \mathcal{C}$ for every n). Clearly $T^{**} = \bigcup_{t \in \mathcal{C}} t$ is a closed subtree of ${}^{\omega>}\omega$, it belongs to D, and there is no problem to prove $T^* \subseteq T^{**}$. The only point left is why xRT^{**}.

Let $\mathcal{C}^\dagger$ be the set of $t \in \mathcal{C}$ such that if $n < \omega, h^{**}(n) \le ht(t)$ then for some $k < ht(t)$ and $n < i(0) < j(1) < \ldots < i(k) < j(k) \le ht(t)$ the statement in

$(*)$ of Definition 2.12 A clause (b) holds (for t and y). Let $\mathcal{C}''$ be the maximal closed tree $\subseteq \mathcal{C}^\dagger$. It is easy to check that $\mathcal{C}'' \in D$, and that T^* induces a branch $\subseteq \mathcal{C}''$, so without loss of generality $\mathcal{C} = \mathcal{C}^\dagger = \mathcal{C}''$.

Now for arbitrarily large n, there are $k < \omega$, and $n < i(0) < j(0) < i(1) < j(1) < \ldots < i(k) < j(k) < \omega$, and for each $\ell < k$ there are $m(\ell) < \omega$, $t_{\ell,0}, \ldots, t_{\ell,m(\ell)} \in \mathcal{C} \cap TTR_{j(\ell)}$ such that $j(\ell) > z(i(\ell) + m(\ell))$ and

$$(\forall t \in \mathcal{C} \cap TTR_{j(k)})[\bigvee_{l,n} t_{l,n} \leq t].$$

By the definition of z, there are $\xi(\ell,0) < \ldots < \xi(\ell, m(\ell)+1)$ such that $i(\ell)+m(\ell)+1 < \xi(\ell,0)$ and $\xi(\ell, m(\ell)+1) < j(\ell)$, and $h^{**}(\xi(\ell,m)) < \xi(\ell,m+1)$. So by the assumption on $\mathcal{C}(= \mathcal{C}^\dagger)$ for each such $\ell < k$, $m < m(\ell)$, there are $k_{\ell,m}$, $\xi(\ell,m) < i(0,\ell,m) < j(0,\ell,m) < i(1,\ell,m) < j(1,\ell,m) < \ldots < i(k_{\ell,m},\ell,m) < j(k_{\ell,m},\ell,m) < \xi(\ell,m+1)$ and $n(\alpha,\ell,m)$ (for $\alpha < k_{\ell,m}$) such that $j(\alpha,\ell,m) > x(i(\alpha,\ell,m) + n(\alpha,\ell,m))$ and $\eta_{\alpha,\beta,\ell,m} \in ({}^{j(\alpha,\ell,m)}\omega) \cap t_{\ell,m}$ (for $\beta < n(\alpha,\ell,m)$) and $(\forall \nu \in t_{\ell,m} \cap {}^{\xi(\ell,m+1)}\omega)[\bigvee_{\alpha,\beta} \eta_{\alpha,\beta} \lhd \nu]$.
Now the set of $i(\alpha,\ell,m), j(\alpha,\ell,m), n(\alpha,\ell,m)$ and $\eta_{\alpha,\ell,m,\beta}$ for $\beta < n(\alpha,\ell,m)$ supplies the required witnesses.

(ζ) Easy (by 2.8 C). $\square_{2.12C}$

2.12D Claim. Assume $V \subseteq V'$ and $\varphi_5^{\mathrm{cm}}(V)$ covers in V'. *Then* for every $\eta \in ({}^\omega 2)^{V'}$ there is an infinite $w \subseteq \omega$ from V and $\langle k_n, \langle i_n(\ell), j_n(\ell) : \ell \leq k_n\rangle : n \in w\rangle$ from V such that:

(a) $n < i_n(0) < j_n(0) < i_n(1) < j_n(1) < \ldots < i_n(k_n) < j_n(k_n) < \min(w \setminus (n+1))$.

(b) for every $n \in w$ for some $\ell \leq k_n$ we have $\eta(i_n(\ell)) = \eta(j_n(\ell))$.

Remark. Only the x defined by $x(\ell) = 2^\ell$ suffices.

Proof. Easy.

2.12E Definition. 1) A forcing notion Q has the PP-property *iff* it is φ_5^{cm}-preserving.

2) A forcing notion Q has the weak PP-property if V, V^Q satisfies the conclusion of 2.12D.

3) A forcing notion Q has the strong PP-property if changing φ_5^{cm} to $\varphi_{5.5}^{\mathrm{cm}}$ in Definition 2.12A by demanding $k = 0$ in $(*)$, we have: $\varphi[V]$ covers in V^Q.

2.12F Claim. For a forcing notion Q:

1) the strong PP-property implies the PP-property.

2) The PP-property implies the weak PP-property.

2.12G Conclusion. For a CS iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha \rangle$, if $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ is proper with the PP-property" *then* P_α is proper with the PP-property.

Proof. By 2.3(2), by 2.12C.

* * *

The following deals with "no Cohen real + no real dominates F ($\subseteq {}^{\omega>}\omega$, see §3)".

2.13A Definition. We define $\varphi = \varphi_6^{cm}$ (a definition of a covering model) by letting $\varphi[V] = (D, R, <)$ where

a) $D = H(\aleph_1)$

b) xRT *iff* T is a perfect nowhere dense tree, $x \in \mathrm{DP}\,({}^{\omega}\omega)$.

c) $< = <_0$

2.13B Observation. Q is φ_6^{cm} preserving iff Q adds no Cohen real.

2.13C Claim. φ_6^{cm} is a 2-directed, fine definition of a covering model for forcing which are φ_1^{cm}-preserving† (= ${}^{\omega}\omega$-bounding) *or* even just not adding a dominating real†† and are proper (or satisfy UP) (caution: not preserved under composition).

† Of course as ${}^{\omega}\omega$-bounded forcing necessarily add no Cohen reals.
†† On this see 3.17(2),(3).

Proof. (α) (a), (b), (c) Trivial

(β) Trivial

(γ) Check, even $(\gamma)_1$ (see 1.3(8)) and $(\gamma)_2$ (of 2.5(2)) hold. E.g. concerning $(\gamma)_2$, given $\langle T_n : n < \omega \rangle$, nowhere dense trees, choose by induction on $i < \omega$, $n_i < \omega$ as follows: $n_0 = 0$, n_{i+1} is minimal n such that $n \in (n_i, \omega)$ and for every $\eta \in {}^{n_i \geq}(n_i + 1)$ there is ν: $\eta \triangleleft \nu \in {}^{n>}n$ such that $\nu \notin \bigcup_{j \leq n_i} T_j$. Let for $\ell < 2$, $T^\ell \overset{\text{def}}{=} \{\eta \in {}^{\omega>}\omega$: for some $i = \ell \mod 2$, and $n \in [n_i, n_{i+1})$ we have $\eta \restriction n \in T_0$, and $\eta \in T_n\}$.

(δ) By 2.5(2). $\square_{2.13C}$

2.13D Conclusion. 1) For a CS iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i \leq \alpha, j < \alpha \rangle$, if $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ is proper not adding a Cohen real (over V^{P_i})" and P_α adds no dominating real over V *then* P_α is proper and adds no Cohen real over V.

2) The property "P purely does not add a Cohen real nor an $\underset{\sim}{\eta} \in {}^\omega\omega$ dominating F" where $F \subseteq {}^\omega\omega$ is fixed not dominated in the old universe, is preserved in limit of iterations as in $0.1_{\theta=\aleph_0}$.

Remark. 1) Note we have $(|D|, \aleph_1)$-covering in §1.

2) What if in 0.1 we use (D), $\leq_{\text{pr}}$ is $=$ (so we use FS iterations satisfying the c.c.c.)? In the limit we add a Cohen real, necessarily the family is empty. We cannot apply it as for P a c.c.c. forcing, $\leq_{\text{pr}}$ is equality so "P purely preserves φ_0^{cm}" always fails.

3) Of course we can interchange using/not using F in parts (1) and (2) of 2.13D.

Proof. 1) By 2.3(1) and 2.13C applied to φ_6^{cm}.

2) Usually using in addition 3.17. $\square_{2.13D}$

* * *

The following deals with "every new real belongs to some old closed set of Lebesgue measure zero".

2.14A Definition. We define φ_7^{cm} (a definition of a covering model) by letting $\varphi[V] = (D, R, <)$ where

(a) $D = H(\aleph_1)$.

(b) xRT means T is a perfect tree, with $\lim T$ having Lebesgue measure zero.

(c) $<=<_0$

So if Q is φ_7^{cm}-preserving then Q adds no random real but not inversely.

2.14B Claim. φ_7^{cm} is a 2-directed fine definition of a covering model for forcing which are purely φ_1^{cm}-preserving ($=$ ${}^\omega\omega$-bounding property) or even just not adding a dominating real (caution! not preserved under composition.)

Proof.

(α) Trivial

(β) Trivial

$(\gamma)^+$ Check

(δ) By 2.5(1). $\square_{2.17D}$

2.14C Conclusion. 1) The property "P purely does not add any real, which does not belong to any old closed measure zero set from V and is purely ${}^\omega\omega$-bounding and has pure $(2,2)$-decidability" is preserved by limit (for iterations as in $0.1_{\theta=2}$) [but not necessarily composition].

2) The property "P purely does not add any real not belonging to any closed old set of measure zero from V and adds no real dominating F" is preserved in limits for iterations as in 0.1, where $F \subseteq {}^\omega\omega$ is a fixed undominated family.

Proof. Like the proof of 2.13C. $\square_{2.14C}$

* * *

The following deals with "every new dense open subset of ${}^{\omega>}\omega$ is included in some old one".

2.15A Definition. Let $\langle \rho^*_\ell : \ell < \omega \rangle$ enumerate ${}^{\omega>}\omega$. Let $T^* \subseteq {}^{\omega>}\omega$ be a perfect tree such that for every $\nu \in \lim T^*$, $A_\nu \overset{\text{def}}{=} \{\rho^*_\ell : \ell < \omega$ and $\nu(2\ell) = 1\}$ is open, and $\rho^*_\ell {}^\wedge \rho^*_{\nu(2\ell+1)} \in A_\nu$ (hence A_ν is dense)$\}$, and such that for every dense open subset A of ${}^{\omega>}\omega$ there is $\nu \in \lim T^*$ such that $A = A_\nu$.
We define φ^{cm}_8 (a definition of a covering model) by letting

$\varphi^{\text{cm}}_8(V) = (D, R, <)$:

(a) $D = H(\aleph_1)$

(b) xRT means that $x \in \mathrm{DP}\,({}^\omega\omega)$ and $T \subseteq T^*$ is perfect satisfying: $\bigcap\{A_\nu : \nu \in \lim T\}$ is dense open.

(c) $<=<_0$ (see 1.4)

2.15B Claim. 1) For $A \subseteq {}^{\omega>}\omega$ there is a closed $T = T_A \subseteq T^*$ such that: if $\eta \in \lim T^*$ then A_η (which is dense open) include A iff $\eta \in \lim T_A$. So $T \in \mathrm{Rang}(R)$ iff for some dense open $A \subseteq {}^{\omega>}\omega$ we have $T \subseteq T_A$

2) A forcing notion Q is φ^{cm}_8 (V)-preserving *iff* every open dense subset of ${}^{\omega>}\omega$ in V^Q include a dense open subset of ${}^{\omega>}\omega$ from V *iff* for some (every) subuniverse $V^\dagger$ of V such that $\varphi^{\text{cm}}_8(V^\dagger)$ covers in V, Q is $\varphi^{\text{cm}}_8(V^\dagger)$ preserving.

3) If Q_0 is $\varphi^{\text{cm}}_8(V)$-preserving and $\Vdash_{Q_0}$ "$\underset{\sim}{Q}_1$ is $\varphi^{\text{cm}}_8(V^{Q_0})$-preserving" *then* $Q_0 * \underset{\sim}{Q}_1$
is $\varphi^{\text{cm}}_8(V)$-preserving.

4) If Q is φ^{cm}_8-preserving *then* Q is ${}^\omega\omega$-bounding.

Proof. 1) - 3) Check.

4) For $h \in ({}^\omega\omega)^{V^Q}$ let $A_h = \{\eta \in {}^{\omega>}\omega :$ for some n, $\bigwedge_{\ell<n} \eta(\ell) = 0$, $\eta(n) \neq 0$ and $\ell g(\eta) \geq n + h(n) + 1\}$. So $A_h \subseteq {}^{\omega>}\omega$ is dense open, so there is $A \subseteq {}^{\omega>}\omega$, dense open, $A \in V, A \subseteq A_h$. Define $g : \omega \to \omega$ in V by: $g(n) = \min\{\ell g(\eta) : \eta \in A, n = \min\{\ell : \eta(\ell) > 0\}\}$. Then $g : \omega \to \omega$, $g \in V$ and $(\forall n) h(n) < g(n)$. $\square_{2.15B}$

2.15C Claim. φ^{cm}_8 is a finest definition of a covering model which is 2-directed.

Proof. $(\alpha), (\beta)$, 2-directed: easy, now we prove more than (γ) (see 2.5(3)).

$(\gamma)_3$ Assume yRT_n, so for each n there is a dense open $A_n \subseteq {}^{\omega>}\omega$ such that $T_n \subseteq T_{A_n}$. We choose by induction on $n < \omega, k_n < \omega$ such that $\bigwedge_{\ell<n} k_\ell < k_n$, and if $\ell \leq n$ there is $\rho \in \bigcap_{m\leq n} A_m$ such that $\rho^*_\ell \lhd \rho$, $\rho \in \{\rho^*_m : m < k_n\}$. Now let $k = 2$, and for $i < k$ let

$$B_i = \{n : \text{ for some } \ell = i \mod 2 \text{ we have } k_\ell \leq n < k_{\ell+1}\}$$

and let

$$T_i = \{\nu \in T^* : \text{ if } n \leq \ell g(\nu) \text{ then } \nu{\restriction}n \in \bigcup\{T_{A_m} : m < n \text{ and } m \in B_i\}.$$

(δ) by 2.5(2) (remember that $(\gamma)_3 \Rightarrow (\gamma)_2$ by 2.5(3)).

$(\varepsilon)^+$, (ζ) left to the reader. $\square_{2.15C}$

Remark. alternatively, instead of 2.15 B,C, look in XVIII §3.

2.15D Conclusion. For CS iteration $\langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha\rangle$, if $\Vdash_{P_i}$ "Q_i is proper and any new open dense subset of ${}^{\omega>}\omega$ includes an old one" *then* P_α is proper and any dense open subset $A \in V^P$ of ${}^{\omega>}\omega$ includes a dense open subset $A \in V$ of ${}^{\omega>}\omega$.

Proof. By 2.3(2) and 2.16C (and 2.16B(2)).

* * *

2.16 Conclusion. For $\theta = \aleph_0$ the property " Q is purely $\varphi_\ell[V]$-preserving with pure $(\theta, 2)$ decidability" is preserved by iteration as in 0.1_θ for $\ell = 1, \ldots, 8$ (e.g. by CS iterations of proper forcing). This is true for $\theta = 2$, $\ell = 1, \ldots, 8$ when $(*)$ of 1.12 holds (recheck).

* * *

The following is an addition from early nineties, inspired by the interest in "adding no Cohen real". It is dual to 2.13, see 2.17 C(1) below.

2.17A Definition. For $Y \subseteq DP({}^\omega\omega)$ with an absolute definition we define $\varphi^{\mathrm{cm}}_{9,Y}$, a definition of a covering model. For a universe V we let $\varphi[V] = (D, R, <)$ be:

(a) $D = H(\aleph_1)$.

(b) xRT iff

(i) $x = \langle x^{[0]}, x^{[1]}, x^{[2]} \rangle$, say $x(3n+i)$ codes $x^{[i]}(n)$, $x^{[i]} \in Y$, $x^{[1]}(n+1) > x^{[1]}(n)$ and the difference is a power of 2, $x^{[2]}[n] \leq \log_2[x^{[1]}(n+1) - x^{[1]}(n)]$,

(ii) $T \in {}^{\omega>}\omega$ closed subtree,

$$\eta \in {}^{\omega>}\omega \ \&\ \eta{\restriction}n \in T \ \&\ \eta(n) \geq x^{[0]}(n) \ \Rightarrow\ \eta \in T.$$

(iii) for each n the following holds:

$(*)_n$ for any $m < (x^{[1]}(n+1) - x^{[1]}(n))/2^{x^{[2]}(n)}$ there is a function $g = g_{n,m}$ with domain the interval $[x^{[1]}(n) + m \cdot 2^{x^{[2]}(n)}, x^{[1]}(n) + (m+1) \cdot 2^{x^{[2]}(n)})$, $(\forall \ell)[g(\ell) < x^{[0]}(\ell)]$ and $[\eta \in T \ \&\ \ell g(\eta) \geq x^{[1]}(n+1) \Rightarrow g_{n,m} \not\subseteq \eta]$. So $g \stackrel{\mathrm{def}}{=} \bigcup\limits_{n,m} g_{n,m}$ belongs to $\prod_{\ell<\omega} x^{[0]}(\ell)$, we call g a witness.

(c) $x < y$ iff $x^{[0]} = y^{[0]}$, $x^{[1]} = y^{[1]}$ and $x^{[2]} <_{\mathrm{dis}} y^{[2]}$ (see 1.4).

Explanation. So what is the meaning of xRT? The interesting part of T is $T' \stackrel{\mathrm{def}}{=} \{\eta{\restriction}\ell : \eta \in (\lim T) \cap \prod\limits_{n<\omega} x^{[0]}(n)\}$ and T is in a way "explicitly nowhere dense" i.e. for some $g \in \prod\limits_{n<\omega} x^{[0]}(n)$, for every $\eta \in \lim(T')$ for every n for many subintervals I of $[x^{[1]}(n), x^{[1]}(n+1))$ we have $g{\restriction}I \neq \eta{\restriction}I$.

2.17B Claim. Assume $V \subseteq V^+$. Then $(\gamma) \Rightarrow (\beta) \Rightarrow (\alpha)$, where:

(α) "for every $f \in DP({}^\omega\omega)^V$ and $g \in (\prod_{\ell<\omega} f(\ell))^{V^+}$ there is $h \in (\prod_{\ell<\omega} f(\ell))^V$ such that $\{\ell : h(\ell) = g(\ell)\}$ is finite.

(β) $\varphi^{\mathrm{cm}}_{9,Y}[V]$ covers in V^+ for $Y = DP({}^\omega\omega)$.

(γ) every covering model from $\varphi^{\mathrm{cm}}_{9,DP({}^\omega\omega)}(V)$ covers in V^+.

2.17C Remark. 1) It is well known that: condition (α) implies there is no Cohen real over V in V^+; and if $V_0 \subseteq V_1 \subseteq V_2$, in V_1 there is a real f in ${}^\omega\omega$ dominating $({}^\omega\omega)^{V^0}$; and in V_2 there is $g \in \prod_{\ell<\omega} f(\ell)$ contradicting (α) *then* in V_2 there is a Cohen real over V_0.

The preservation theorem below implies a Cohen real is not added in limits.

2) Note that making $x^{[0]}$ smaller makes being in $\lim T$, xRT, harder.

3) The absoluteness requirements can be restricted as usual.

4) What we deduce below is complimentary in a sense to 2.13 A-C.

5) Why in 2.17A the $2^{x^{(2)}(n)}$? Of course a more general notion will use norms (see [RoSh:470]).

6) If Y is closed enough then in 2.17B we have $(\gamma) \Leftrightarrow (\beta)$.

2.17D Claim. 1) Assume

(i) Y is a subset of $DP({}^\omega\omega)$,

(ii) for every $x \in Y$ there is $y \in Y$ and there are $\langle \ell^*_n : n < \omega\rangle$ such that:

(a) $\lim\limits_n \ell^*_n = \infty$

(b) $1 \leq \ell^*_n \leq \ell^*_{n+1}$, ℓ^*_n a power of 2 (for technical reasons)

(c) $x^{[1]} = y^{[1]}$

(d) $x^{[0]} = y^{[0]}$

(e) $y^{[2]}(n) = x^{[2]}(n) - \log_2(\ell^*_n) \geq 0$

(iii) for $<$ from 2.17A(c) $(Y, <)$ is dense with no minimal member.

Then $\varphi^{\mathrm{cm}}_{9,Y}[V]$ is a fine covering model.

2) Assume (i) Y is an absolute definition of a subset of $DP({}^\omega\omega)$,

(ii) clause (1)(ii) holds absolutely,

(iii) clause (1)(iii) holds absolutely.

Then $\varphi^{\mathrm{cm}}_{9,Y}$ is a fine definition of a covering model.

Proof. We check the condition in definition 2.2.

(α) (a)(b)(c) Check.

(β) Check.

$(\gamma)^+$ We check on $\varphi^{\mathrm{cm}}_{9,Y}[V]$.

So let $x_n < x_{n+1} < x^\dagger < y$ be given, $x_n R T_n$. Now any thin enough infinite $w \subseteq \omega$ will work as:

$\otimes$ $\bigcup_{\ell < \ell(*)} T_\ell$ satisfies $(*)_n$ from (b)(iii) of Definition 2.17A for y if $\bigwedge_\ell x R T_\ell$, $x^{[0]} = y^{[0]}$, $x^{[1]} = y^{[1]}$ and $2^{x^{[2]}(n)/y^{[2]}(n)}$ is $\geq \ell(*)$.

(δ) We use 2.4(3) (and for checking the demand (iv)*** there we use assumption (ii)).

Let $x \in \mathrm{Dom}(R)$ be given and we shall define y and $\mathbf{B}$ as required in clause (iv)*** of 2.4(3). So let y be as defined by clause (ii) of 2.17D(1). Now we define the Borel function $\mathbf{B}$; we let $\mathbf{B}(\langle \eta_\alpha : \alpha \leq \omega \rangle) = \nu$ (where $\eta_\alpha \in {}^\omega\omega$, $\eta_\alpha \restriction \alpha = \eta_\omega \restriction \alpha$, $\nu \in {}^\omega\omega$) if:

($\oplus_1$) Let $n < \omega$, m, $(x^{[1]}(n+1) - x^{[1]}(n))/2^{x^{[2]}(n)}$

- (a) $\langle \eta_\omega(x^{[1]}(n) + m \cdot 2^{x^{[2]}(n)} + i) : i < 2^{x^{[2]}(n)} \rangle$ is equal to $\langle \nu(y^{[1]}(n) + m \cdot 2^{x^{[2]}(n) \cdot \ell^*_n} + i) : i < 2^{x^{[2]}(n)} \rangle$
- (b) if $k < \ell^*_n - 1$ *then* $\langle \eta_k(x^{[1]}(n) + m \cdot 2^{x^{[2]}(n)} + i) : i < 2^{x^{[2]}(n)} \rangle$ is equal to $\langle \nu(y^{[1]}(n) + m \cdot 2^{x^{[2]}(n) \cdot \ell^*_n} + (k+1) \cdot 2^{x^{[2]}(n)} + i) : i < 2^{x^{[2]}(n)} \rangle$.

So assume T_1 satisfying $y R T_1$ is given and we should define appropriate T_1. As $y R T_1$, there are functions $g^1_{n,m}$ for $n < \omega$, $m < (y^{[1]}(n+1) - y^{[1]}(n))/2^{y^{[2]}(n)}$ witnessing it. For $n < \omega$, $m < (y^{[1]}(n+1) - y^{[1]}(n))/2^{x^{[2]}(n)}$ we define a function $g^0_{n,m}$ as follows. Its domain is of course $(x^{[1]}(n) + m \cdot 2^{x^{2]}(n)}, x^{[1]}(n) + (m+1) \cdot 2^{x^{[2]}(n)})$ and for each $k < \ell^*_n$, we have $g^1_{n, \ell^*_n \cdot m + \ell} \subseteq g^0_{n,m}$.
The checking is straight. $\square_{2.17D}$

Remark. Note that for many pairs (x_0, x_1) from T_1, $x_1 R T_1$ we can produce T_0, $x_0 R T_0$, where T_1 in a way codes T_0.

2.17E Conclusion. For $\varphi^{\mathrm{cm}}_{9,Y}$ as in 2.17D(1), for CS iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \delta, j < \delta \rangle$, if $\Vdash_{P_i}$ "Q_i is proper" and P_i preserve $\varphi^{cm}_{9,Y}[V]$ for $i < \delta$ *then* P_δ preserve $\varphi^{cm}_{9,Y}[V]$.

§3. Preservation of Unboundedness

3.1 Notation. 1) ψ may denote an absolute definition of a two-place relation on ${}^\omega\omega$ which we denote $R^\psi[V]$ (so when extending the universe, we reinterpret R, but we know that the interpretations are compatible). We write xRy instead of $R(x,y)$. Sometimes ψ is an absolute definition of a three-place relation R on ${}^\omega\omega$ and then we write xR^zy instead of $R(x,y,z)$.

Let $\bar{R}$ denote $\langle R_n : n < \omega\rangle$ (each R_n as above) so $\bar{R}^m = \langle R_n^m : n < \omega\rangle$. We identify $\langle R : n < \omega\rangle$ with R.

Remember $\mathcal{S}_{<\kappa}(A) = \{B \subseteq A : |B| < \kappa\}$ and if κ is regular uncountable then $\mathcal{D}_{<\kappa}(A)$ is the filter on $\mathcal{S}_{<\kappa}(A)$ generated by the sets $G(M) = \{|N| : N \prec M, ||N|| < \kappa$ and $N \cap \kappa$ is an ordinal$\}$ for M a model with universe A and $< \kappa$ relations.

3.2 Definition. 1) For $F \subseteq {}^\omega\omega$ and a two place relation R on ${}^\omega\omega$, we say that F is R-bounding if $(\forall g \in {}^\omega\omega)(\exists f \in F)[gRf]$.

2) $F \subseteq {}^\omega\omega$ is $\bar{R}$-bounding if it is R_n-bounding for each n (where $\bar{R} = \langle R_n : n < \omega\rangle$).

3) For $F \subseteq {}^\omega\omega$, $\bar{R}$ (each R_n two place) and $S \subseteq \mathcal{S}_{<\aleph_1}(F)$ the pair $(F, \bar{R})$ is S-*nice* if:

α) F is $\bar{R}$-bounding.

β) For any $N \in S$, for some $g \in F$, for every $n_0, m_0 < \omega$ player II has a winning strategy for the following game and, moreover, the strategy is absolute. The game is defined for each countable set N (but only $N \cap F$ is needed) and it lasts ω moves.

In the kth move: *player I* chooses $f_k \in {}^\omega\omega, g_k \in F \cap N$, such that $f_k \restriction m_{\ell+1} = f_\ell \restriction m_{\ell+1}$ for $0 \le \ell < k$ and $f_k R_{n_k} g_k$ and then *player* II chooses $m_{k+1} > m_k$ and $n_{k+1} > n_k$.

In the end player II wins if $\left(\bigcup_{k<\omega} f_k \restriction m_k\right) R_{n_0} g$, (or if player I can't choose in the k'th move he lose).

4) We say $(F, \bar{R})$ is $S/\mathcal{D}_{\aleph_0}(F)$-nice if: for some $C \in \mathcal{D}_{\aleph_0}(F)$, we have: $(F, \bar{R})$ is $(S \cap C)$-nice.

5) We omit S when this holds for some $S \in \mathcal{D}_{\aleph_0}(F)$.

3.3 Notation. $<^*$ is the partial order on ${}^\omega\omega$ defined as: $f <^* g$ iff for all but finitely many $n < \omega, f(n) < g(n)$. In this case we say that g dominates f. We say that g dominates a family $F \subseteq {}^\omega\omega$ if g dominates every $f \in F$.

3.4 Definition. 1) A family $F \subseteq {}^\omega\omega$ is dominating if every $g \in {}^\omega\omega$ is dominated by some $f \in F$.
2) A family $F \subseteq {}^\omega\omega$ is unbounded (or undominated) if no $g \in {}^\omega\omega$ dominates it.

3.5 Definition. 1) A forcing notion P is almost ${}^\omega\omega$-bounding if: *for every* P-name $\underset{\sim}{f}$ of a function from ω to ω and $p \in P$ *for some* $g : \omega \to \omega$ (from V!) *for every* infinite $A \subseteq \omega$ (again A from V) there is $p', p \leq p' \in P$ such that:

$$p' \Vdash_P \text{"for infinitely many } n \in A, \underset{\sim}{f}(n) < g(n)\text{"}$$

2) A forcing notion P is weakly bounding (or F-weakly bounding, where $F \subseteq ({}^\omega\omega)^V$)) if $({}^\omega\omega)^V$ (or F) is an unbounded family in V^P.

3.6 Claim.

1) If a forcing notion P is weakly bounding, and $\underset{\sim}{Q}$ ($\in V^P$) is almost ${}^\omega\omega$-bounding, *then* their composition $P * \underset{\sim}{Q}$ is weakly bounding.
2) If Q is almost ${}^\omega\omega$-bounding, $F \subseteq {}^\omega\omega$ an unbounded family (from V) *then* F is still an unbounded family in V^Q.
3) If Q is adding λ Cohens (i.e. $Q \overset{\text{def}}{=} \{f : f$ a partial finite function from λ to $\{0,1\}\}$ ordered by inclusion) *then* Q is almost ${}^\omega\omega$-bounding.

Proof. 1) By part (2) (apply it in V^P to $F = ({}^\omega\omega)^V$ and the forcing notion Q).
2) Assume $p \in Q$ forces that $\underset{\sim}{f}$ dominates F and we shall get a contradiction. Let $g \in ({}^\omega\omega)^V$ be as in Definition 3.5(1). As in V, F is unbounded, for some $f^* \in F$ we have $\{n < \omega : g(n) < f^*(n)\}$ ($\in V$) is infinite, so choose this set as A, so by Definition 3.5(1) we know that for some p':

(a) $p \leq p' \in Q$

(b) $p' \Vdash_Q$ "for infinitely many $n \in A$, $\underset{\sim}{f}(n) < g(n)$ (hence by A's definition $\underset{\sim}{f}(n) < g(n) < f^*(n)$)"

and this contradicts $p \Vdash_Q$ "$\underset{\sim}{f}$ dominates F".

3) Easy. $\square_{3.6}$

3.7 Definition. $R^{\psi_0} = \psi_0(V)$ is: fRg iff $\{n : f(n) \leq g(n)\}$ is infinite.

3.8 Claim. A forcing notion P (in V) is weakly bounding ($\equiv$ adds no dominating real) *iff* $\Vdash_P$ "F is R-bounding" where $F = ({}^\omega\omega)^V$, $R = R^{\psi_0}$. $\square_{3.8}$

3.9 Claim. Let $R = R^{\psi_0}$ and $F \subseteq {}^\omega\omega$ be an R-bounding set, such that $(\forall f_0, \dots, f_n, \dots \in F)(\exists g \in F)[\bigwedge_{n<\omega} f_n <^* g]$. *Then* (F, R) is nice.

Proof. We have to describe g and an absolute winning strategy for N (and n_0, m_0). Choose $g \in F$ such that $(\forall f \in N)[f \in F \Rightarrow f <^* g]$. As for the strategy, n_ℓ is irrelevant, we just choose $m_{k+1} = \min\{m :$ there are at least k numbers $i < m$ such that $g(i) > f_k(i)\}$. $\square_{3.9}$

3.10 Claim. Suppose that $\mathcal{P} \subseteq [\omega]^{\aleph_0}$ is a P-filter (i.e. it is a filter containing the cobounded subsets and for any $A_n \in \mathcal{P}$ $(n < \omega)$ for some $A^* \in \mathcal{P}$ we have $(\forall n)[A^* \subseteq_{ae} A_n]$) and $\mathcal{P}$ has no intersection (i.e. there is no $X \in [\omega]^{\aleph_0}$ such that $X \subseteq_{ae} A$ for every $A \in \mathcal{P}$; recall that $X \subseteq_{ae} A$ means "$X \setminus A$ is finite"). Let R be:

$$xRy \text{ iff } x \notin [\omega]^{\aleph_0} \text{ or } y \notin [\omega]^{\aleph_0} \text{ or } x \not\subseteq_{ae} y.$$

(We identify $x \subseteq \omega$ with its characteristic function. The case "$y \notin [\omega]^{\aleph_0}$" will be irrelevant.)

Then

1) $(\mathcal{P}, R)$ is nice.

2) Let Q be a proper forcing notion. $\mathcal{P}$ is R–bounding in V^Q iff $\Vdash_Q$ " the filter $\mathcal{P}$ generates is a P-filter with no intersection " (i.e. every $q \in Q$ forces one statement iff it forces the other).

Proof. 1) Clause α) of Definition 3.2(3) is obvious as "$\mathcal{P}$ has no intersection" (see above). In (β) choose $g = A^* \in \mathcal{P}$ such that

$$(\forall A \in N)[A \in \mathcal{P} \Rightarrow A^* \subseteq_{ae} A].$$

Again, the least obvious point is the winning strategy; again n_k is irrelevant and player II chooses $m_k = \min\{m : f_k \cap m \setminus g \text{ has power} > k\}$.

2) Left to the reader. $\square_{3.10}$

3.10A Remark. We can use ${}^{\omega}\lambda$ instead ${}^{\omega}\omega$.

Sometimes we need a more general framework (but the reader may skip it, later replacing H_z, R^z_n by F, R_n).

3.11 Notation. If H is a set of (ordered) pairs, let $\mathrm{Rang}(H) = \{y : (\exists x)[\langle x, y\rangle \in H]\}$ and $\mathrm{Dom}(H) = \{x : (\exists y)[\langle x, y\rangle \in H]\}, H_x = \{y : \langle x, y\rangle \in H\}$.

We shall treat a set F (from e.g. Definition 3.2) as the following set of pairs: $\{\langle 0_\omega, x\rangle : x \in F\}$ where 0_ω is the function with domain ω and constant value 0 (so e.g. 3.13 applies to 3.2 too).

3.12 Definition. 1) For a set $H \subseteq {}^{\omega}\omega \times {}^{\omega}\omega$, and $\bar{R}$ (an ω-sequence of three place relations written as xR^zy) and $S \subseteq \mathcal{S}_{<\aleph_1}(H)$ we say that $(H, \bar{R})$ is S-nice if:

α) H is $\bar{R}$-bounding which means: for every $z \in \mathrm{Dom}(H)$, H_z is $\bar{R}^z$-bounding, i.e. $(\forall n)(\forall f \in {}^{\omega}\omega)(\exists g \in H_z)[fR^z_n g]$ letting $\bar{R}^z = \langle R^z_n : n < \omega\rangle$.

β) For any $N \in S$, $z \in \mathrm{Dom}(H \cap N)$ and for every $n_0, m_0 < \omega$ for some $g \in H_z$ and $z_0 \in \mathrm{Dom}(H) \cap N$ player II absolutely wins the following game which lasts ω moves.

In the kth move: *player* I chooses $f_k \in {}^{\omega}\omega, g_k \in \mathrm{Rang}(H \cap N)$ such that $f_k \restriction m_{\ell+1} = f_\ell \restriction m_{\ell+1}$ for $0 \leq \ell < k$ and $f_k R^{z_k}_{n_k} g_k$; then *player* II chooses $m_{k+1} > m_k, n_{k+1} > n_k$ and $z_{k+1} \in \mathrm{Dom}(H \cap N)$.

At the end of play, player II wins iff $(\bigcup_k f_k \restriction m_{k+1})R^z_{n_0} g$.

2) $(H, \bar{R})$ is $S/\mathcal{D}_{\aleph_0}(H)$-nice if for some $C \in \mathcal{D}_{\aleph_0}(H)$ we have: $(H, \bar{R})$ is $(S \cap C)$-nice.

3.13 Lemma.

1) Suppose

(i) $\bar{Q} = \langle P_j, \underset{\sim}{Q}_i : i < \delta, j \leq \delta \rangle$ is an iteration as in 0.1, for[†] $\mathbb{I}$

(ii) $S \subseteq \mathcal{S}_{<\aleph_1}(H)$ is stationary in V, and if we are in one of the cases (C), (E), (F) of 0.1, then $|H| = \aleph_1$

(iii) $(H, \bar{R})$ is $S/\mathcal{D}_{<\aleph_1}(H)$-nice

(iv) for every $i < \delta$, in V^{P_i} we have: H is $\bar{R}$-bounding

(v) all Q_i have pure $(\aleph_0, 2)$-decidability (see Definition 1.9)

(vi) $|H| = \aleph_1$ or at least

$$\left(\forall a \in [V^{P_\delta}]\right) \left[|a| \leq \aleph_0 \ \& \ a \subseteq H \Rightarrow \left(\exists b \in (\mathcal{S}_{\aleph_0}(H))^V\right)[a \subseteq b]\right].$$

Then in V^{P_δ}, H is $\bar{R}$-bounding.

2) We can weaken (v) to

$(\text{v})^-$ all Q_i has pure $(2,2)$-decidability

provided that for some fixed $f^* \in {}^\omega\omega$, $\left[(\exists i)[f(i) > f^*(i)] \Rightarrow f R^z_n g\right]$ for any $z \in \text{Dom}(H)$ and $g \in H_z$.

3) Assume $\bar{R} = \langle R_n : n < \omega \rangle$ a decreasing sequence of (absolute definitions of) three place relations on ${}^\omega\omega$, $F \subseteq {}^\omega\omega$ is $\bar{R}$ bounding (i.e. we are in the context of Definition 3.2 not 3.11, 3.12). Assume further (i), (ii), (iii), (iv) and (vi) from (1), replacing H by F and

$(\text{v})_f$ all Q_i have pure feeble $(\aleph_0, 2)$-decidability (see Definition 3.14 below). *Then* in V^{P_δ}, F is $\bar{R}$-bounding.

4) Assume, as in (2) that for some fixed $f^* \in {}^\omega\omega$, $[(\exists i) f(i) > f^*(i) \Rightarrow f R_n g]$ for any f, $g \in {}^\omega\omega$. The results of (3) holds if we replace $(\text{v})_f$ by $(\text{v})^-_f$ meaning replacing there $(\aleph_0, 2)$ by $(2,2)$.

[†] So the reader may think on CS of proper forcing so $\leq_{\text{pr}} = \leq$. The $\mathbb{I}$ is from clause (F) there, so can be ignored for the cases (A)-(E), e.g. the two cases just mentioned.

3.13A Remark.

1) You can read the proof with $n_0 = 0, F$ instead H, R instead $R^{z_n}_n$ (see 3.11).
2) The proof gives somewhat more than the lemma, i.e. it applies to more cases. "H is $\bar{R}$-bounding" means that (α) of 3.12 holds.
3) We can weaken 3.12(1)(β) to " in no generic extension of V, no strategy of player I is a winning strategy"(and 3.13 still holds). The proof is similar, only we choose the G_k in $V^{\mathrm{Levy}(\aleph_0, 2^{|P_\omega|})}$.
4) Part (3) (or (4)) of 3.13 is suitable for FS iteration of c.c.c. forcing by 3.16(4) below.

3.14 Definition. 1) A forcing notion Q has pure feeble (θ_1, θ_2)-decidability *if*: for every $p \in Q$ and Q-name $\underset{\sim}{\tau}$ satisfying $p \Vdash_Q$ "$\underset{\sim}{\tau} < \theta_1$" there are $a \subseteq \theta_1, |a| < \theta_2$ and $q, p \leq_{\mathrm{pr}} q \in Q$ such that q weakly decides $\underset{\sim}{\tau} \in a$; where
2) $q \in Q$ weakly decides $\underset{\sim}{\tau} \in a$ (or any other statement) if no pure extension of q decides this is false.
3) A forcing notion Q has pure weak (θ_1, θ_2)-decidability if for each $p \in Q$ in the following game, player II has a winning strategy.

In the n'th move player I chooses $\underset{\sim}{\tau}_n$, a Q-name of an ordinal $< \theta_1$ and player II chooses $a_n, a_n \subseteq \theta_1$, $|a_n| < \theta_2$. In the end player II wins the play if for every $n < \omega$ there is q_n, $p \leq_{\mathrm{pr}} q_n \in Q$, q_n weakly decides $\bigwedge_{\ell < n} \underset{\sim}{\tau}_\ell \in a_\ell$.

3.15 Proof of 3.13 (1). We speak mainly on cases (A) and (F) of 0.1(1). W.l.o.g. $\mathrm{cf}(\delta) = \aleph_0$ or for every $i < \delta$ we have $\Vdash_{P_i}$ "$\mathrm{cf}(\delta) > \aleph_0$" (by 3.16 below we have associativity; use a maximal antichain of conditions deciding and restrict yourselves above one member; then if necessary use renaming.)

If $\mathrm{cf}(\delta) > \aleph_0$, then any real in V^{P_δ} belongs to V^{P_j} for some $j < \delta$ (see III 4.1B(2), (or X or XIV or XV); hence there is nothing to prove, so we shall assume $\mathrm{cf}(\delta) = \omega$. By III, 3.3 or XV 1.7, w.l.o.g. $\delta = \omega$.

Suppose $p \in P_\omega$, $z \in \mathrm{Dom}(H)$, $n_0 < \omega$ and $\Vdash_{P_\omega}$ "$\underset{\sim}{f} \in {}^\omega\omega$" ; we shall find r, $p \leq_{\mathrm{pr}} r \in P_\omega$ and $g \in H_z$ such that $r \Vdash_{P_\omega}$ "$\underset{\sim}{f} R^z_{n_0} g$". Let $m_0 < \omega$. Let N be a countable elementary submodel of $(H(\lambda), \in)$ (λ regular large enough) to which

$\langle P_j, \underset{\sim}{Q}_i : i < \omega, j \leq \omega\rangle, p, \underset{\sim}{f}, z, S, H$ belong as well as the parameters involved the definitions of the R_n's. The set of such N belongs to $\mathcal{D}_{<\aleph_1}(H(\lambda))$, hence for some such N, $N \cap H \in S$ (and N is $\mathbb{I}$-suitable for case (F) of 0.1).

By 1.11 w.l.o.g. for each $n < \omega, \underset{\sim}{f}(n)$ is a P_n-name, and we let $p = \langle p_n^0 : n < \omega\rangle$ where $\Vdash_{P_n}$ "$\underset{\sim}{p}_n^0 \in \underset{\sim}{Q}_n$". Let $g \in H_z$ and $z_0 \in N \cap \mathrm{Dom}(H)$ be as in clause (β) of Definition 3.12 (for $N \cap H$ and z, n_0, m_0).

We shall now, by induction on $k < \omega$, define $q_k, \underset{\sim}{p}_k, \underset{\sim}{g}_k, \underset{\sim}{z}_k, \underset{\sim}{m}_k, \underset{\sim}{n}_k$ such that

(a) $q_k \in P_k$ is (N, P_k)-generic (for (A) of 0.1(1)) or (N, P_k)-semi-generic (for (F) of 0.1(1)) and $q_k \Vdash_{P_k}$ "$N[\underset{\sim}{G}_{P_k}] \cap H = N \cap H$"

(b) $q_k \restriction n = q_n$ for $n < k$

(c) $\underset{\sim}{p}_k \in P_\omega$, in fact is a P_k-name of a member of P_ω

(d) $\underset{\sim}{p}_k \restriction k \leq_{\mathrm{pr}} q_k$

(e) $\underset{\sim}{p}_{k+1} \restriction k = \underset{\sim}{p}_k \restriction k$ and $\underset{\sim}{p}_k \leq_{\mathrm{pr}} \underset{\sim}{p}_{k+1}$

(f) $q_k \Vdash_{P_k}$ "$\underset{\sim}{p}_k \in N[\underset{\sim}{G}_{P_k}]$" i.e. $\underset{\sim}{p}_k$ is a P_k-name of a member of $N[\underset{\sim}{G}_{P_k}] \cap (P_\omega / \underset{\sim}{G}_{P_k})$

(g) $\underset{\sim}{z}_k$ is a P_k-name of a member of $\mathrm{Dom}(H) \cap N$

(h) $\underset{\sim}{m}_k < \underset{\sim}{m}_{k+1}$ and $\underset{\sim}{n}_k < \underset{\sim}{n}_{k+1}$

(i) $\underset{\sim}{m}_k$, $\underset{\sim}{n}_k$ are P_k-names of natural numbers

Note that (a) implies that $N \cap H$ belongs to the club of $\mathcal{S}_{<\aleph_1}(H)$ involving "$(H, \bar{R})$ is $S/\mathcal{D}_{<\aleph_1}(H)$-nice".

For $k = 0$ we let $q_0 = \emptyset, p_0 = p$.

For $k+1$, we work in $V[G_k]$, G_k a generic subset of P_k satisfying $q_k \in G_k$. So $p_k = \underset{\sim}{p}_k[G_k] \in N[G_k]$ and $p_k \restriction k \in G_k$. In $N[G_k]$ we can find an $\leq_{\mathrm{pr}}$-increasing sequence of conditions $p_{k,i} \in P_\omega / G_k$ for $i < \omega$, such that $p_{k,0} = \underset{\sim}{p}_k[G_k]$ and $p_{k,i} \in N[G_k]$, moreover even $\langle p_{k,i} : i < \omega\rangle \in N[G_k]$ and $p_{k,i}$ forces values for $\underset{\sim}{f}(j)$ for $j \leq i$. So for some function $f_k \in N[G_k]$ we have $f_k \in {}^\omega\omega$ and $p_{k,i} \Vdash_{P_\omega/P_k}$ "$\underset{\sim}{f} \restriction i = f_k \restriction i$". As $N[G_k] \prec (H(\lambda)[G_k], \in)$ (see III 2.11), for some $g_k \in N[G_k] \cap H_{\underset{\sim}{z}_k} = N \cap H_{z_k}$, we have $N[G_k] \models$ "$f_k R^{z_k}_{n_k} g_k$"†. Now we use the absolute strategy (from Definition 3.6(2) for $N \cap H$) to choose

† Really n_k, g_k are P_k-names so we should have written $\underset{\sim}{n}_k[G_k]$ but ignore this.

$z_{k+1}, n_{k+1}, m_{k+1}$ (the strategy's parameters may not be in N, but the result is) and we want to have $p_{k+1} = p_{k,m_{k+1}}$. However all this was done in $V[G_k]$, so we have only suitable P_k-names which is O.K. In the end, let $r \in P_\omega$ be defined by $r \restriction k = q_k \restriction k$ for each k; by requirement (b) we know that r is well defined and belongs to P_ω. Suppose $r \in G_\omega \subseteq P_\omega, G_\omega$ generic over V. As in the proof of the preservation of properness we can prove by induction on k that $\underset{\sim}{p}_k \leq_{\mathrm{pr}} r$ for each k. Then in $V[G_\omega]$ we have made a play of the game from Definition 3.12(1)(β), player II using his winning strategy so $((\bigcup_k \underset{\sim}{f}_k \restriction k)[G_\omega])R^z_{n_0} g$ holds in $V[G_\omega]$, but clearly $p_{k,n_k} \leq_{\mathrm{pr}} \underset{\sim}{p}_{k+1} \leq_{\mathrm{pr}} r$ hence $p_{k,n_k} \in G_\omega$ hence $(\underset{\sim}{f} \restriction m_k)[G_\omega] = (\underset{\sim}{f}_k \restriction m_k)[G_\omega]$. Consequently $\underset{\sim}{f}[G_\omega] = (\bigcup_k \underset{\sim}{f}_k \restriction k)[G_\omega]$ and $\underset{\sim}{f}[G_\omega]R^z_{n_0} g$ holds in $V[G_\omega]$ and r forces the required information .

Proof of 3.13(2): Similar.

Proof of 3.13(3): Like 3.13(1). We use freely 3.16 below, but note that no harm is caused if player II increases m_k, n_k (not z_k!). A play (or an initial segment of the play) in which player II do this is said to weakly follow the strategies. Now the strategy in use is to weakly follow all possible subplays. I.e. above (in the proof of 3.13(1)) we, by induction on $k < \omega$, choose q_k, $\underset{\sim}{p}_k$, $\underset{\sim}{g}_k$, $\langle (\underset{\sim}{z}_v, \underset{\sim}{m}_v, \underset{\sim}{n}_v) : k \in v \subseteq k+1 \rangle$: and $\underset{\sim}{m}_k$, $\underset{\sim}{n}_k$ such that:

(a) - (f) and (h) as before

(g)$'$ $\underset{\sim}{z}_v$ is a P_k-name of a member of $\mathrm{Dom}(H) \cap N$

(i) $\underset{\sim}{m}_v$, $\underset{\sim}{n}_v$ are P_k-names of natural numbers, and

$$k = \max(v) \Rightarrow \underset{\sim}{m}_v < \underset{\sim}{m}_{v\setminus\{k\}} \ \&\ \underset{\sim}{n}_v < \underset{\sim}{n}_{v\setminus\{k\}}$$

(j) $\underset{\sim}{m}_k = \max\{m_v + k : v \subseteq k+1\}$, $\underset{\sim}{n}_k = \max\{\underset{\sim}{n}_v + k : v \subseteq k+1\}$.

In the induction step, $p_{k,i}$ ($i < \omega$), f_k are chosen such that: $p_k \leq_{\mathrm{pr}} p_{k,0}$, $p_{k,i} \leq_{\mathrm{pr}} p_{k,i+1}$ and no pure extension of $p_{k,i}$ in P_ω/G_k forces $\underset{\sim}{f}\restriction i \neq f_k \restriction i$. Now for each v such that $k \in v \subseteq k+1$ we pretend that the play so far involve only player I choosing $\langle (f_\ell, g_\ell) : \ell \in v \rangle$ and player II choosing $\langle (\underset{\sim}{m}_{v\cap(\ell+1)}, \underset{\sim}{n}_{v\cap(\ell+1)}, \underset{\sim}{z}_{v\cap(\ell+1)}) : \ell \in v \setminus \{k\} \rangle$ and player's II given winning strategy dictates $(\underset{\sim}{m}_v, \underset{\sim}{n}_v, \underset{\sim}{z}_v)$. Lastly $\underset{\sim}{m}_{k+1}$, $\underset{\sim}{n}_{k+1}$ are computed by clause (j).

We have defined a name for a strategy; we can show that it is forced that unboundedly often we have made the right move, so moving to the appropriate subplay we are done.

Proof of 3.13(4): Similar. $\square_{3.13}$

3.16 Claim. 1) For $(\theta_1, \theta_2) \in \{(2,2), (\aleph_0, 2)\}$ the property "Q has pure feeble (θ_1, θ_2)-decidability" is preserved by iteration as in 0.1.

2) Similarly† for "pure weak".

3) Q has pure feeble (θ_1, θ_2)-decidability if Q has pure weak decidability.

4) If Q has feeble pure $(\theta^*, 2)$-decidability and θ^* is uncountable and $\leq_{\rm pr}$ is equality (as we do for FS iteration of c.c.c. forcing) or $\theta^* \geq 2$ and $\leq_{\rm pr}$ is $\leq^Q$ (as for CS iteration of proper forcing) *then* Q has pure feeble $(\theta, 2)$-decidability for every θ.

5) For $(\theta_1, \theta_2) \in \{(n, 2) : 2 \leq n < \omega\}$, every Q has pure feeble (θ_1, θ_2)-decidability.

Proof. 1) We copy the proof of 1.10, changing (iii) (in the proof of case 5 $(\alpha = \omega)$) to

(iii)$'$ first for $n < \omega$ we define a P_{n+1}-name $\underset{\sim}{s}_n$: for $G_{n+1} \subseteq P_{n+1}$ generic over V, $\underset{\sim}{s}_n[G_{n+1}]$ is $k+1$ if there is $r \in P_\omega / G_{n+1}$ such that $\mathrm{Dom}(r) = [n+1, \omega)$, $P_\omega / G_{n+1} \models$ "$p \restriction [n+1, \omega] \leq_{\rm pr} r$" and r weakly decides $\underset{\sim}{t} = k$, i.e. for no $r', r \leq_{\rm pr} r' \in P_\omega / G_{n+1}$ does $p' \Vdash_{P_\omega / G_{n+1}}$ "$\underset{\sim}{t} \neq k$"; if there is no such r, then $\underset{\sim}{s}_n[G_{n+1}] = 0$.

Second let $q_n \in \underset{\sim}{Q}_n[G_n]$ be such that $p_n \leq_{\rm pr} q_n$ and q_n weakly decides the value of $\underset{\sim}{s}_n$, (i.e. of $\underset{\sim}{s}_n / G_n$) (if $\theta_1 = 2$, use Definition 3.13A twice).

Also in the end we prove by downward induction on $m \leq n(*)$ that $(r \restriction m) \cup \{q_m\}$ weakly decides $\underset{\sim}{s}_m = \ell$.

2) Similar proof (using 3.16(1)).

3) Read the definitions.

4) Straight.

† Alternatively use XIV §2.

5) Easy. $\square_{3.16}$

* * *

We now give some applications. Concerning 3.17 if you want also "no Cohen", see 2.13.

3.17 Conclusion. 1) The property "P is weakly bounding" i.e. "P does not add a dominating real over V" is preserved in limit (for iterations as in $0.1_{\theta=2}$, see 0.1(3)) provided that $\mathfrak{b}^V = \aleph_1$ in the non-proper case.
2) If $F \subseteq {}^\omega\omega$ is not $<^*$-bounded then "P does not add a $<^*$-bound to F" is preserved in limit (for iterations as in $0.1_{\theta=2}$) provided that e.g. $|F| = \aleph_1$ in the non proper-cases.
3) In parts (1)+(2) we can use iterations as in 0.1 with pure feeble $(\aleph_0, 2)$-decidability.

Proof. 1), 2) Let $\bar{Q}$ be such an iteration, $F = ({}^\omega\omega)^V$ for 3.17(1), given for 3.17(2) and R is defined by ψ_0 (see Definition 3.7). By 3.9 (F, R) is a nice pair in V. Even for every $i < \ell g(\bar{Q})$, in V^{P_i} the set F is still unbounded and every countable subset of F in V^{P_i} is included in a countable subset of F from V; hence by 3.9 (F, R) is a nice pair even in V^{P_i}. By 3.13(3) this is true also in V^{P_δ}(where $\delta = \ell g(\bar{Q})$.
3) Similar proof (to that of 3.13(1)) or by 3.13(3)). $\square_{3.17}$

3.18 Lemma. The property "P_α purely adds no random real over V" is preserved under limits for iterations as in $0.1_{\theta=2}$ or just by iteration as in 0.1 with every $\underset{\sim}{Q}_i$ having pure feeble $(\aleph_0, 2)$-decidability (see 3.16(4)).

Remark. Concerning the successor case see XVIII 3.20(i). Before we prove 3.18 we need some definitions and claims. Now for $T \subseteq {}^{\omega>}\omega$, and $\eta \in {}^{\omega>}2$ we let $T^{\langle\eta\rangle} = \{\nu : \eta\hat{\ }\nu{\restriction}[\ell g(\eta), \omega) \in T\}$. Note that Lemma 3.18 includes the case of FS iterations.

3.19 Definition. 1) We let ψ_1 be as follows:
$xR^{\psi_1}y$ *iff* y is a perfect subtree of ${}^{\omega>}2$ with positive Lebesgue measure, $x \in {}^{\omega}2$ and $(\forall n < \omega)(\forall \rho \in {}^{n}2)[\rho\hat{\ }(x\restriction[n,\omega)) \notin \lim y]$.
2) Let H_1^V be $\{\langle y_1, y_2\rangle : y_1, y_2$ are perfect subtrees of ${}^{\omega>}2$ with positive Lebesgue measure such that: $\lim y_2 \subseteq \{\eta \in {}^{\omega}2$: for some $n < \omega$ and $\rho \in {}^{n}2$ we have $\rho\hat{\ }(\eta\restriction[n,\omega)) \in \lim y_1\}$ $\}$

3.20 Claim. 1) H_1^V is an $\aleph_1$-directed partial order.
2) Suppose $V \subseteq V_1$, and for any countable $a \subseteq H_1^V$ from V_1 there is a countable $b \subseteq H_1^V, a \subseteq b \in V_1$ (and $R = R^{\psi_1}$). The following are equivalent:

(i) no real in V_1 is random over V

(ii) $\mathrm{Dom}(H_1^V)$ is R-bounding in V_1

(iii) $(\mathrm{Dom}(H_1^V), R)$ is nice in V_1 (here Definition 3.2(3) is the relevant one, with $\mathrm{Dom}(H_1^V)$ here having the role of F there).

Proof. 1) Easy
2) (i) $\Rightarrow$ (ii): Let $x \in ({}^{\omega}2)^{V_1}$. As x is not random over V there is a Borel set $B \in V$ of Lebesgue measure 0 such that $x \in B$ (i.e. x belongs to the V_1-interpretation of B). Without loss of generality B is closed under $=^*$ (i.e. if $\eta_1, \eta_2 \in {}^{\omega}2$ and $\eta_1 =^* \eta_2 (\equiv \bigvee\limits_{n<\omega} \eta_1\restriction(n,\omega) = \eta_2\restriction[n,\omega))$ then $\eta_1 \in B \equiv \eta_2 \in B$). There is $T \subseteq {}^{\omega>}2$ perfect, $T \in V$ such that $\lim T$ has positive measure and $\lim(T) \cap B = \emptyset$. So it is enough to prove that xRT, i.e. $(\forall n < \omega)[x \notin \lim T^{\langle n\rangle}]$ where $T^{\langle n\rangle} \stackrel{\text{def}}{=} \{\eta :$ for some $\rho \in T$ we have $\ell g(\rho) = \ell g(\eta)$ and $(\forall \ell)(n \le \ell < \ell g\rho \rightarrow \rho(\ell) = \eta(\ell))\}$ i.e. $x \in {}^{\omega}2 \setminus \bigcup_{n<\omega} \lim(T^{\langle n\rangle})$, but this follows from $x \in B$.

(ii) $\Rightarrow$ (iii): Condition (α) of Definition 3.2(3) is clear. For condition (β) let $N \prec (H(\chi), \in)$ be countable, $z_0 \in N \cap \mathrm{Dom}(H_1)$, so for some a we have $N \cap H_1^V \subseteq a \subseteq H_1^V$, $a \in V$, $V \models |a| = \aleph_0$. So there is $T \in \mathrm{Dom}(H_1^V)$, such that ${}^{\omega}2 \setminus \bigcup_{n<\omega} T^{\langle n\rangle}$ contains all $2^{\omega} \setminus \bigcup_{n<\omega} \lim(T_1^{\langle n\rangle})$ for $T_1 \in N \cap \mathrm{Dom}(H_1^V)$, hence it contains all Borel measure zero sets from V which are in N.

We have to give the winning strategy for player II.
In stage k, f_k, g_k are given $f_k R g_k$, $g_k \in N \cap \mathrm{Dom}(H_1^V)$, so g_k is a perfect subtree of ${}^{\omega>}2$ of positive Lebesgue measures. Then $\rho\hat{\ }(f_k\restriction[n,\omega) \notin \lim g_k$ for $\rho \in {}^{n}2$,

$n < \omega$ and $(g_k, T) \in H_1$; together with the choice of T we know that for $n < \omega$ and $\rho \in {}^n2$ we have $\rho\hat{}(f_k{\restriction}[n,\omega)) \notin \lim T$.

Choose $m_{k+1} > m_k$ large enough such that: for every $n \le m_k$, $\rho \in {}^n2$ we have: $\rho\hat{}\, f_k \restriction [n, m_{k+1}) \notin T$.

(iii) $\Rightarrow$ (i): Immediate. $\square_{3.20}$

3.21 *Proof of lemma 3.18*

Let $F \stackrel{\text{def}}{=} \mathrm{Dom}(H_1^V)$. Let $\bar{Q}$ be an iteration as in 0.1, $\ell g(\bar{Q}) = \delta$, and in no $V^{P_i}(i < \delta)$ is there a real random over V. So by the claim 3.20(2) we know that $(F, \bar{R})$ is nice in V^{P_i}. Hence by 3.13(3) it is nice in V^{P_δ}, hence by claim 3.20(2) in V^{P_δ} there is no real random over V. $\square_{3.18}$

We now give an application of 3.17, taken from [Sh:207], Lemma 3.22 is proved in §6 (see 6.13). On history see introduction to §6.

3.22 Lemma. There is a forcing notion Q such that

(a) Q is proper

(b) Q is almost ${}^\omega\omega$-bounding.

(c) $|Q| = 2^{\aleph_0}$

(d) In V^Q there is an infinite set $A^* \subseteq \omega$ such that for every infinite $B \subseteq \omega$ from V we have $A^* \cap B$ is finite or $A^* \setminus B$ is finite.

3.22A Remark. For 3.23 it is enough to prove 3.22 assuming CH.

3.23 Theorem. Assume $V \models$ CH.

1) For some forcing notion P^*, P^* is proper, satisfies the $\aleph_2$-c.c., and

(*) In V^{P^*}, $2^{\aleph_0} = \aleph_2$, there is an unbounded family of power $\aleph_1$, but no splitting family (see below) of power $\aleph_1$.

2) We can also demand that in V^{P^*} there is no MAD of power $\aleph_1$ (see Definition 3.24(2)).

3.24 Definition. 1) $\mathcal{P}$ is a splitting family if $\mathcal{P} \subseteq [\omega]^{\aleph_0}$ (= the family of infinite subsets of ω) and for every $A \in [\omega]^{\aleph_0}$ for some $B \in \mathcal{P}$ we have: $|A \cap B| = |A \setminus B| = \aleph_0$.

2) A family $\mathcal{A}$ is MAD (maximal almost disjoint) if:

(a) $\mathcal{A}$ is a subset of $[\omega]^{\aleph_0}$

(b) for any distinct $A, B \in \mathcal{A}$ the intersection $A \cap B$ is finite

(c) $\mathcal{A}$ is maximal under (a) + (b).

3) Let $\mathfrak{b} = \min\{|F| : F \subseteq {}^\omega\omega$ is not dominated$\}$ where "F not dominated" means that for every $g \in {}^\omega\omega$ for some $f \in F$ we have $\neg f \leq^* g$. Let $\mathfrak{d} = \min\{|F| : F \subseteq {}^\omega\omega$ is dominating$\}$ where "F is dominating" means that for every $g \in {}^\omega\omega$ for some $f \in F$ we have $g <^* f$. Let $\mathfrak{s} = \min\{|\mathcal{P}| : \mathcal{P} \subseteq [\omega]^{\aleph_0}$ is a splitting family (see above)$\}$

Proof of 3.23. 1) We define a countable support iteration of length $\aleph_2$: $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \omega_2 \rangle$ with (direct) limit $P^* = P_{\omega_2}$. Now each $\underset{\sim}{Q}_\alpha$ is the Q from 3.22 for V^{P_α}, so $V^{P_\alpha} \models$ "$|\underset{\sim}{Q}_\alpha| = 2^{\aleph_0}$". As $V \models$ CH we can prove by induction on $\alpha < \omega_2$ that $\Vdash_{P_\alpha}$ "CH" (see III, Theorem 4.1). We also know that P^* satisfies the $\aleph_2$-c.c. (see III, Theorem 4.1). If $\mathcal{P}$ is a family of subsets of ω of power $\leq \aleph_1$ in V^{P^*} then for some $\alpha < \omega_2, \mathcal{P} \in V^{P_\alpha}$, and forcing by Q_α gives a set A^*_α exemplifying $\mathcal{P}$ is not a splitting family by clause (d) of 3.22. So from all the conclusions of 3.23 only the existence of an undominated family of power $\aleph_1$ remains. Now we shall prove that $F = ({}^\omega\omega)^V$ is as required. By 3.8 it is enough to show

(*) $\Vdash_{P_{\omega_2}}$ "F is R^{ψ_0}-bounding" (see Definition 3.7).

Now note: F has power $\aleph_1$ as $V \models$ CH. We prove that F is R^{ψ_0}-bounding in V^{P_α} by induction on $\alpha \leq \omega_2$. For $\alpha = 0$ this is trivial; $\alpha = \beta + 1$: as $\underset{\sim}{Q}_\beta$ is almost ${}^\omega\omega$-bounding (see 3.22 clause (b)) and by Fact 3.6(1); if $\mathrm{cf}(\alpha) \geq \aleph_0$ by Conclusion 3.17(1).

2) Similar. We use a countable support iteration $\langle P_j, \underset{\sim}{Q}_i : i < \omega_2, j \leq \omega_2 \rangle$ such that:

(a) for every $i < \omega_2$, and MAD $\langle A_\alpha : \alpha < \omega_1 \rangle \in V^{P_i}$, for some $j > i$, *either* $\underset{\sim}{Q}_{2j}$ = adding $\aleph_1$ Cohen reals, and $\underset{\sim}{Q}_{2j+1} = \{p \in \underset{\sim}{Q}^{V^{P_{2j+1}}} :$

$p \geq p_{2j+1}\}$ where in $V^{P_{2j+1}}$ we have $p_{2j+1} \Vdash_{\underset{\sim}{Q}}$ "$\langle A_\alpha : \alpha < \omega_1 \rangle$ is not a MAD family"

or $\underset{\sim}{Q}_{2j}$ = adding $\aleph_1$-Cohen reals, $\underset{\sim}{Q}_{2j+1} = Q[I_{2j+1}]$ where I_{2j+1} is the ideal (of subsets of ω) which $\langle A_\alpha : \alpha < \omega_1 \rangle$ and the cofinite sets generate (on $Q[I]$ see Definition 6.10).

(b) For j even $\underset{\sim}{Q}_j$ is adding $\aleph_1$ Cohen reals.

(c) For j odd, $\underset{\sim}{Q}_j$ is $\underset{\sim}{Q}$, or $\{p \in \underset{\sim}{Q} : p \geq p_j\}$, or it is $Q[I_{\underset{\sim}{\mathcal{A}}}]$ where $\underset{\sim}{\mathcal{A}}$ is a P_{j-1}-name of a MAD family of cardinality $\aleph_1$ and $V^{P_j} \models [\Vdash_{\underset{\sim}{Q}}$ "$\mathcal{A}_j$ is MAD"]

(d) for $\aleph_2$ ordinals j, Q_{2j+1} is $Q^{V^{P_{2j+1}}}$.

There is no problem to carry out the definition. Each $\underset{\sim}{Q}_j$ is almost ${}^\omega\omega$-bounding by 3.22 (i.e. see 6.13, when $\underset{\sim}{Q}_j = Q^{V^{P_j}}$), by 6.22 (when $\underset{\sim}{Q}_j = Q[I_{\mathcal{A}_j}]$ we can apply it to $V^{P_{j-1}}$ as $\underset{\sim}{Q}_{j-1}$ is adding $\aleph_1$ Cohens so 6.15 applies and the second possibility in 6.22 fails by clause (c) above) and 3.6(3) (if j is even i.e. Q_j is adding $\aleph_1$ Cohens). So as in part (1), P_{ω_2} preserves "$({}^\omega\omega)^V$ unbounded". Also $\mathfrak{s} = \aleph_2$ and $2^{\aleph_0} = \aleph_2$ are proved as part (1) by clause (d). Lastly assume $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is a MAD family, $|\mathcal{A}| = \aleph_1$, so for some i, $\mathcal{A} \in V^{P_i}$. So there is j as in clause (a). Work over $V_0 = V^{P_{2j}}$ so $\underset{\sim}{Q}_{2j}$ is adding $\aleph_1$ Cohens. If $p \Vdash_{\underset{\sim}{Q}_{2j} * \underset{\sim}{Q}}$ "$\mathcal{A}$ is not MAD" for some $p \in \underset{\sim}{Q}_{2j} * \underset{\sim}{Q}$ *then* w.l.o.g. $p \in \underset{\sim}{Q}$ (as $\underset{\sim}{Q}_{2j}$ is homogeneous) and we use the second possibility in clause (c). If not, we use the third possibility of clause (c). $\square_{3.23}$

* * *

We add the following in Summer'92 after a question of U. Abraham. In the proof of the consistency of "there is no P-point" below (§4) we use the "PP-property" (see 2.12A-F). We actually prove a stronger property called "the strong PP-property" which implies the "PP-property" which we have proved is preserved, so Abraham asked whether it itself is preserved. The following variants would have sufficed for the purpose of §4 which was the reason of existence.

3.25 Lemma. Assume that $\mathsf{f} : \omega \to \omega + 1 \setminus \{0, 1\}$, $\mathsf{h} : \omega \to \omega \setminus \{0, 1\}$ and $F \subseteq \prod_{\mathsf{f},\mathsf{h}} \stackrel{\mathrm{def}}{=} \{f : \mathrm{Dom}(f) = \omega$ and $f(n)$ a subset of $\mathsf{f}(n)$ of cardinality $< \mathsf{h}(n)$, $\lim\limits_{n \to \infty} (|f(n)|/\mathsf{h}(n)) = 0\}$ are such that:

(∗) for any countable $A \subseteq F$ there is $f \in F$ such that $\bigwedge\limits_{g \in A} (\forall^* n)[g(n) \subseteq f(n)]$.

Let R be defined as: gRf *iff*

(a) $g, f \in \prod_{\mathsf{f},\mathsf{h}}$ (i.e. we consider a member of ${}^\omega\omega$ as coding such sequences)

(b) $(\exists^* n) g(n) \subseteq f(n)$.

Let $S \subseteq \mathcal{S}_{<\aleph_2}(F)$ be stationary and assume F is R-bounding.

Then (F, R) is S-nice. (Hence, we have a preservation theorem for a limit).

Proof. Check Definition 3.2(3). Part (α), F is R-bounding, should be clear. For clause (β), given $N \in S$, let $f_N \in F$ be such that $f \in N \cap F \Rightarrow (\forall^* n)[f(n) \subseteq f_N(n)]$ (it exists by the assumption (∗)). The winning strategy is clear: choose m_{k+1} such that $\{i < m_{k+1} : f_k(i) \subseteq g_k(i) \subseteq f_N(i)\}$ has at least k members. $\square_{3.25}$

But of course it is nicer to have also preservation for composition of two forcing notions.

3.26 Lemma. 1) Let $\mathsf{f} : \omega \to \omega + 1 \setminus \{0, 1\}$, $\mathsf{h}^t : \omega \to \omega \setminus \{0\}$ for $t \in \mathbb{Q}$ be such that for $s < t$ (from $\mathbb{Q}$) we have $0 = \lim\limits_{n \to \infty} (\mathsf{h}^s(n)/\mathsf{h}^t(n))$ and for each $t \in \mathbb{Q}$ the set $\prod_{\mathsf{f},\mathsf{h}^t}$ satisfies (∗) of 3.25. The following property is preserved by iterations as in $0.1_{\theta=2}$ and as in 0.1 with each Q_i having pure feeble $(2,2)$-decidability:

$(*)_1$ (a) Q is purely ${}^\omega\omega$-bounding.

(b) *for every* $s < t$ from $\mathbb{Q}$, $f \in V^Q$ such that $f \in \prod_{\mathsf{f},\mathsf{h}^s}$ and $k_0 < k_1 < \dots$ (so $\langle k_i : i < \omega \rangle \in V$) *for some* $g \in V$ such that $g \in \prod_{\mathsf{f},\mathsf{h}^t}$ we have $(\exists^* i)\ [\bigwedge_{\ell \in [k_i, k_{i+1})} f(\ell) \subseteq g(\ell)]$.

1A) So e.g. in (1), if $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha \rangle$ is CS iteration, $\Vdash_{P_i}$ "Q_i is proper satisfying $(*)_1$" then P_α is proper satisfying $(*)_1$.

2) We can replace $(*)_1$, by

$(*)_2$ (a) Q is purely ${}^\omega\omega$-bounding

(b) *for every* $s < t$ from $\mathbb{Q}$ and $f \in V^Q$, such that $f \in \prod_{\mathbf{f},\mathbf{h}^s}$ *for some* $g \in V$, $g \in \prod_{\mathbf{f},\mathbf{h}^t}$ and for every infinite $A \in ([\omega])^{\aleph_0})^V$, for infinitely may $i \in A$, $f(i) \subseteq g(i)$.

3) In (1) we can assume $F = ({}^\omega\omega)^{V'}$ (for some $V' \subseteq V$, a just reasonably closed) is unbounded $\aleph_1$-directed by $<^*$ and replace $(*)_1$ by

$(*)_3$ (a) Q purely preserves "F is unbounded"

(b) like (b) of $(*)_1$ for $\langle k_i : i < \omega \rangle \in F$.

Proof. Similarly to the previous Lemma one may deal with limit cases using 3.13 for the respective variant of clause (b) (and by 2.8 for $(*)_1$(a), $(*)_2$(a), 3.17 for $(*)_3$(a)). So now it suffices to prove this for iteration of length two: $P_2 = Q_0 * \underset{\sim}{Q}_1$ (so $P_1 = Q_0$, P_0 is trivial). First we prove part (2). Let $f \in (\prod_{\mathbf{f},\mathbf{h}^s})^{V^{P_2}}$, $s < t$ from $\mathbb{Q}$. Choose $t' \in (s,t)_{\mathbb{Q}}$. Applying $(*)_2$ to f, s, t', V^{P_1}, V^{P_2} we can find $f' \in [\prod_{\mathbf{f},\mathbf{h}^{t'}}]^{V^{P_1}}$ satisfying the requirements there on g. Next we apply $(*)_2$ on f', t', t, V, V^{P_1} and get $g \in [\prod_{\mathbf{f},\mathbf{h}^t}]^V$.

Now for any infinite $A \subseteq \omega$, $A \in V$, by the choice of g we know that $A' =: \{i \in A : f'(i) \subseteq g(i)\} \in V^{Q_0}$ is infinite. Hence by the choice of f' we know that $A'' \overset{\text{def}}{=} \{i \in A' : f(i) \subseteq f'(i)\}$ is infinite and clearly it belongs to V^{P_2}. Putting together $A'' = \{i \in A : f(i) \subseteq f'(i) \subseteq g(i)\}$ is infinite. So g is as required.

Now we prove part (1), so we are given $s < t$ from $\mathbb{Q}$ and $\langle k_i : i < \omega \rangle \in V$ (strictly increasing) and $f \in (\prod_{\mathbf{f},\mathbf{h}^s})^{V^{P_2}}$. Let $t' \in (s,t)_{\mathbb{Q}}$. Applying $(*)_1$, to f, s, t', $V^{P_1}, V^{P_2}, \langle k_i : i < \omega \rangle$ we get $f' \in (\prod_{\mathbf{f},\mathbf{h}^{t'}})^{V^{P_1}}$ satisfying the requirements on g in $(*)_1$. So $A =: \{i :$ for every $\ell \in [k_i, k_{i+1})$ we have $f(\ell) \subseteq f'(\ell)\}$ is infinite. As Q_0 is ${}^\omega\omega$-bounding there is a sequence $\ell(0) < \ell(1) < \ldots$ (in V) such that $A \cap [\ell(i), \ell(i+1)) \neq \emptyset$ for every $i < \omega$. Let $k'_i = k_{\ell(i)}$ and apply $(*)_1$ to $f', t', t, V, V', \langle k'_i : i < \omega \rangle$ and get g, which is as required.

The proof of part (3) is similar. $\square_{3.26}$

3.26A Remark. In 3.25 we have the requirement $F \cap \prod_{\mathbf{f},\mathbf{h}}$ satisfies $(*)$ of 3.25. We can work as 3.26(2) and weaken it to:

$(*)^-$ for any $s < t$ and countable $A \subseteq \prod_{\mathbf{f},\mathbf{h}^t}$ there is $f \in \prod_{\mathbf{f},\mathbf{h}^t}$ such that $\bigwedge_{g \in A} (\forall^* n)[g(n) \subseteq f(n)]$.

§4. There May Be No *P*-Point

We define the forcing notion $P(F)$ (introduced by Gregorief) which, for an ultrafilter F, adds a set A such that $\omega \setminus A$, $A \neq \emptyset \bmod F$, see definition 4.1. If F is a P-point (see definition 4.2A) this forcing is α-proper for every $\alpha < \omega_1$, and has the PP-property. Our point is that $P(F)^\omega$ enjoys all these properties and in addition $\Vdash_{P(F)^\omega}$ "F cannot be completed to a P-point ". We will argue in the following way: as we use $P(F)^\omega$, we can define a new subset A_n of ω such that $\Vdash_{P(F)^\omega}$ "$A_n \in \underset{\sim}{E}$", where $\underset{\sim}{E}$ is an extension of F to an ultrafilter in the generic extension, but for each $g \in {}^\omega\omega \cap V$ we have $\Vdash_{P(F)^\omega}$ "$\bigcap_{n<\omega}(A_n \cup g(n)) \equiv \emptyset \bmod F$".

We originally (see the presentation in Wimmer [Wi]) use the stronger version of the PP-property, but there were problems with the preservation theorem i.e., in that version the essential forcing was not an iteration.

Note that, we continue to add reals after forcing with $P(F)^\omega$, so in fact we prove the above described argument works with Q instead of $P(F)^\omega$ provided $P(F)^\omega \lessdot Q$, Q has the PP-property. So the importance of proving that this property of Q is preserved is clear. The iteration in the end is standard.

The proof presented in [Wi] uses not exactly $P(F)^\omega$. Rather we note that if Q satisfies the c.c.c. then for any P-point F_0 in V^P there is $F_1 \stackrel{\text{def}}{=} \{A \subseteq \omega : A \in V, \Vdash_Q \text{“}A \in \underset{\sim}{F}_0\text{”}\}$ which is a filter enjoying some of the properties of $F_0 : \mathcal{P}(\omega)/F_1$ (in V) is a Boolean algebra satisfying the c.c.c. and (if Q has the ${}^\omega\omega$-bounding property), for every $A_n \in F_1$ there is $A \in F_1$, $\bigwedge_{n<\omega} A \subseteq_{ae} A_n$. Let $\{F^i : i < \aleph_2\}$ (assuming G.C.H.) list all such filters in V. Now the product P with countable support of all the $P(F^i)^\omega$ satisfies: in V^P, no F^i can be extended to a P-point by an argument as mentioned above. However to close the proof we need "P satisfies the c.c.c. ", which fails. But we replace P by a subset which satisfies it and still has the desirable other properties. I expect that the proof can be modified to have $2^{\aleph_0} > \aleph_2$ (but this was not carefully checked), whereas for the present proof we do not know how to do this.

4.1 Definition. For a filter D on a set I (we always assume all co-finite subsets of I are in D), we define the following forcing notions ordered by inclusion:

1) $P(D) = \{f : f$ is a function from B to $\{0,1\}$ for some $B = \emptyset \bmod D$, i.e. $B \subseteq I, I \setminus B \in D\}$,

2) $P^\dagger(D) = \{f \in P(D) : f^{-1}(\{1\})$ is finite $\}$,
$P'(D) = \{f : f$ a function from B to $\{0,1\}$, $B \neq I \bmod D\}$,
$P''(D) = \{f \in P'(D) : f^{-1}(\{1\})$ is finite $\}$.

4.1A Remark. Mathias [Mt3] used $P''(D)$ for the filter D of co-finite subsets of ω; Silver used $P^\dagger(D)$ for the filter D of cofinite subsets of ω and for an ultrafilter D, Gregorief used $P(D)$ for an ultrafilter D, and proved that it collapses $\aleph_1$ iff it is not a P-point.

4.2 Lemma. If F is a P-point (see below) *then* $Q = P(F)^\omega$ is proper (in fact α-proper for every $\alpha < \omega_1$) and has the PP-property.

Proof. It will follow from 4.3 and 4.4. $\square_{4.2}$

4.2A Definition.

1) A filter F on I is called a P-filter or P-point filter *if* (it contains all co-finite subsets of I and) for every $A_n \in F$ (for $n < \omega$) there is $A \in F$ such that $A \subseteq_{ae} A_n$ for every n. Just "a P-point" means an ultrafilter.

2) We call F fat if for every family of finite pairwise disjoint $w_n \subseteq I$ (for $n < \omega$) there is an infinite $S \subseteq \omega$ such that $\bigcup_{n \in S} w_n = \emptyset \bmod F$. (Clearly every P-point is fat.)

3) F is a Ramsey ultrafilter (on I) if for every $h : I \to \omega$ there is $A \in F$ such that $h \restriction A$ is a constant or $1-1$ (and F contains all co-finite subsets of I). Note that a Ramsey ultrafilter is a P-filter.

4.3 Fact. Assume F is a fat P-filter on ω. Let F^* be $\{A \subseteq \omega \times \omega :$ for every n for some $B \in F$, $A \cap (\{n\} \times \omega) = \{n\} \times B\}$. *Then* F^* is a fat P-filter on $\omega \times \omega$, and the forcing notion $P(F)^\omega$ is isomorphic to $P(F^*)$.

Proof of the Fact.
First condition: F^* is a filter on $\omega \times \omega$ including all co-finite sets. Check.
Second condition: F^* is a P-filter. Let $A_k \in F^*$, then $A_k \cap (\{n\} \times \omega) = \{n\} \times B_{k,n}$ for some $B_{k,n} \in F$. As F is a P-filter there is $B^* \in F$, $B^* \subseteq_{ae} B_{k,n}$ for every k, n. Let $B_n^* \stackrel{\text{def}}{=} B^* \cap \bigcap_{k<n} B_{k,n}$ and $A^* \stackrel{\text{def}}{=} \bigcup_{n<\omega}(\{n\} \times B_n^*)$. Clearly $B_n^* \in F$ (as we have assumed that F is a filter), hence $A^* \in F^*$. Now $A^* \setminus A_k \subseteq \bigcup_{n<\omega}[\{n\} \times (B_n^* \setminus B_{k,n})] \subseteq \bigcup_{n\leq k}[\{n\} \times (B_n^* \setminus B_{k,n})]$, but $B_n^* \setminus B_{k,n} \subseteq B^* \setminus B_{k,n}$, hence it is finite. Therefore $A^* \setminus A_k$ is finite and hence $A^* \subseteq_{ae} A_k$. But k is arbitrary, so A^* is as required.

Third condition: F^* is fat. Let $w_n \subseteq \omega \times \omega$ be finite and pairwise disjoint for $n < \omega$. We define by induction on n infinite sets $S_n \subseteq \omega$, $S_{n+1} \subseteq S_n$ such that $\{i < \omega : \langle n, i\rangle \in \bigcup_{\ell \in S_n} w_\ell\} = \emptyset \bmod F$. We can do this with no problem, and let $k(0) = \mathrm{Min}(S_0)$, $k(n) = \mathrm{Min}(S_n \setminus \{k(0) \dots k(n-1)\})$. As every cofinite subset of ω belongs to F, it is easy to check $\bigcup_{n<\omega} w_{k(n)} = \emptyset \bmod F^*$.
So we have established the first conclusion of 4.3.

The isomorphism of $P(F)^\omega$ and $P(F^*)$ is trivial, for $p = \langle f_0, f_1, f_2 \dots\rangle \in P(F)^\omega$, let $H(p) \in P(F^*)$ be $H(p)(\langle n, k\rangle) = f_n(k)$. $\square_{4.3}$

So it suffices (for proving Lemma 4.2) to prove:

4.4 Lemma. If F is a fat P-filter on a countable set *then* $P(F)$ is proper (in fact α-proper for every $\alpha < \omega_1$) and has the PP-property.

4.4A Remark. We will really prove the strong PP-property, see remark to 2.12A and Def 2.12E(1),(3).

Proof of 4.4. W.l.o.g. F is a filter on ω. So let $p_0 \in P(F)$, $\{p_0, F\} \in N \prec (H(\lambda), \in)$, N is countable and λ is large enough. Now, before proving properness, we prove:

4.5 Crucial Fact. For every $p \in P(F)$ and every $P(F)$-name $\underset{\sim}{t}$ of an ordinal and $n < \omega$, there is $q \in P(F)$, $p \leq q$, such that $q{\restriction}n = p{\restriction}n$ and for every $g : n \to 2$, there is an ordinal α_g such that $(q{\restriction}[n,\infty)) \cup g \Vdash_{P(F)}$ "$\underset{\sim}{t} = \alpha_g$".

Proof. Let g_i (for $i < 2^n$) be a list of all functions $g : n \to 2$. We shall define by induction, an increasing sequence of conditions $p_i \in P(F)$ (so $p_i \leq p_{i+1}$), for $i \leq 2^n$. Let $p_0 = p$, and if p_i is defined let

$$p_i^\dagger = (p_i{\restriction}[n,\infty)) \cup g_i.$$

Clearly $p_i^\dagger \in P(F)$ hence there are α_{g_i} and $p_i'' \in P(F), p_i^\dagger \leq p_i''$ such that $p_i'' \Vdash_{P(F)}$ "$\underset{\sim}{t} = \alpha_{g_i}$". Let $p_{i+1} = p_i \cup p_i''{\restriction}[n,\omega)$. Clearly $p_i \leq p_{i+1} \in P(F)$, and $(p_{i+1}{\restriction}[n,\omega)) \cup g_i \Vdash$ "$\underset{\sim}{t} = \alpha_{g_i}$". So $p_{(2^n)}$ is as required from q. So we have proved Fact 4.5. $\square_{4.5}$

Before we prove 4.4 we also note

4.6. Fact. Assume F is a fat P-filter (on ω). If $p_n \leq p_{n+1}$ (for $n < \omega$), $p_n \in P(F)$ *then* there is $q \in P(F)$, $q \geq p_0$ such that $q \geq p_n{\restriction}[n,\infty)$ for infinitely many n.

Proof of Fact 4.6. Let A_n be the domain of p_n, so $A_n = \emptyset \bmod F$ hence $\omega \setminus A_n \in F$. As F is a P-filter there is A, $\omega \setminus A \in F$, such that for every n, $(\omega \setminus A) \subseteq_{ae} (\omega \setminus A_n)$ i.e., $A_n \subseteq_{ae} A$. Hence there are $k_n < \omega$ such that $(A_n \setminus [0,k_n)) \subseteq A$. w.l.o.g. $A_0 = \mathrm{Dom}(p_0) \subseteq A$ and $k_n > n$.

Now we shall choose natural numbers $\ell(0) < \ell(1) < \ell(2) < \ldots$, and want to choose them such that

$$q = p_0 \cup \bigcup_n (p_{\ell(n)}{\restriction}[\ell(n),\omega)) \in P(F)$$

So as q is a function from a subset of ω to 2, (because $p_n \leq p_{n+1}$) we only have to take care of the demand $\mathrm{Dom}(q) = \emptyset \bmod F$. Note that $\mathrm{Dom}(q) \setminus A = \bigcup_{n<\omega}(\mathrm{Dom}(p_{\ell(n)}) \setminus A \setminus [0, \ell(n)) = \bigcup_{n<\omega}(A_{\ell(n)} \setminus A \setminus [0, \ell(n))) = \bigcup_n (A_{\ell(n)} \cap [\ell(n), k_{\ell(n)}) \setminus A)$ (remember $A_n \setminus [0, k_n) \subseteq A, A_0 \subseteq A$).

Now let $w_\ell = [\ell, k_\ell)$, (for $\ell < \omega$) which is a finite set. As $\mathrm{Min}(w_\ell) = \ell$, there is an infinite $S \subseteq \omega$ such that $\{w_\ell : \ell \in S\}$ is a family of pairwise disjoint sets. Since F is fat there is an infinite $S_1 \subseteq S$ such that $\bigcup\{w_\ell : \ell \in S_1\} = \emptyset \bmod F$. So let $\{\ell(n) : n < \omega\}$ be a list of the members of S_1, $\ell(n) < \ell(n+1)$. Then $q \in P(F)$ which proves Fact 4.6. $\square_{4.6}$

Continuation of the Proof of 4.4. Let $\{\underset{\sim}{\tau}_i : i < \omega\}$ be a list of all $P(F)$-names of ordinals which belong to N. Using the crucial fact 4.5 we can define by induction on n, $p_n \in P(F) \cap N$, $p_n \leq p_{n+1}$ (p_0 is already defined) such that:

($*$) if $g : n \to 2 = \{0, 1\}$ and $\ell < n$ then for some ordinal $\alpha(g, \ell)$

$$(p_n{\restriction}[n, \omega)) \cup g \Vdash_{P(F)} \text{“}\underset{\sim}{\tau}_\ell = \alpha(g, \ell)\text{”}.$$

Applying the Fact 4.6 for the sequence $\langle p_n : n < \omega \rangle$ constructed above with the property ($*$) we obtain a $q \in P(F)$ such that $q \geq p_0$ and $q \geq p_n{\restriction}[n, \omega)$ for infinitely many n. For such an n we have $(p_n{\restriction}[n, \omega)) \cup q{\restriction}[0, n) \leq q$, hence $q \Vdash$ “for $\ell < n$, $\underset{\sim}{\tau}_\ell = \alpha(g, \ell)$ for some function $g : n \to 2$ extending $q{\restriction}[0, n)$”. So q is $(N, P(F))$-generic, and as $q \geq p_0$, we have proved that $P(F)$ is proper. In fact not only q is $(N, P(F))$-generic, but even $q{\restriction}[n, \omega)$ for any $n < \omega$ is, as every $g \cup q{\restriction}[n, \omega)$ is, for $g : n \to \{0, 1\}$.

Let us prove $P(F)$ is α-proper for any countable α, by induction on α. Let $\langle N_i : i \leq \alpha \rangle$ be as in V.3.1, $p \in P(F)$ and $\{p, F\} \in N_0$; we shall prove that not only is there $q \geq p$, $(N_i, P(F))$-generic for every $i \leq \alpha$, but also $q{\restriction}[n, \omega)$ is $(N_i, P(F))$-generic (for $i \leq \alpha, n < \omega$). If $\alpha = 0$ we have proved this, if α is a successor use the induction hypothesis. So assume α is limit and let $\alpha = \bigcup_{n<\omega} \alpha_n$, $\alpha_n < \alpha_{n+1}$. By the induction hypothesis we can define $q_n \in N_{\alpha_n+1}$, such that for every $i \leq \alpha_n$ and $k < \omega$ we have $q_n{\restriction}[k, \infty)$ is $(N_i, P(F))$-generic and $q_0 \geq p, q_{n+1} \geq q_n$.

Apply Fact 4.6 to $p_0, q_0, q_1, \dots$ and get q as required, remember that as q is $(N_i, P(F)))$-generic for every $i < \alpha$, it is also $(N_\alpha, P(F))$-generic.

¿From Lemma 4.4 only the strong PP-property remains to be shown (see on it 2.12, particularly 2.12E(3); this suffices by 2.12F(1)). So let $x \in {}^\omega\omega$ diverge to infinity, and let N be as above, $\underset{\sim}{f} \in N$ be a $P(F)$-name of a member of ${}^\omega\omega$, and w.l.o.g. we can assume that for $n < \omega$, $\underset{\sim}{\tau}_{2n} = \underset{\sim}{f}(n)$ where $\{\underset{\sim}{\tau}_n : n < \omega\}$ was a list of the names of ordinals used in the proof of the properness of $P(F)$. When we define the p_n, by induction on n, make one change in $(*)$ above (in this proof): instead of considering $\ell < n$, we consider the ℓ such that $\ell \leq 2x(n+1+2^{n+1}) + 2$. So we have:

$(*)'$ if $g : n \to 2$ and $\ell \leq 2(x(n+1+2^{n+1}) + 1)$ then for some ordinal $\alpha(g, \ell)$, $p{\restriction}[n, \omega) \cup g \Vdash_{P(F)}$ "$\underset{\sim}{t}_\ell = \alpha(g, \ell)$".

Now we let $k_n = 0$, $m_n(0) = 2^n$, $i_n(0) = n+1$, $j_n(0) = x(n+1+2^n)+1$. Then, by $(*)'$, we have $p_n{\restriction}[n, \omega) \Vdash_{P(F)}$ "$\underset{\sim}{f}{\restriction}j_n \in \{h_{g,j_n} : g : n \to 2\}$", where $h_{g,j_n}(\ell) = \alpha(g, 2\ell)$. So $p_n{\restriction}[n, \omega)$ allows $\underset{\sim}{f}{\restriction}j_n(0)$ at most 2^n possibilities which is $m_n(0)$. As $q \geq p_n{\restriction}[n, \omega)$ for infinitely many n, and for each such n, q "allows" $\underset{\sim}{f}{\restriction}j_n$ less than $m_n(0)+1$ possibilities, clearly $k_n = 0, m_n(0), i_n(0), j_n(0)$ witness n is as required in 2.12E(3) (i.e. 2.12A(b)$(*)$ with $k = 0$), so we have finished.

$\square_{4.4,4.2}$

4.7 Lemma. Suppose F is a P-point and $P(F)^\omega \lessdot P$ and P has the PP-property (or just it is ${}^\omega\omega$-bounding and has the weak PP-property, see Definition 2.12E).

Then in V^P, F cannot be extended to a P-point.

Proof. Suppose $p \in P$ forces that $\underset{\sim}{E}$ is an extension of F to a P-point (in V^P). Let $\langle \underset{\sim}{r}_n : n < \omega \rangle$ be the sequence of reals which $P(F)^\omega$ introduces (i.e. $\underset{\sim}{r}_n(i) = \ell$ iff for some $\langle f_0, f_1, \dots \rangle \in \underset{\sim}{G}_{P(F)^\omega}$ we have $f_n(i) = \ell$). Define a P-name:

$\underset{\sim}{h}(n)$ is 1 if $\{i < \omega : \underset{\sim}{r}_n(i) = 1\} \in \underset{\sim}{E}$ and

$\underset{\sim}{h}(n)$ is 0 if $\{i < \omega : \underset{\sim}{r}_n(i) = 0\} \in \underset{\sim}{E}$.

So $p \Vdash_P$ "$\underset{\sim}{h} \in {}^{\omega}2$". Now as P have the PP-property, by 2.12D (and see 2.12E), there is $p_1 \geq p$, ($p_1 \in P$), and for each $n < \omega$ there are $k(n) < \omega$, $i_n(0) < j_n(0) < i_n(1) < j_n(1) < \ldots < i_n(k(n)) < j_n(k(n))$, and $j_n(k(n)) < i_{n+1}(0)$ such that:

$$p_1 \Vdash_P \text{" for every } n < \omega \text{ for some } \ell \leq k(n) \text{ we have } \underset{\sim}{h}(i_n(\ell)) = \underset{\sim}{h}(j_n(\ell))\text{"}.$$

Now define the following P-names:

$$\underset{\sim}{A}_n = \{m < \omega : \text{ for some } \ell \leq k(n), \underset{\sim}{r}_{i_n(\ell)}(m) = \underset{\sim}{r}_{j_n(\ell)}(m)\}.$$

4.7A Fact. $p_1 \Vdash_P$ "$\underset{\sim}{A}_n \in \underset{\sim}{E}$".

This is true because p_1 forces that for some $\ell \leq k(n)$ we have $\underset{\sim}{h}(i_n(\ell)) = \underset{\sim}{h}(j_n(\ell))$ and by the definition of $\underset{\sim}{h}$ we know:

$$p \Vdash_P \text{"}\{m < \omega : \underset{\sim}{r}_{i_n(\ell)}(m) = \underset{\sim}{h}(i_n(\ell))\} \in \underset{\sim}{E}\text{"}$$
$$p \Vdash_P \text{"}\{m < \omega : \underset{\sim}{r}_{j_n(\ell)}(m) = \underset{\sim}{h}(j_n(\ell))\} \in \underset{\sim}{E}\text{"}.$$

Putting together these three things (and $p \leq p_1$) we get $p_1 \Vdash_P$ "$\{m < \omega :$ for some $\ell \leq k(n)$ we have $r_{i_n(\ell)}(m) = \underset{\sim}{h}(i_n(\ell)) = \underset{\sim}{h}_j(j_n(\ell)) = r_{j_n(\ell)}(m)\} \in \underset{\sim}{E}$" but this set is included in $\underset{\sim}{A}_n$, hence $p_1 \Vdash$ "$\underset{\sim}{A}_n \in \underset{\sim}{E}$". So Fact 4.7A holds. $\square_{4.7A}$

So $p_1 \Vdash$ "$\{\underset{\sim}{A}_n : n < \omega\} \subseteq \underset{\sim}{E}$", but as $p_1 \Vdash$ "$\underset{\sim}{E}$ is a P-point" for some $\underset{\sim}{g}$ we also have $p_1 \Vdash_P$ "$\underset{\sim}{g} \in {}^{\omega}\omega$ and $\bigcap_{n<\omega}(\underset{\sim}{A}_n \cup [0, \underset{\sim}{g}(n)]) \in \underset{\sim}{E}$". Now as P has the PP-property by 2.12B(1) (or 2.12F(4)), it has the ${}^{\omega}\omega$-bounding property, hence there is p_2, $p_1 \leq p_2 \in P$ and $g \in {}^{\omega}\omega$ (in V) such that $p_2 \Vdash_P$ "$\underset{\sim}{g}(n) \leq g(n)$ for every n". Hence

$$p_2 \Vdash_P \text{"} \bigcap_{n<\omega} (\underset{\sim}{A}_n \cup [0, g(n)]) \in \underset{\sim}{E}\text{"}$$

and therefore

(*) $\quad p_2 \Vdash_P$ "$\bigcap_{n<\omega}(\underset{\sim}{A}_n \cup [0, g(n)]) \neq \emptyset \bmod F$.".

As $p_2 \in P$ and $P(F)^{\omega} \lessdot P$, there is $q = \langle f_0, f_1, \ldots \rangle \in P(F)^{\omega}$ such that p_2 is compatible (in P) with any $q^{\dagger}, q \leq q^{\dagger} \in P(F)^{\omega}$. As F is a P-point, there is $A^* \subseteq \omega$ (in V) such that $A^* = \emptyset \bmod F$ and $\mathrm{Dom}(f_i) \subseteq_{ae} A^*$ for every i. Choose, by induction on $n < \omega$, $\alpha_n < \alpha_{n+1} < \omega$, such that $\alpha_n > g(n)$ and for $\ell \leq k(n)$:

$$(\mathrm{Dom}(f_{i_n(\ell)}) \cup \mathrm{Dom}(f_{j_n(\ell)})) \setminus [0, \alpha_n) \subseteq A^*.$$

Now we shall define $q^\dagger = \langle f^\dagger_\ell : \ell < \omega \rangle$, $q \leq q^\dagger \in P(F)^\omega$. Let: $f^\dagger_{i_n(\ell)} = f_{i_n(\ell)} \cup 0_{[\alpha_n, \alpha_{n+1}) \setminus A^*}$ and $f^\dagger_{j_n(\ell)} = f_{j_n(\ell)} \cup 1_{[\alpha_n, \alpha_{n+1}) \setminus A^*}$ (where 0_B is the function with domain B and constant value $0, 1_B$ defined similarly.) Otherwise $f^\dagger_m = f_m$.

Plainly, $f^\dagger_m$ is a function (by the definition of α_n), its domain is the same as that of f_m plus a finite subset of ω, hence $\mathrm{Dom}(f^\dagger_m) \subseteq \omega$, $\mathrm{Dom}(f^\dagger_m) = \emptyset \bmod F$. Also $f_m \subseteq f^\dagger_m$, hence $q \leq q^\dagger = \langle f^\dagger_0, f^\dagger_1, \ldots \rangle \in P(F)^\omega$. Clearly $q^\dagger \Vdash_P$ "$\underset{\sim}{A}_n$ is disjoint to of $[\alpha_n, \alpha_{n+1}) \setminus A^*$" (by the definition $\underset{\sim}{A}_n$ and $f^\dagger_{i_n(\ell)}, f^\dagger_{j_n(\ell)}$).

Also $g(n) < \alpha_n$, hence

$$q^\dagger \Vdash_P \text{“}\underset{\sim}{A}_n \cup [0, g(n)] \setminus A^* \text{ is disjoint to } [\alpha_n, \alpha_{n+1})\text{”}$$

and thus

$$q^\dagger \Vdash_P \text{“} \bigcap_{n<\omega} (\underset{\sim}{A}_n \cup [0, g(n)] \setminus A^*) \text{ is disjoint to } \bigcup_n [\alpha_n, \alpha_{n+1}) = [\alpha_0, \omega)\text{”}.$$

Consequently (as $A^* = \emptyset \bmod F$ and $[0, \alpha_0]$ is finite)

$$q^\dagger \Vdash_P \text{“} \bigcap_{n<\omega} (\underset{\sim}{A}_n \cup [0, g(n)]) = \emptyset \bmod F\text{”}.$$

By the choice of q we know that p_2, $q^\dagger$ are compatible in P so let $p_3 \in P$ be a common upper bound of p_2, $q^\dagger$, hence $p_3 \Vdash_P$ "$\bigcap_{n<\omega} (\underset{\sim}{A}_n \cup [0, g(n)]) = \emptyset \bmod F$" which contradicts $(*)$. $\square_{4.7}$

4.8 Theorem. It is consistent with ZFC $+2^{\aleph_0} = \aleph_2$ that there is no P-point.

Proof. It is left to the reader, or see the 5.13, where a similar proof is carried out. $\square_{4.8}$

4.9 Claim. Assume iteration $\overline{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \delta, j < \delta \rangle$ is as in 0.1, E is a non principal ultafilter in V which is a P-point (i.e. if $A_n \in E$ for $n < \omega$ then for some $A \in E$ we have $A \setminus A_n$ finite for each $n < \omega$) and if P_δ is not proper, E is generated by $\leq \aleph_1$ sets. If E is (pedantically generates) an ultrafilter in V^{P_i} for each $i < \delta$ (and δ is a limit ordinal) *then* E is a P-point in V^{P_δ}.

Remark. We weaken the assumption on E (in V) to

$(*)_0$ $E \subseteq \mathcal{P}(\omega)$ and $\mathrm{fil}(E) = \{A \subseteq \omega : (\exists B \in E) A \supseteq B\}$ is a non principal ultrafilter on ω, which is a P-point.

Proof.

We shall use 1.17 (see Definition 1.16, for our family of forcing notions). Let $k^* = 2$, $D_\ell = H(\chi)^V$, and

xR_0T *iff* ($x \in ({}^\omega\omega) \cap D$ and) for some $A = A^0_T \in E$ for every $\eta \in \lim(T)$ we have $\{n < \omega : \eta(n) = 1\} \supseteq A$

xR_1T *iff* for some $A = A^1_T \in E$ for every $\eta \in \lim(T)$ we have $\{n < \omega : \eta(n) = 1\} \cap A = \emptyset$

(so the x's are not important).

Then clearly $(\overline{D}, \overline{R}) = (\langle D_0, D_1 \rangle, \langle R_0, R_1 \rangle)$ is a weak covering 2-model, in particular it covers in V; use the standard $<_i$. Note

$(*)_1$ $(\overline{D}, \overline{R})$ covers in V^P *iff* E generates an ultrafilter in V^P

$(*)_2$ if in V^P the family E generates an ultrafilter then E generates a P-point in V^P (of course provided that P is proper or P preserves $\aleph_1$ and $|E| = \aleph_1$ or $\Vdash_P$ "$\mathcal{S}_{<\aleph_1}(|E|)^V$ is cofinal in $\mathcal{S}_{<\aleph_1}(|E|)^{V^P}$").

We next prove that $(\overline{D}, \overline{R}, \overline{<})$ is a fine covering 2-model. We check Definition 1.16(2).

Clauses (α), (β) are trivial.

Clause (γ)(a): Let $k < 2$, xR_kT_n. By symmetry let $k = 0$ so xR_0T_n. So let $A_n = A^k_{T_n}$, so $A_n \in E$. We can find $B \in E$ such that $B \subseteq A_0$, $B \subseteq_{ae} A_n$ for each n. Let $T^* = \{\eta \in {}^\omega\omega : (\forall i \in B)\eta(i) = 1\}$ so the choice $A^k_{T^*} = B$ exemplifies xR_kT^* (for any x), so it suffices to prove the inclusion from (γ)(a). Toward this let for $i < \omega$ let $m_i \in (i, \omega)$ be such that $B \setminus m_i \subseteq A_i$, and let $w \subseteq \omega$ be infinite such that if $i < \omega$, $j \in w$, $i \leq \max(w \cap j)$ then $m_i < j$. Now suppose $\eta \in {}^\omega\omega$ and $i \in w \Rightarrow \eta{\restriction}(\min(w \setminus (i+1)) \in \bigcup_{j<i, j \in w} T_j \cup T_0$, and we are going to prove $\eta \in \lim(T^*)$. This means that we should prove $A' = \{n < \omega : \eta(n) = 1\} \supseteq B$; so let $n \in B$; if n is smaller than the second member of w, then applying the assumption for $i =$ the first member of w we get $\eta{\restriction}(n+1) \in T_0$ which implies the desired conclusion. If not let $i_0 < i_1 \leq n < i_2$, where i_0, i_1, i_2 are successive members of w. So by the assumption $\eta{\restriction}i_2 \in \bigcup_{j \leq i_0} T_j \cup T_0$, now if $\eta{\restriction}i_2 \in T_0$ we are done, so assume $\eta{\restriction}i_2 \in T_j$, $j \leq i_0$, hence $m_j \leq i_1 \leq n$. Then $A^k_{T_j} \cap [m_j, \infty) \supseteq B \cap [m_j, \infty) \supseteq \{n\}$, hence as $\eta \in T_j$ we get $\eta(n) = 1$ as required.

Clause (γ)(b): We can find $B_i \in E$ such that $\{n < \omega : \eta_i(n) = k\} \supseteq B_i$, then we can find $B \in E$ such that $B \setminus B_i$ is finite say $\subseteq [0, m_i)$ for $i < \omega$. Choose $\langle n_j : j < \omega \rangle$ as in the proof of (γ)(a); by symmetry w.l.o.g. $\bigcup_j [n_{2j}, n_{2j+1})$ belongs to the ultrafilter which E generated. So $B \cap \bigcup_j [n_{2j}, n_{2j+1}) \cap \{i : \eta(i) = k\}$ belongs to this ultrafilter, hence it includes some $B' \in E$. Consequently, $\{i < \omega : \eta_{n_{2j}}(i) = k\} \supseteq B'$ for each $j < \omega$ and $\{i < \omega : \eta(i) = k\} \supseteq B'$, as required.

Clause (δ): Straight by $(*)_1$, $(*)_2$. $\square_{4.9}$

4.10 Remark. 1) Mekler [Mk84] considers the generalizations to finitely additive measures $\mu : \mathcal{P}(\omega) \to [0,1]_{\mathbb{R}}$, generalizing this proof to prove the consistency of "the parallels of P-points do not existence". Though there the PP-property fails, he showed that ${}^\omega\omega$-bounding suffices. Still we felt the PP-property is inherently interesting.

2) Baumgartner [B6] was interested in ultrafilters with properties which weaken "being P-points". Answering his question we prove that if in the iteration above we use unboundedly often random real then there is no P-point (see above), and there is a measure zero ultrafilter (see [B6]).

3) The question of whether there are always NWD-ultrafilters (see van Douwen [vD81], and [B6]) is answered negatively in [Sh:594], generalizing the proof here (and continuing the "use of E" from [Sh:407]). There the PP-property is used.

§5. There May Exist a Unique Ramsey Ultrafilter

Usually it is significantly harder to prove that there is a unique object than to prove there is none. The proof is similar to the one in the previous section, but here we are destroying other Ramsey ultrafilter (in fact "almost" all other P-points) while preserving our precious Ramsey ultrafilter. By a similar proof we can construct a forcing notion P such that e.g. in V^P there are exactly two Ramsey ultrafilters (in both cases up to the equivalence induced by the Rudin Keisler order) or any other number.

More exactly we shall prove the consistency of "there is a unique Ramsey ultrafilter F_0 on ω, up to permutation of ω, moreover for every P-point F, $F_0 \leq_{RK} F$".

Note that if there is a unique P-point it should be Ramsey; however, concerning the question of the existence of a unique P-point we return to it in XVIII §4.

Our scheme is to start with a universe with a fixed Ramsey ultrafilter F_0, to preserve its being an ultrafilter and even a Ramsey ultrafilter. Our ultrafilter will be generated by $\aleph_1$ sets. Now in each stage we shall try to destroy a given P-point F such that $F_0 \not\leq_{RK} F$. The forcing from §4 does not work, but if we use a version of it in the direction of Sacks forcing it will work.

5.1 Claim.

1) If F is a P-point in V, P is a proper forcing notion and $\Vdash_P$ "F generates an ultrafilter " *then* it (more exactly the one it generates) is a P-point in V^P.

2) If the ultrafilter F is Ramsey in V, and P is ${}^\omega\omega$-bounding, proper and $\Vdash_P$ "F generates an ultrafilter", *then* in V^P, F still generates a Ramsey ultrafilter.

Proof.

1) As for being a P-filter, let $p \Vdash_P$ "$\{\underset{\sim}{A}_n : n < \omega\}$ is included in the ultrafilter

which F generates ". So w.l.o.g. $p \Vdash_P$ "$\underaccent{\tilde}{A}_n \in F$", and by properness for some q, $p \leq q \in P$, and $A_{n,m} \in F$ (for $n, m < \omega$) we have $q \Vdash_P$ " for each n, $\underaccent{\tilde}{A}_n \in \{A_{n,m} : m < \omega\}$". As F is a P-point in V and $\{A_{n,m} : n, m < \omega\} \subseteq F$ belong to V, there is $A \in F$ which is almost included in every $A_{n,m}$ hence in each $\underaccent{\tilde}{A}_n$; (note: e.g., if F is generated by $\aleph_1$ sets, then "P does not collapse $\aleph_1$" is sufficient instead "P is proper").

2) As F generates a P-point in V^P, the following will suffice: let $0 = \underaccent{\tilde}{n}_0 < \underaccent{\tilde}{n}_1 < \underaccent{\tilde}{n}_2 \ldots$ and $p \in P$; then we can find $A \in F$ and $q \geq p$ such that $q \Vdash$ "$A \cap [\underaccent{\tilde}{n}_i, \underaccent{\tilde}{n}_{i+1})$ has at most one element for each i" (i.e. F is a so called Q-point). Remember P has the ${}^\omega\omega$-bounding property. So there are $h \in {}^\omega\omega \cap V$, and $q \geq p$ such that $q \Vdash_P$ " $(\forall i)\underaccent{\tilde}{n}_i < h(i)$". W.l.o.g. h is strictly increasing.

Define n_i^* (in V by induction on i): $n_0^* = 0, n_{i+1}^* = h(n_i^* + 1) + 1$. Now for no i, j we have $\underaccent{\tilde}{n}_i[G] \leq n_j^* < n_{j+1}^* < \underaccent{\tilde}{n}_{i+1}[G]$. [Why? Assume this holds and, of course, $i < j$; as $\underaccent{\tilde}{n}_\ell < \underaccent{\tilde}{n}_{\ell+1}$, clearly $\ell \leq \underaccent{\tilde}{n}_\ell[G]$ hence

$$n_{j+1}^* > h(n_j^* + 1) \geq h(\underaccent{\tilde}{n}_i[G] + 1) \geq h(i + 1) \geq \underaccent{\tilde}{n}_{i+1}[G]$$

(remember h is strictly increasing), a contradiction]. Also F is an ultrafilter in $V[G]$, by the assumption. As in V, F is a Ramsey ultrafilter and $\langle n_i^* : i < \omega\rangle \in V$, there is $A \in F$ such that $A \cap [n_i^*, n_{i+1}^*)$ has at most one element for each i. Let $G \subseteq P$ be generic over V be such that $q \in G$. Checking carefully in $V[G]$ we see that for every i we have $A \cap [\underaccent{\tilde}{n}_i[G], \underaccent{\tilde}{n}_{i+1}[G])$ has at most two elements and in this case they are necessarily successive members of A. Let $A_0 = \{k \in A : |A \cap k| \text{ is even}\}$, so either A_0 or $A \setminus A_0$ belong to the ultrafilter which F generates, and both are as required. $\square_{5.1}$

5.2 Lemma.

1) "F generates an ultrafilter in V^Q which is a P-point, Q is proper" is preserved by countable support iteration for F a P-point.

2) "F generates an ultrafilter in V^Q which is Ramsey + Q is ${}^\omega\omega$-bounding + Q is proper " is preserved by countable support iteration for F a Ramsey ultrafilter.

Proof. 1) By 4.9 and see 5.1(1).

2) Combine (1), 5.1(2) and 2.8. $\square_{5.2}$

5.3 Definition. For F a filter on ω, let $\mathrm{SP}(F)$ be $\{T : T$ is a perfect tree $\subseteq {}^{\omega>}2$ and for some $A \in F$, for every $n \in A$, $\eta \in T \cap {}^n2$ implies $\eta\char`^\langle 0\rangle \in T$ & $\eta\char`^\langle 1\rangle \in T\}$. The order is the inverse inclusion. We denote the maximal such A by $\mathrm{spt}(T)$.

5.3A Remark.

1) So $\mathrm{SP}(F)$ is a "mixture" of $P(F)$ and Sacks forcing and $\mathrm{SP}^*(F)$ (defined below) is half way between $\mathrm{SP}(F)$ and $\mathrm{SP}(F)^\omega$.
2) Remember $T_{[\eta]} \overset{\text{def}}{=} \{\nu \in T : \nu \trianglelefteq \eta$ or $\eta \trianglelefteq \nu\}$ for any $\eta \in T$ and $T^{[n]} \overset{\text{def}}{=} \{\eta \in T : \ell g(\eta) = n\}$ for any $n < \omega$.

5.4 Definition. Let $T^\otimes_n \overset{\text{def}}{=} \bigtimes_{\ell<n}({}^\ell 2)$, $T^\otimes \overset{\text{def}}{=} \bigcup_{n<\omega} T^\otimes_n$ ordered by the being initial segment, i.e. for $f \in T^\otimes_n$ and $g \in T^\otimes_m$ we set $f \lhd g$ iff $f(i) = g(i)$ for each $i < n$. (Note $f(i) \in {}^i2$). For a filter F on ω, let $\mathrm{SP}^*(F)$ be

$$\{T : T \text{ is a perfect tree } \subseteq T^\otimes \text{ and for every } k < \omega \text{ we have } \mathrm{spt}_k(T) \in F\},$$

where

$$\mathrm{spt}_k(T) = \{n < \omega : \text{for every } \eta \in T^{[n]} (= T \cap T^\otimes_n) \text{ and } \nu \in {}^k2 \text{ there is } \rho \in {}^n2, \eta\char`^\langle\rho\rangle \in T^\otimes_{n+1} \cap T \text{ such that } \rho\restriction k = \nu\}.$$

The order is the inverse inclusion.

5.5 Claim. Let F be a filter on ω and Q be $\mathrm{SP}(F)$ or $\mathrm{SP}^*(F)$.

1) If $T \in Q, T^{[n]} = \{\eta_1, \ldots, \eta_k\}$ (with no repetition), $T_\ell = T_{[\eta_\ell]}, T_\ell^\dagger \in Q$, $T_\ell \leq T_\ell^\dagger$ (i.e. $T_\ell^\dagger \subseteq T_\ell$) *then* $T \leq T^\dagger \stackrel{\text{def}}{=} \bigcup_{\ell=1}^k T_\ell^\dagger \in Q$ and $T^\dagger \Vdash$ "for some $\ell \in \{1, \ldots, k\}$ we have $T_\ell^\dagger \in \underset{\sim}{G}_Q$".
2) If $\underset{\sim}{t}$ is a P-name of an ordinal $T \in Q$ and $n < \omega$ *then* there are $T^\dagger$, $T \leq T^\dagger \in Q$ and A such that $T^\dagger \Vdash_Q$ "$\underset{\sim}{t} \in A$" and $|A| \leq |T^{[n]}|$, and $\bigcup_{\ell \leq n} T^{[\ell]} \subseteq T^\dagger$. Moreover for each $\eta \in T^{[n]}, T^\dagger_{[\eta]}$ determines $\underset{\sim}{t}$.

Proof. 1) Observe that $\mathrm{spt}_j(T^\dagger) \supseteq \bigcap_{1 \leq \ell \leq k} \mathrm{spt}_j(T_\ell) \setminus (n+1)$.
2) For each $\eta \in T^{[n]}$ there is T^η, $T_{[\eta]} \leq T^\eta$ such that T^η decides the value $\underset{\sim}{t}$. Now amalgamate the T^η together by applying part 1). $\square_{5.5}$

5.6 Lemma. Let F be a P-point ultrafilter on ω. *Then*

1) $\mathrm{SP}(F)$ is proper, in fact α-proper for every $\alpha < \omega_1$, and has the strong PP-property; and so is $\mathrm{SP}(F)^\omega$
2) $\mathrm{SP}^*(F)$ is also proper, α-proper for every $\alpha < \omega_1$ and has the strong PP-property.

Proof. Similar to the proof of 4.4. For its proof we shall use the following theorem, of Galvin and McKenzie, (but later we shall prove a similar theorem in detail (5.11)); note that we use only the "only if" direction.

5.7 Theorem. Let F be an ultrafilter on ω. Then F is a P-point [Ramsey ultrafilter] iff in the following game player I has no winning strategy:

in the n-th move:
player I chooses $A_n \in F$
player II choose $w_n \subseteq A_n, w_n$ is finite [a singleton].
In the end player II wins if $\bigcup_{n<\omega} w_n \in F$.

Proof of 5.6 from 5.7. We just have to define a strategy for player I, (in the game from 5.7): playing on the side with the conditions in the forcing. From the two forcing listed in the lemma we concentrate on proving only the properness of $\mathrm{SP}^*(F)$ (the other have similar proofs and this is the only one we shall use). Let $N \prec (H(\chi), \in, <^*_\chi)$ be countable with $F \in N$, so $\mathrm{SP}^*(F) \in N$; and let

$T \in \mathrm{SP}^*(F) \cap N$ and let $\langle \mathcal{I}_n : n < \omega \rangle$ be a list of the dense subsets of $\mathrm{SP}^*(F)$ which belong to N. We shall define now a strategy for player I. In the n'th move player I chooses "on the side" condition $T_n \in \mathrm{SP}^*(F) \cap N$ in addition to choosing $A_n \in F$ and player II chooses finite $w_n \subseteq A_n$. For $n = 0$, player I chooses $T_0 = T$ and $A_0 = \omega$.

For $n > 0$, for the n'th step player I, using 5.5, chooses $T_n \in \mathrm{SP}^*(F) \cap N$, such that $T_{n-1} \leq T_n$, $T_{n-1}^{[k_n]} = T_n^{[k_n]}$, where $k_n \overset{\text{def}}{=} \max[\bigcup\{w_{n'} : n' < n\} \cup \{n\}] + n + 1$ and $(\forall \eta \in T_n^{[k_n]})$ $((T_n)_{[\eta]} \in \mathcal{I})$. Then player I plays $A_n = \mathrm{spt}_n(T_n)$. Note that whatever are the choices of player II, we have $T_n \in N$ and we can let player I choose T_n as the first one which is as required by the well ordering $<^*_\chi$. As F is a P-point, by 5.7 there is a play in which he uses the strategy described above and player II wins the play; this will give us the desired sequence of conditions. Indeed, $T = \bigcap_{n<\omega} T_n \in \mathrm{SP}^*(F)$ satisfies $\mathrm{spt}_n(T) \supseteq \bigcup\{w_k : k \in [n, \omega)\}$ (for each $n < \omega$) and hence T belongs to $\mathrm{SP}^*(F)$. $\square_{5.6}$

Similar argument is carried out in more detail in the proof of 5.12.

5.8 Lemma. 1) If F is a P-point ultrafilter, $\mathrm{SP}(F)^\omega \lessdot Q$, and Q has the PP-property *then* in V^Q, F cannot be extended to a P-point ultrafilter.
2) If F is a P-point ultrafilter, $SP^*(F) \lessdot Q$, Q has the PP-property *then* in V^Q, F cannot be extended to a P-point ultrafilter.

Proof. The proof is almost identical with the proof of 4.7, so we do not carry out it in detail. (In fact we get the variant with weaker assumption as proved in 4.7).

This is particularly true for part (1). For part (2) copy the proof of 4.7, replacing $P(F)$ by $\mathrm{SP}^*(F)$ and defining $\underset{\sim}{r}_n$ as:

$\underset{\sim}{r}_n(i) = \ell$ *iff* $i \leq n \Rightarrow \ell = 0$ and

$$i > n \Rightarrow (\exists T \in \underset{\sim}{G}_{\mathrm{SP}^*(F)})(\exists \eta \in T^{\otimes}_{i+1})[T = T_{[\eta]} \ \& \ (\eta(i))(n) = \ell].$$

This is done up to and including the choice of p_2 (i.e. $(*)$ in the proof of 4.7).

As $p_2 \in P$ and $\mathrm{SP}^*(F) \lessdot P$ clearly there is $q \in \mathrm{SP}^*(F)$ such that p_2 is compatible in P with any q' satisfying $q \le q' \in \mathrm{SP}^*(F)$. For $k < \omega$, as $q \in \mathrm{SP}^*(F)$ by Definition 5.4 we know that $\mathrm{spt}_k(q) \in F$, so as F is a P-point there is $B^* \in F$ such that $B^* \setminus \mathrm{spt}_k(q)$ is finite for every $k < \omega$. Choose by induction on $n < \omega$, $\alpha_n < \omega$ such that $\alpha_n < \alpha_{n+1}$, $\alpha_n > g(n)$ and $\alpha_n > j_n(k(n))$ and $B^* \setminus \mathrm{spt}_{j_n(k(n))+1}(q) \subseteq [0, \alpha_n)$. Define $q' \stackrel{\text{def}}{=} \{\eta : \eta \in q$ and for every $m < \omega$ we have: if $\alpha_n \le m < \ell g(\eta)$, $m < \alpha_{n+1}$ and $m \in \mathrm{spt}_{j_n(k(n))+1}(q)$ then for each $\ell \le k(n)$ we have $(\eta(m))(i_n(\ell)) = 0$ and $(\eta(m))(j_n(\ell)) = 1\}$.

Now

(a) $q' \subseteq T^{\otimes}$ is closed under initial segments and $\langle\rangle \in q'$

[Why? Read the definition of q']

(b) q' has no $\lhd$-maximal element

[Why? Assume $\eta \in q' \cap T^{\otimes}_m$. If $m < \alpha_0$ then any $\nu \in \mathrm{Suc}_q(\eta)$ belongs to q'. So let $\alpha_n \le m < \alpha_{n+1}$; if $m \notin \mathrm{spt}_{j_n(k(n))+1}(q)$ again any $\nu \in \mathrm{Suc}_q(\eta)$ belongs to q', so assume $m \in \mathrm{spt}_{j_n(k(n))+1}(q)$, which means

$$(\forall \eta' \in q \cap T^{\otimes}_m)(\forall \rho \in {}^{j_n(k(n))+1}2)(\exists \nu)[\eta'{}^\frown\langle\nu\rangle \in q \ \&\ \nu \restriction j_n(k(n)) + 1 = \rho].$$

Apply this for η' and for the $\rho^* \in {}^{j_n((k(n))+1}2$ defined by $\{\ell < j_n(k(n)) + 1 : \rho^*(\ell) = 1\} = \{j_n(\ell) : \ell \le k(n)\}$, and find ν satisfying $\rho^* \trianglelefteq \nu$ and such that $\eta{}^\frown\langle\nu\rangle \in \mathrm{Suc}_q(\eta)$ and even $\eta{}^\frown\langle\nu\rangle \in \mathrm{Suc}_{q'}(\eta)$.]

(c) If $\alpha_n \le m < \alpha_{n+1}$, $m \in \mathrm{spt}_{j_n(k(n))+1}(q)$ then $m \in \mathrm{spt}_{i_n(0)}(q')$.

[Why? Same proof as of clause (b) noting that for any $\rho_1 \in {}^{i_n(0)}2$ we can find ρ^* such that $\rho_1 \lhd \rho^* \in {}^{j_n(k(n))+1}2$, such that for $m \in [i_n(0), j_n(k(n)) + 1)$, we have $\rho^*(m) = 1 \Leftrightarrow m \in \{j_n(\ell) : \ell \le k(n)\}$]

(d) Let $k < \omega$, then $\mathrm{spt}_k(q') \in F$.

[Why? Choose $n(*)$ such that $k < i_{n(*)}(0)$. Now if $m \in B^* \setminus \alpha_{n(*)}$ then for some n, $n(*) \le n < \omega$ and $\alpha_n \le m < \alpha_{n+1}$ hence $m \in \mathrm{spt}_{j_n(k(n))+1}(q)$ and so by clause (c) we have $m \in \mathrm{spt}_{i_n(0)}(q')$. But $\mathrm{spt}_\ell(q')$ decreases with ℓ and $k < i_{n(*)}(0) \le i_n(0)$, so $m \in \mathrm{spt}_k(q')$. Together $B^* \setminus \alpha_{n(*)} \subseteq \mathrm{spt}_k(q')$, but the former belongs to F.]

(e) $q' \Vdash_{\mathrm{SP}^*(F)}$ "$\bigcap\limits_{n<\omega} (\underset{\sim}{A}_n \cup [0, g(n)))$ is disjoint to $B^* \setminus \alpha_0$"

[Why? Because if $\alpha_n \leq m < \alpha_{n+1}$ and $m \in B^*$ then: by the definitions of $\underset{\sim}{r}_{i_n(\ell)}$, $\underset{\sim}{r}_{j_n(\ell)}$ ($\ell \leq k(n)$) and $\underset{\sim}{A}_n$ (which is $\{\alpha < \omega :$ for some $\ell \leq k(n)$, $r_{i_n(\ell)}(\alpha) = r_{j_n(\ell)}(\alpha)\}$) we know $m \notin \underset{\sim}{A}_n$, also $m \geq \alpha_n > g(n)$, together this suffices.]

Now q', p_2 are compatible members of P (see the choice of q and remember $q \leq q' \in SP^*(F)$), so let $p_3 \in P$ be such that $p_2 \leq p_3$, $q' \leq p_3$. So by clause (e) the condition p_3, being above q', forces that $\bigcap\limits_{n<\omega} (\underset{\sim}{A}_n \cup [0, g(n)))$ is disjoint to a member of F. So as $p_2 \leq p_3$ clearly p_2 cannot force $\bigcap\limits_{n<\omega} (\underset{\sim}{A}_n \cup [0, g(n))) \neq \emptyset \mod F$. But this contradicts the choice of p_2. $\square_{5.8}$

We now state some well known basic facts on the Rudin-Keisler order on ultrafilters.

5.9 Definition.

1) Let F_1, F_2 be ultrafilters on I_1, I_2, respectively. We say $F_1 \leq_{RK} F_2$ *if*: there is a function f from I_2 to I_1 such that $f(I_2) = \{f(i) : i \in I_2\} \in F_1$ and: $A \in F_1$ iff $f^{-1}(A) \in F_2$

2) In this case we say $F_1 = f(F_2)$, if $|I_1| \leq |I_2|$ we can assume w.l.o.g. f is onto I_1.

5.9A Remark. We shall use only ultrafilters on ω, which are not principal, i.e. in $\beta(\omega) \setminus \omega$ in topological notation.

It is known (see e.g. [J])

5.10 Theorem.

1) $\leq_{RK}$ is a quasi-order.

2) An ultrafilter F on ω is minimal *iff* it is Ramsey (minimal means $F^\dagger \leq_{RK} F \Rightarrow F \leq_{RK} F^\dagger$ (see part (4)).

3) If F is a P-point, $F^\dagger \leq_{RK} F$ *then* $F^\dagger$ is a P-point.

4) If $F^1 \leq_{RK} F^2 \leq_{RK} F^1$, *then* there is a permutation f of ω such that $F_2 = f(F_1)$.

Proof. Well known. $\square_{5.10}$

5.11 Lemma. Suppose F_0, F_1 are ultrafilters on ω. *Then* the condition (A) and condition (B) below are equivalent.

(A) F_1 is a P- point, F_0 is a Ramsey ultrafilter, and not $F_0 \leq_{RK} F_1$.

(B) in the following game player I has no winning strategy:

in the n-th move, n even:

player I chooses $A_n \in F_0$

player II chooses $k_n \in A_n$

in the n-th move, n odd:

player I chooses $A_n \in F_1$

player II chooses a finite set $w_n \subseteq A_n$

In the end player II wins if

$$\{k_n : n < \omega \text{even}\} \in F_0 \text{ and } \bigcup\{w_n : n < \omega \text{ odd }\} \in F_1.$$

Proof. $\underline{\neg(\text{A}) \Rightarrow \neg(\text{B}):}$ If F_1 is not a P-point or F_0 is not Ramsey then player I can win by 5.7. (I.e., if F_1 is not a P-point, then are $B_n \in F_1$ for $n < \omega$ such that for no $B \in F_1$ do we have $B \setminus B_n$ is finite for every n, now player I has a strategy guaranteeing: for n odd, $A_n = \bigcap_{\ell \leq (n-1)/2} B_\ell \setminus (\sup \bigcup\{w_\ell : \ell < n \text{ odd}) + 1)$, this is a winning strategy. If F_0 is not a Ramsey ultrafilter there are $B_n \in F_0$ for $n < \omega$ such that for no $k_n \in B_n$ (for $n < \omega$) do we have $\{k_n : n < \omega\} \in F_0$, now player I has a strategy guaranteeing $A_{2n} = B_n$, this is a winning strategy.) So we can assume F_1 is a P-point and F_0 is Ramsey, so by $\neg$(A) necessarily $F_0 \leq_{RK} F_1$, hence some $h : \omega \to \omega$ witnesses $F_0 \leq_{\mathrm{RK}} F_1$. Then player I can play such that $\bigcup\{h^{-1}(k_n) : n \in \omega\}$ and $\bigcup\{w_n : n \in \omega\}$ will be disjoint. So one of them is not in F_1, thus player I wins.

$\underline{(\text{A}) \Rightarrow (\text{B}):}$ Suppose H is a wining strategy of player I. Let λ be big enough, $N \prec (H(\lambda), \in), \{F_0, F_1, H\} \in N$ and N is countable. As F_ℓ is a P-point there is $A^*_\ell \in F_\ell$ such that $A^*_\ell \subseteq_{ae} B$ for every $B \in F_\ell \cap N$.

Now we can find an increasing sequence $\langle M_n : n < \omega \rangle$ of finite subsets of $N, N = \bigcup_{n<\omega} M_n$ such that it increases rapidly enough; more exactly:

α) $H, F_0, F_1 \in M_0, M_n \in M_{n+1}$; also can demand $x \in M_n$ & x finite $\Rightarrow x \subseteq M_n$; also $M_n \cap \omega$ is an initial segment of ω,

β) if $\varphi(x, a_0, \ldots)$ is a formula of length $\leq 1000 + |M_n|$ with parameters from $M_n \cup \{M_n\}$ satisfied by some $x \in N$, then it is satisfied by some $x \in M_{n+1}$,

γ) for $\ell = 0, 1$ if $B \in F_\ell \cap N$, $B \in M_n$ then $B \cup M_{n+1} \supseteq A^*_\ell$,

δ) $M_0 \cap \omega = \emptyset$.

Let $u_{n+1} = (M_{n+1} \setminus M_n) \cap \omega$. So $\langle u_n : n < \omega \rangle$ forms a partition of ω. As F_ℓ is an ultrafilter, there are $S_\ell \subseteq \omega$ such that $\bigcup\{u_n : n \in S_\ell\} \in F_\ell$, and $n < m$ & $\{n, m\} \subseteq S_\ell \Rightarrow m - n \geq 10$.

Can we demand also $n \in S_0$, $m \in S_1$ implies the absolute value of $n - m$ is ≥ 5? For the S_0, S_1 we have, for each $n \in S_0$ there is at most one $m \in S_1$ such that $|n - m| \leq 4$ and vice versa. So in the bad case there are $S^\dagger_\ell \subseteq S_\ell$, $f : S^\dagger_0 \to S^\dagger_1$ one to one and onto, $n - 4 \leq f(n) \leq n + 4$, $\bigcup\{u_n : n \in S^\dagger_\ell\} \in F_\ell$ for $\ell = 0, 1$; moreover, for any $S^*_\ell \subseteq S^\dagger_\ell$,

$$\bigcup\{u_n : n \in S^*_0\} \in F_0 \quad \text{iff} \quad \bigcup\{u_n : n \in S^*_1\} \in F_1$$

provided that $S^*_1 = f(S^*_0)$. Also as F_0 is a Ramsey ultrafilter, there are $k_n \in u_n$ (for $n \in S^\dagger_0$) such that $\{k_n : n \in S^\dagger_0\} \in F_0$. So the function $f^* : \omega \to \omega$ defined by $f^*(\ell) = k_n$ for $\ell \in u_{f(n)}$, $n \in S^\dagger_0$, and $f^*(\ell) = 0$ otherwise, exemplifies $F_0 \leq_{RK} F_1$, contradiction.

So without loss of generality

$(*)$ for $n \in S_0, m \in S_1$ we have $n - m$ has absolute value ≥ 5,

$(**)$ there are $k^*_n \in u_n \cap A^*_0$ (for $n \in S_0$) such that $\{k^*_n : n \in S_0\} \in F_0$ (because F_0 is Ramsey.)

It is also clear that by (γ) above, as $A^* \in F_1$:

(∗∗∗) For $n \in S_1$ let $v_n \stackrel{\text{def}}{=} u_n \cap \bigcap\{A : A \in F_1 \cap M_{n-2}\}$. Then

$$\bigcup\{v_n : n \in S_1\} \in F_1,$$

also $h^*_\ell \in \bigcap\{A : A \in F_0 \cap M_{n_2}\}$.
[Simply note $u_n \cap A^* \subseteq v_n$ and w.l.o.g. $\min(S_\ell) > 2$].

Now there is no problem to define by induction on $\ell < \omega$, $n_\ell < \omega$ and an initial segment $\bar{t}^\ell$ of length ℓ of a play of the game (both increasing) such that: the initial segment belong to M_{n_ℓ}; and every k^*_n will appear among the k's which player II have chosen in the play if $n \leq n_\ell$, $n \in S_0$; and every v_n will appear among the w's player II have chosen in the play if $n \leq n_\ell$, $n \in S_1$; and n_ℓ has the form $n^* + 2$ with $n^* \in S_0 \cup S_1$; and player I uses his strategy. But in the play we produce player II wins, contradiction. $\square_{5.11}$

5.12 Main Lemma. Suppose F_0 is a Ramsey ultrafilter (on ω), F is a P-point, and $Q = \mathrm{SP}^*(F)$, and $\Vdash_Q$ "F_0 is not an ultrafilter" *then* $F_0 \leq_{RK} F$.

Proof. Let $T_0 \in Q$, $\underset{\sim}{A}$ be a Q-name, $T_0 \Vdash_Q$ "$\underset{\sim}{A} \subseteq \omega$ and $\omega \setminus \underset{\sim}{A}, \underset{\sim}{A} \neq \emptyset$ mod F_0", and w.l.o.g. $\Vdash_Q$ "$\underset{\sim}{A} \subseteq \omega$", (such $T_0, \underset{\sim}{A}$ exists as after forcing with Q, F_0 will no longer generate an ultrafilter). Note that by the choice of T_0, $\underset{\sim}{A}$ for any $T \geq T_0$:

$$\{n : \text{ for some } T^\dagger \geq T, T^\dagger \Vdash_Q \text{“} n \in \underset{\sim}{A}\text{” and for some } T^\dagger \geq T, T^\dagger \Vdash_Q \text{“} n \notin \underset{\sim}{A}\text{”}\}$$

belongs to F_0.

Now we use the game defined in Lemma 5.11. We shall describe a winning strategy for player I. During the play, player I in his moves defines also $T_n \in Q$ preserving the following:

(∗) (a) $T_{n+1} \geq T_n$
(b) $T_n \Vdash_Q$ "$k_\ell \in \underset{\sim}{A}$" for ℓ even, $\ell < n$
(c) $T^{[m(n)]}_{n+1} = T^{[m(n)]}_n$ where $m(n) = 1 + \max[\bigcup\{w_\ell : \ell \text{ odd}, \ell < n\} \cup \{n\}]$
(d) for $\ell < n$ odd we have: $w_\ell \subseteq \mathrm{spt}_\ell(T_n)$ (see Definition 5.4)

(e) for n even, for the play from 5.7 player I chooses

$$A_n \subseteq \{k : T_n \nVdash \text{“}k \notin \underset{\sim}{A}\text{”}\}$$

(f) for n odd, for the play from 5.7 player I chooses $A_n = \mathrm{spt}_{m(n)}(T_n)$.

More exactly, player I chooses T_{n+1} in the n-th move after player II's move (see below more).

This is enough, as if in the end $\bigcup\{w_\ell : \ell < \omega \text{ odd }\} \in F$, then $T \stackrel{\text{def}}{=} \bigcap_n T_n \in Q$, because for each $\ell < \omega$, we have $n > \ell \Rightarrow \mathrm{spt}_\ell(T_{n+1}) \subseteq \mathrm{spt}_\ell(T_n)$ and $\mathrm{spt}_{\ell+1}(T_n) \subseteq \mathrm{spt}_\ell(T_n)$ so by clauses (c)+(d)
$(*)$ $\ell < m \leq k \Rightarrow w_k \subseteq \mathrm{spt}_\ell(T_m)$.
Hence $\mathrm{spt}_\ell(T) \supseteq \bigcap_{m>\ell} \mathrm{spt}_\ell(T_m) \supseteq \bigcup_{m\geq\ell} w_m \in F$ (as all cofinite subsets of ω belong to F). Now T forces $\{k_\ell : \ell < \omega \text{ even }\} \subseteq \underset{\sim}{A}$ (remember clause (b)), so $\{k_\ell : \ell < \omega \text{ even }\} \notin F_0$ by the hypothesis on $T_0, \underset{\sim}{A}$ (as $\{k_\ell : \ell < \omega\} \in V$, and $T_0 \leq T$, $T \Vdash_P$ “$\{k_\ell : \ell < \omega\} \subseteq \underset{\sim}{A}$” so $\{k_\ell : \ell < \omega\} \in F_0$ implies: $T \Vdash_Q$ “$\omega \setminus \underset{\sim}{A} = \emptyset \mod F$”, a contradiction). So the strategy defined above is a winning strategy for player I hence by Lemma 5.11, $F_0 \leq_{RK} F$. So it remains to show that player I can carry out the strategy i.e. can preserve $(*)$. Note that T_0 is defined.

Case 1: n even > 0: Player I lets $m(n) < \omega$ be $\max[\bigcup\{w_\ell : \ell < n \text{ odd}\} \cup \{n\}] + 1$, and let $T_n^{[m(n)]} = \{\eta_0, \ldots, \eta_{s(n)}\}$ with no repetition. For each η_ℓ ($\ell \leq s(n)$) clearly $(T_n)_{[\eta_\ell]}$ is $\geq T_0$ and belongs to Q, hence

$$A_\ell^n = \{k < \omega\text{: there are } T'_{\ell,k}, T''_{\ell,k} \geq (T_n)_{[\eta_\ell]}, \text{ such that } T'_{\ell,k} \Vdash_Q \text{“}k \in \underset{\sim}{A}\text{”, and } T''_{\ell,k} \Vdash_Q \text{“}k \notin \underset{\sim}{A}\text{”}\}$$

belong to F_0.

Now: player I plays $A_n = \bigcap_{\ell \leq s(n)} A_\ell^n$ which is clearly a legal move.

Player II chooses some $k_n \in A_n$.

Player I (“on the side”) lets $T_{n+1} = \bigcup_{\ell \leq s(n)} T'_{\ell,k_n}$ (it is as required in $(*)$).

Case 2: n odd: Player I lets $A_n = \mathrm{spt}_{m(n)}(T_n)$ (note $Q = \mathrm{SP}^*(F)$). Note T_n has just been chosen.

Player II chooses a finite $w_n \subseteq A_n$ and player I lets on the side $T_{n+1} = T_n$. $\square_{5.12}$

5.13 Theorem. It is consistent with ZFC $+2^{\aleph_0} = \aleph_2$ that, up to a permutation on ω, there is a unique Ramsey ultrafilter on ω. Moreover any P-point is above it (in the Rudin-Keisler order).

Proof. We start with a universe satisfying $2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2$ and $\diamondsuit_{\{\delta<\aleph_2:\mathrm{cf}(\delta)=\aleph_1\}}$. There is a Ramsey ultrafilter F in V. We shall use a CS iterated forcing $\langle P_i, \underset{\sim}{Q}_i : i < \omega_2\rangle$ such that each Q_i is proper, has the PP-property (hence is ${}^\omega\omega$-bounding), has cardinality continuum and forces that F still generates an ultrafilter. So by 5.1, 5.2, F remains a Ramsey ultrafilter in V^{P_i} for $i \leq \omega_2$ and also we can show by induction on $i < \omega_2$, that in V^{P_i}, CH holds and P_i has cardinality $\aleph_1$; so by VIII §2 below P_{ω_2} satisfies the $\aleph_2$-chain condition. If $F_1 \in V[G_{\omega_2}]$ ($G \subseteq P_{\omega_2}$ generic) is a P-point, not above F, then there is a $p \in P_{\omega_2}$ forcing $\underset{\sim}{F}_1$ is a name of such ultrafilter, and for a closed unbounded set of $\delta < \aleph_2$, $\mathrm{cf}(\delta) = \aleph_1$ implies that $\underset{\sim}{F}{}^1_\delta \stackrel{\mathrm{def}}{=} \underset{\sim}{F}_1 \cap \mathcal{P}(\omega)^{V^{P_\delta}} \in V^{P_\delta}$ and p forces that $\underset{\sim}{F}{}^1_\delta$ is a P-point not above F (in V^{P_δ}).

Now, by the diamond $\diamondsuit_{\{\delta<\aleph_2:\mathrm{cf}(\delta)=\aleph_1\}}$ we can assume that for some such δ, $\underset{\sim}{Q}_\delta = \mathrm{SP}^*(\underset{\sim}{F}{}^1_\delta)$.

Now by 5.12 forcing with Q_δ (over V^{P_δ}) preserves "F (generates) an ultrafilter", by 5.6(2) Q_δ has the PP-property hence (by 2.12B) Q_δ is ${}^\omega\omega$-bounding and trivially Q_δ has cardinality continuum; so Q_i is as required. Now as each Q_j ($i < j < \omega_2$) has the PP-property, P_{ω_2}/P_δ has the PP-property (by 2.12C+2.3). So by lemma 5.12 we know $\underset{\sim}{F}{}^1_\delta$ cannot be completed to a P-point in $V^{P_{\omega_2}}$. $\square_{5.13}$

§6. On the Splitting Number $\mathfrak{s}$ and Domination Number $\mathfrak{b}$ and on $\mathfrak{a}$

For a survey on this area, see van Douwen [D] and Balcar and Simon [BS].

Nyikos has asked us whether there may be (in our terms) an undominated family $\subseteq {}^\omega\omega$ of power $\aleph_1$ while there is no splitting family $\subseteq [\omega]^{\aleph_0}$ of power $\aleph_1$. He observed that it seems necessary to prove, assuming CH, the existence of a P-point without a Ramsey ultrafilter below it (in the Rudin-Keisler order).

In the third section we have proved a preservation lemma for countable support iterations whose first motivation is that no new $f \in {}^\omega\omega$ dominates all old ones, and prove (3.23(1)) the consistency of ZFC$+2^{\aleph_0} = \aleph_2 + \mathfrak{d} = \mathfrak{s} > \mathfrak{b}$ where $\mathfrak{d}$ is the minimal power of a dominating subfamily of ${}^\omega\omega$ (see 3.24(3)), and $\mathfrak{s}$ is the minimal power of splitting subfamily of $[\omega]^{\aleph_0}$ (see Def 3.24(1)) and $\mathfrak{b}$ is the minimal power of an undominated subfamily of ${}^\omega\omega$ (see Definition 3.24(2)).

However one point was left out in Sect. 3: the definition of the forcing we iterate, and the proof of its relevant properties: that it adds a subset $\underset{\sim}{r}$ of ω such that $\{A \in V : A \subseteq \omega, \underset{\sim}{r} \subseteq^* A\}$ is an ultrafilter of the Boolean algebra $\mathcal{P}(\omega)^V$; but in a strong sense it does not add a function $\underset{\sim}{f} \in {}^\omega\omega$ dominating all old members of ${}^\omega\omega$; this was promised in 3.22. Note that Mathias forcing adds a subset $\underset{\sim}{r}$ of ω as required above, but also adds an undesirable $\underset{\sim}{f}$. This is done here; its definition takes some space. This forcing notion makes the "old" $[\omega]^{\aleph_0}$ an unsplitting family. The proof of this is quite easy, but we have more trouble proving the "old" ${}^\omega\omega$ is not dominated. ¿From the forcing notion (and, in fact, using a simpler version), we can construct a P-point as above.

Then A. Miller told us he is more interested in having in this model "no MAD of power $\leq \aleph_1$" (MAD stands for "a maximal almost disjoint family of infinite subsets of ω") (i.e. $\mathfrak{s}$, $\mathfrak{a} > \aleph_1 = \mathfrak{b}$). A variant of our forcing can "kill" a MAD family and the forcing has the desired properties if we first add $\aleph_1$ Cohen reals (see 3.23(2), 6.16). We also like to prove the consistency of ZFC$+2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 + \aleph_2 = \mathfrak{s} > \mathfrak{a} = \mathfrak{b} = \aleph_1$, where $\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A}$ a maximal family of almost disjoint subsets of $\omega\}$ (see Definition 3.24(2)). In the seventh section we show that in the model we have constructed (in the proof of 3.23(1)) there is a MAD (maximal family of pairwise disjoint infinite subsets of ω) of power $\aleph_1$ (hence $\mathfrak{a} = \aleph_1$). This answers a question of Balcar and Simon:

they defined

$$\mathfrak{a}_s = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a maximal family of almost disjoint subsets of } \omega \times \omega, \text{ which are graphs of partial functions from } \omega \text{ to } \omega\}.$$

They have proved $\mathfrak{s} \le \mathfrak{a}_s$ and $\mathfrak{a} \le \mathfrak{a}_s \le 2^{\aleph_0}$, so our result implies that $\mathfrak{a} < \mathfrak{a}_S$ is consistent.

In the eighth section we prove the consistency (with ZFC $+2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$) of $\aleph_1 = \mathfrak{h} < \mathfrak{a} = \mathfrak{b} = \aleph_2$ (where $\mathfrak{h}$ is the minimal cardinal κ for which $\mathcal{P}(\omega)/\text{finite}$ is a $(\kappa, 2^{\aleph_0})$-distributive Boolean algebra).

The relations between the cardinals above are described by the following diagram.

$$\begin{array}{ccccccccc}
 & & \mathfrak{s} & \longrightarrow & \mathfrak{d} & \longrightarrow & & & 2^{\aleph_0} \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
\aleph_1 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{b} & \rightarrow & \mathfrak{a} & \rightarrow & \mathfrak{a}_s
\end{array}$$

(where arrow means "$\le$ is provable is ZFC") (see [D] and [Sh:207] for results not mentioned above, and two other cardinal invariants); sections 6, 7, 8 represent material from [Sh:207] (revised).

* * *

Now we turn to the definition of the forcing we iterate and the proof of its relevant properties: that it adds a subset $\underset{\sim}{r}$ of ω such that $\{A \in V : A \subseteq \omega, \underset{\sim}{r} \subseteq_{ae} A\}$ is an ultrafilter in the Boolean algebra $\mathcal{P}(\omega)^V$; but in a strong sense (that is, almost ${}^\omega\omega$-bounding) it does not add a function $\underset{\sim}{f} \in {}^\omega\omega$ dominating all old members of ${}^\omega\omega$.

More on such forcing notions see [RoSh:470].

6.1 Definition. 1) Let K_n be the family of pairs (s, h), s a finite set, h a partial function from $\mathcal{P}(s)$ (you can think of $h(t)$ when not defined as -1) to $n+1$ such that:

(a) $h(s) = n$

(b) if $h(t) = \ell + 1$ (so $t \subseteq s$), $t = t_1 \cup t_2$ then $h(t_1) \ge \ell$ or $h(t_2) \ge \ell$ and $|t| > 1$.

We may add

(c) if $t_1 \subseteq t_2$ are in $\mathrm{Dom}(h)$ then $h(t_1) \leq h(t_2)$.

2) $K_{\geq n}, K_{\leq n}, K_{n,m}$ are defined similarly, and $K = \bigcup_{n<\omega} K_n$.

We call s the domain of (s,h) and write $a \in (s,h)$ instead of $a \in s$. We call (s,h) standard if s is a finite subset of the family of hereditarily finite sets. We use the letter t to denote such pairs. We call (s,h) simple if $h(t) = [\log_2(|t|)]$ for $t \subseteq s$. If $t = (s,h) \in K$, let $\mathrm{lev}(t) = \mathrm{lev}(s,h)$ be the unique $n < \omega$ such that $t \in K_n$.

6.2 Definition. 1) Suppose $(s_\ell, h_\ell) \in K_{s(\ell)}$ for $\ell \in \{0,1\}$. We say $(s_0,h_0) \leq^d (s_1,h_1)$ (or (s_1,h_1) refines (s_0,h_0)) if:
$s_0 = s_1$ and $[t_1 \subseteq t_2 \subseteq s_0 \ \&\ h_1(t_1) \leq h_1(t_2) \Rightarrow h_1(t_1) \leq h_0(t_1) \leq h_0(t_2)]$ (so $\mathrm{lev}(s_0,h_0) \geq \mathrm{lev}(s_1,h_1)$ and $\mathrm{Dom}(h_1) \subseteq \mathrm{Dom}(h_0)$).
2) We say $(s_0,h_0) \leq^e (s_1,h_1)$ if for some $s_0' \in \mathrm{Dom}(h_0)$, $(s_0', h_0 \restriction \mathcal{P}(s_0')) = (s_1,h_1)$
3) We say $(s_0,h_0) \leq (s_1,h_1)$ if for some (s',h'), $(s_0,h_0) \leq^e (s',h') \leq^d (s_1,h_1)$.

6.3 Fact. The relations $\leq^d$, $\leq^e$, $\leq$ are partial orders of K. $\square_{6.3}$

6.4 Definition.

1) Let L_n be the family of pairs (S,H) such that:
   a) S is a finite tree with a root called $\mathrm{root}(S)$.
   b) H is a function whose domain is $\mathrm{in}(S)$ =the set of non-maximal points of S and with values H_x for $x \in \mathrm{in}(S)$.
   c) For $x \in \mathrm{in}(S)$, $(\mathrm{Suc}_S(x), H_x) \in K_{\geq n}$, where $\mathrm{Suc}_S(x)$ is the set of immediate successors of x in S, so $H_x(\mathrm{Suc}_S(x)) \geq n$.
2) We say $(S^0,H^0) \leq (S^1,H^1)$ if $S^0 \supseteq S^1$, they have the same root, $\mathrm{in}(S^1) = S^1 \cap \mathrm{in}(S^0)$ and for every $x \in \mathrm{in}(S^1)$, $(\mathrm{Suc}_{S^0}(x), H^0_x) \leq (\mathrm{Suc}_{S^1}(x), H^1_x)$ and of course $\mathrm{Suc}_{S^1}(x) = \mathrm{Suc}_{S^0}(x) \cap S^1$.
3) Let $\mathrm{int}(S) \stackrel{\mathrm{def}}{=} S \setminus \mathrm{in}(S)$, $\mathrm{lev}(S,H) = \max\{n : (S,H) \in L_n\}$, $x \in (S,H)$ means $x \in S$. A member of L_n is standard if $\mathrm{int}(S) \subseteq \omega$ and $\mathrm{in}(S)$ consists of hereditarily finite sets not in ω. Let for $x \in S$, $(S,H)^{[x]} = (S^{[x]}, H \restriction S^{[x]})$ where $S^{[x]}$ is $S \restriction \{y \in S : S \models x \leq_S y\}$.

4) For $\mathbf{t} \in L_n$ let $\mathbf{t} = (S^{\mathbf{t}}, H^{\mathbf{t}})$ and let $\text{lev}(\mathbf{t}) = \max\{n : \mathbf{t} \in L_n\}$
5) We say $\mathbf{t}^1, \mathbf{t}^2 \in \bigcup_{n<\omega} L_n$ are disjoint *if*: $S^{\mathbf{t}^1} \cap S^{\mathbf{t}^2} = \emptyset$.
6) Let $\text{int}(\mathbf{t}) = \text{int}(S^{\mathbf{t}})$.
7) Let $L = \bigcup\limits_{n<\omega} L_n$

6.5 Fact. The relation $\leq$ is a partial order of $L = \bigcup_n L_n$. $\square_{6.5}$

6.6 Fact. If $(S, H) \in L_n$ then $(S', H') \stackrel{\text{def}}{=} \text{half}(S, H)$ belongs to $L_{[(n+1)/2]}$ and $(S, H) \leq (S', H')$ where $S' = S$, $H'_x(A) = [H_x(A) - \text{lev}(S, H)/2]$ where $[x]$ is the largest integer $\leq x$ and $\text{Dom}(H'_x) = \{A : H_x(A) \geq \text{lev}(S, H)/2\}$. $\square_{6.6}$

6.7 Fact. If $(S, H) \in L_{n+1}$, $\text{int}(S) = A_0 \cup A_1$ *then* there is $(S^1, H^1) \geq (S, H)$, $(S^1, H^1) \in L_n$ such that $[\text{int}(S^1) \subseteq A_0$ or $\text{int}(S^1) \subseteq A_1]$.

Proof. Easy by induction on the height of the tree (using clause (b) of Def 6.1(1)). $\square_{6.7}$

6.8 Definition. We define the forcing notion Q:

1) $p \in Q$ *if* $p = (w, T)$ where w is a finite subset of ω, T is a countable (infinite) set of pairwise disjoint standard members of L and $T \cap L_n$ is finite for each n, moreover for simplicity the convex hulls of the $\text{int}(\mathbf{t})$ for $\mathbf{t} \in T$ are pairwise disjoint; let $\text{cnt}(T)$ and $\text{cnt}(p)$ mean $\bigcup_{(H,S)\in T} \text{int}(S, H)$. Writing $T = \{\mathbf{t}_n : n < \omega\}$ we mean $\langle \min(\text{int}(\mathbf{t}_n)) : n < \omega \rangle$ strictly increasing.
2) Given $\mathbf{t}_1 = (S_1, H_1), \ldots, \mathbf{t}_k = (S_k, H_k)$ all from L such that $S_i \cap S_j = \emptyset$ $(i \neq j)$, and given $\mathbf{t} = (S, H)$ from L, we say $\mathbf{t}$ is *built* from $\mathbf{t}_1, \ldots, \mathbf{t}_k$ if: there are incomparable nodes $a_1, \ldots, a_k$ of S such that every node of S is comparable with some a_i, and such that, letting $S(a_i) = \{b \in S : b \geq_S a_i\}$ we have $(S_i, H_i) = (S(a_i), H{\restriction}S(a_i))$.
3) $(w^0, T^0) \leq (w^1, T^1)$ iff: $w^0 \subseteq w^1 \subseteq w^0 \cup \text{cnt}(T^0)$, and, letting $T^0 = \{\mathbf{t}^0_0, \mathbf{t}^0_1, \ldots\}$, $T^1 = \{\mathbf{t}^1_0, \mathbf{t}^1_1 \ldots\}$, there are finite, nonempty pairwise disjoint subsets of ω, B_0, $B_1, \ldots$, and there are $\hat{\mathbf{t}}_i \geq \mathbf{t}^0_i$ for all $i \in \bigcup_j B_j$ such that

for each n only finitely many of the $\hat{\mathbf{t}}_i$ are inside L_n and such that for each j, letting $B_j = \{i_1, \dots, i_k\}$, $\mathbf{t}^1_j$ is built from $\hat{\mathbf{t}}_{i_1}, \dots, \hat{\mathbf{t}}_{i_k}$.

4) We call (w, T) standard if $T = \{\mathbf{t}_n : n < \omega\}$, $\max(w) < \min[\mathrm{int}(\mathbf{t}_n)]$, $\max[\mathrm{int}(\mathbf{t}_n)] < \min[\mathrm{int}(\mathbf{t}_{n+1})]$ and $\mathrm{lev}(\mathbf{t}_n)$ is strictly increasing (and writing $T = \{\mathbf{t}_n : n < \omega\}$ we mean this).

6.9 Definition. For $p = (w, T)$ we write $w = w^p$, $T = T^p$. We say q is a pure extension of p ($p \leq_{\mathrm{pr}} q$) if $q \geq p$, $w^q = w^p$. We say p is pure if $w^p = \emptyset$, and $p \leq^* q$ means omitting finitely many members of T^q makes $q \geq p$.

The following generalization will be used later.

6.10 Definition. 1) For an ideal I of $\mathcal{P}(\omega)$ (which includes all finite sets) let $Q[I]$ be the set of $p \in Q$ such that for every $A \in I$, for infinitely many $\mathbf{t} \in T^p$, $\mathrm{int}(\mathbf{t}) \cap A = \emptyset$. The main case is $I =$ family of finite subsets of ω (then $Q[I] = Q$).

2) Let $Q'[I]$ be $\{p \in Q :$ there is q such that $Q \models p \leq q$ and $q \in Q[I]\}$ (so $Q[I]$, $Q'[I]$ are equivalent).

6.10A Remark. 1) So if $p = (w, \{\mathbf{t}_n : n < \omega\}) \in Q[I]$ then $p \leq (w, \{\mathrm{half}(\mathbf{t}_n) : n < \omega\}) \in Q[I]$.

2) More generally if $p = (w, \{\mathbf{t}_n : n < \omega\}) \in Q[I]$ and $h : \omega \to \omega$ is a function from ω to ω going to ∞ (i.e. $\liminf_{n<\omega} h(n) = \infty$) and $\mathbf{t}'_n \geq \mathbf{t}_n$, or even $\mathbf{t}'_n \geq \mathbf{t}_n^{[x_n]}$ and $\mathrm{lev}(\mathbf{t}'_n) \geq h(\mathrm{lev}(\mathbf{t}_n))$ then $(w, \{\mathbf{t}'_n : n < \omega\}) \in Q[I]$.

6.11 Fact. 0) Q is a partial order.

1) If $p \in Q$ and $\underset{\sim}{\tau}_n$ (for $n < \omega$) are Q-names of ordinals, then there is a pure standard extension q of p such that: letting $T^q = \{\mathbf{t}_\ell : \ell < \omega\}$, for every $n < \omega$ and $w \subseteq \max[\mathrm{int}(\mathbf{t}_n)] + 1$, if we let $q^n_w = (w, \{\mathbf{t}_\ell : \ell > n\})$, then for $k \leq n$:

$$q^n_w \text{ forces a value on } \underset{\sim}{\tau}_k \textit{ iff } \text{some pure extension of } q^n_w \text{ forces a value on } \underset{\sim}{\tau}_k.$$

Moreover if $T^p = \{\mathbf{t}^0_n : n < \omega\}$, we can demand $\bigwedge_{\ell<n^*} \mathbf{t}_\ell = \mathbf{t}^0_\ell$ but then the demand on n, w above is for $n \geq n^* - 1$ only.

2) Q is proper (in fact α-proper for every $\alpha < \omega_1$).

3) $\Vdash_Q$ "$\{n : (\exists p \in \underset{\sim}{G}_Q)[n \in w^p]\}$ is an infinite subset of ω which $\mathcal{P}(\omega)^V$ does not split."

Proof. Easy (for (3) use 6.7, see more in 6.16(3)). $\square_{6.11}$

6.12 Lemma. Let q, $\underset{\sim}{\tau}_n$ be as in 6.11(1). Then for some pure standard extension r of q, letting $T^r = \{\mathbf{t}'_n : n < \omega\}$, (lev($\mathbf{t}'_n$) strictly increasing, of course and) the following holds.

(∗) *For every* $n < \omega$, $w \subseteq [\max(\text{int}(\mathbf{t}'_{n-1})) + 1]$, and $\mathbf{t}''_n \geq \mathbf{t}'_n$ (so we ask only lev($\mathbf{t}''_n$) ≥ 0) *there is* $w' \subseteq \text{int}(\mathbf{t}''_n)$, such that $(w \cup w', \{\mathbf{t}'_\ell : \ell > n\})$ forces a value on $\underset{\sim}{\tau}_m$ for $m \leq n$ (we let $\max \text{int}(\mathbf{t}'_{-1})$ be $\max(w^q \cup \{-1\})$).

This lemma follows easily from claim 6.14 (see below) (choose by it the $\mathbf{t}'_n$ by induction on n) and is enough for a proof of Lemma 3.22, which we now present.

6.13 Proof of Lemma 3.22. By 6.11(2), clause (a) (of 3.22 i.e. Q is proper) holds (more fully use the last clause of 6.11(1) to get a sequence of conditions as needed); and by 6.11(3) clause (d) (of 3.22 i.e. inducing an ultrafilter on the old $\mathcal{P}(\omega)$) holds; and clause (c) (of 3.22 i.e. $|Q| = 2^{\aleph_0}$) is trivial. For proving clause (b) (i.e. Q is almost ${}^\omega\omega$-bounding, see Definition 3.5(1)) let $\underset{\sim}{f} \in {}^\omega\omega$ and $p \in Q$ be given. Let $\underset{\sim}{\tau}_n = \underset{\sim}{f}(n)$, apply 6.11(1) to get q and then apply (on q, $\underset{\sim}{\tau}_n$ $(n < \omega)$) 6.12 getting $r = (w^p, \{\mathbf{t}'_n : n < \omega\}) \geq q$. We have to define $g \in {}^\omega\omega$ (as required in Definition 3.5(1)). Let $g(n) = \max\{k + 1 :$ for some $w \subseteq [\max(\text{int}(\mathbf{t}'_n)) + 1]$ we have $(w, \{\mathbf{t}'_\ell : \ell > n) \Vdash$ "$\underset{\sim}{f}(n) = k$"$\}$. Let A be any infinite subset of ω, and we define $p' = (w^p, \{\mathbf{t}'_n : n \in A\})$, so $p' \geq r \geq p$. We have to show that $p' \Vdash_Q$ "for infinitely many $n \in A$, $\underset{\sim}{f}(n) < g(n)$". So it is enough, given $n_0 < \omega$ and p^2, $p' \leq p^2 \in Q$ to find $n \in A \setminus n_0$ and p^3 such that $p^2 \leq p^3 \in Q$ and $p^3 \Vdash_Q$ "$\underset{\sim}{f}(n) < g(n)$". So assume that $n_0 < \omega$ and

$p' \le p^2 \in Q$, and $p^2 = (w^2, T^2)$, $T^2 = \{\mathbf{t}^2_n : n < \omega\}$, and w.l.o.g. for some $i(*) > n_0$ for every n we have $\min[\mathrm{int}(\mathbf{t}^2_n)] > \max[\mathrm{int}(\mathbf{t}'_{i(*)})] > \sup(w^2)$. As $p^2 \ge p'$, we can find $k < \omega$, $i_1 < \cdots < i_k$ from A, and $\mathbf{t}^*_{i_\ell} \ge \mathbf{t}'_{i_\ell}$ such that $\mathbf{t}^2_0$ is built from $\mathbf{t}^*_{i_1}, \ldots, \mathbf{t}^*_{i_k}$; by the previous sentence $i_1 > i(*)$. By $(*)$ from 6.12 (as $w^2 \subseteq \max[\mathrm{int}(\mathbf{t}'_{i(*)})] + 1$, and $i(*) < i_1$ and r from 6.12 is standard), there is $w'' \subseteq \mathrm{int}(\mathbf{t}^*_{i_k})$ (hence $w'' \subseteq \mathrm{int}(\mathbf{t}^2_0)$) such that $p^3 = (w^2 \cup w'', \{\mathbf{t}'_j : j \in (i_k, \omega)\})$ forces a value, say m to $\underset{\sim}{f}(i_k)$, so by the definition of g clearly $m < g(i_k)$. But clearly p^2, p^3 have a common upper bound: $p^4 = (w^2 \cup w'', \{\mathbf{t}^2_n : n \in (n_4, \omega)\})$ for every $n_4 < \omega$ large enough (really $n_4 = 0$ is O.K.!). So we are done. $\square_{3.22}$

6.14 Claim. Let $(\emptyset, T)$ be a pure condition, and let W be a family of finite subsets of $\mathrm{cnt}(T)$ so that

$(*)$ for every $(\emptyset, T') \ge (\emptyset, T)$, there is a $w \subseteq \mathrm{cnt}(T')$ such that $w \in W$.

Let $k < \omega$. Then there is $\mathbf{t} \in L_k$ appearing in some $(\emptyset, T') \ge (\emptyset, T)$ such that:

$$\mathbf{t}' \ge \mathbf{t} \quad \Rightarrow \quad (\exists w \in W)[w \subseteq \mathrm{int}(\mathbf{t}')].$$

Proof. Let T^* be arbitrary such that $(\emptyset, T) \le (\emptyset, T^*) \in Q$, and $T^* = \{\mathbf{t}_n : n < \omega\}$. For notational simplicity, without loss of generality let W be closed upward.

Stage A: There is n such that for every $\mathbf{t}'_\ell \ge$half$(\mathbf{t}_\ell)$ (for $\ell < n$) we have $\bigcup_{\ell<n} \mathrm{int}(\mathbf{t}'_\ell) \in W$. This is because the family of $\langle \mathbf{t}'_\ell : \ell < n \rangle$, $n < \omega$, $\mathbf{t}'_\ell \ge$half$(\mathbf{t}_\ell)$ form an ω-tree with finite branching and for every infinite branch $\langle \mathbf{t}'_\ell : \ell < \omega \rangle$ by $(*)$ there is an initial segment $\langle \mathbf{t}'_\ell : \ell < n \rangle$ with $\bigcup_{\ell<n} \mathrm{int}(\mathbf{t}'_\ell) \in W$. [Why? Define $(S^\ell, H^\ell) \in L$ such that $S^\ell = S^{\mathbf{t}'_\ell}$ and $H^\ell_x(A) = H^{\mathbf{t}_\ell}_x(A)$ (and not $H^{\mathbf{t}'_\ell}_x(A)$!) when $x \in \mathrm{in}(S^\ell)$, $A \subseteq \mathrm{Suc}_{S_\ell}(x)$, so letting $T' =: \{(S^\ell, H^\ell) : \ell < \omega\}$ we have: $\mathrm{lev}(S^\ell, H^\ell) \ge \mathrm{lev}(\mathbf{t}_\ell)/2^\ell - 1/2$ and $(\emptyset, T^*) \le (\emptyset, T')$. Now apply $(*)$ remembering W is upward closed.] By König's lemma we finish.

Stage B: There are $n(0) < n(1) < n(2) < \ldots$ such that for every m and $\mathbf{t}'_\ell \ge$ half$(\mathbf{t}_\ell)$ for $n(m) \le \ell < n(m+1)$, the set $\bigcup\{\mathrm{int}(\mathbf{t}'_\ell) : n(m) \le \ell < n(m+1)\} \in W$. The proof is by repeating stage A (changing T^*).

Stage C: There are $m(0) < m(1) < \dots$ such that: if $i < \omega$, for every function h with domain $[m(i), m(i+1))$ such that $h(j) \in [n(j), n(j+1))$ and $\mathbf{t}'_\ell \geq \mathrm{half}(\mathbf{t}_\ell)$ for all relevant ℓ then $\bigcup\{\mathbf{t}'_{h(j)} : j \in [m(i), m(i+1))\}$ belongs to W.

The proof is parallel to that of stage B; as there it is enough, assuming $m(i^*)$ was chosen, to find appropriate $m(i^*+1) > m(i^*)$. The set of branches corresponds to $\{\langle \mathbf{t}'_\ell : \ell \in [m(i^*), \omega)\rangle$: for some function $h \in \prod_{\ell \in [m(i^*), \omega)} [n(\ell), n(\ell+1))$ for every $\ell \in [m(i^*), \omega)$, $\mathbf{t}'_\ell \geq \mathrm{half}(\mathbf{t}_{h(\ell)})\}$. So if the conclusion fails i.e. for every $m > m(i^*)$ if we assign $m(i^*+1) = m$, for some function h_m with domain $[m(i^*), m)$, $h(\ell) \in [n(\ell), n(\ell+1))$ and $\langle \mathbf{t}^m_\ell : \ell \in [m(i^*), m)\rangle$, where $\mathbf{t}^m_\ell \geq \mathrm{half}(\mathbf{t}_{h_m(\ell)})$ the desired conclusion fails. So by König's lemma we can find $h \in \prod_{\ell \in [m(i^*), \omega)} [n(\ell), n(\ell+1))$, $\langle \mathbf{t}'_\ell : \ell \in [m(i^*), \omega)\rangle$ such that for every $m' \in [m(i^*), \omega)$ for infinitely many $m \in [m', \omega)$ we have

$$\ell \in [m(i^*), m') \Rightarrow h_m(\ell) = h(\ell) \ \&\ \mathbf{t}'_\ell = \mathbf{t}^m_\ell.$$

As before using $\langle \mathbf{t}'_{h(\ell)} : \ell < \omega\rangle$ we can contradict the assumption $(*)$.

Stage D: We define a partial function H from finite subsets of ω to ω: let $H(u) \geq 0$ if for every $\mathbf{t}'_\ell \geq \mathrm{half}(\mathbf{t}_\ell)$ (for $\ell \in u$) we have $(\bigcup_{\ell \in u} \mathrm{int}(\mathbf{t}'_\ell)) \in W$ and let $H(u) \geq m+1$ if $[u = u_1 \cup u_2 \Rightarrow H(u_1) \geq m \vee H(u_2) \geq m]$.

We have shown that $H([n(i), n(i+1))) \geq 0$, and $H([n(m(i)), n(m(i+1)))) \geq 1$, (for the later, assuming $u = [n(m(i)), n(m(i+1))) = u_1 \cup u_2$ we have that: either u_1 contains an interval $[n(j), n(j+1))$ for some $j \in [m(i), m(i+1))$ *or* u_2 has a member in each such interval so it contains $\{h(j) : j \in [m(i), m(i+1))\}$ for some $h \in \prod_{\ell=m(i)}^{m(i+1)-1} [n(\ell), n(\ell+1))$; now apply stage B to show that in the first case $H(u_1) \geq 0$ and Stage C to show that in the second case $H(u_2) \geq 0$).

It clearly suffices to find u, $H(u) \geq k$. [We then define $\mathbf{t} = (S, H)$ as follows: $S = \bigcup_{\ell \in u} S^{\mathbf{t}_\ell} \cup \{u\}$, u is the root with set of immediate successors being $\{\mathrm{root}(\mathbf{t}_\ell) : \ell \in u\}$; and the order restricted to $S^{\mathbf{t}_\ell}$ is as in $\mathbf{t}_\ell$; and for $x \in S^{\mathbf{t}_\ell}$ we have $H^{\mathbf{t}}_x = H^{\mathrm{half}(\mathbf{t}_\ell)}_x$ and $H^{\mathbf{t}}_u(A) \stackrel{\mathrm{def}}{=} H(\{\ell : \mathrm{root}(S^{\mathbf{t}_\ell}) \in A\})$.] We prove the existence of such u by induction on k, (e.g. simultaneously for all T',

$(\emptyset, T') \geq (\emptyset, T))$. This is done by repeating the proof above (alternatively, we just repeat 2^k times getting an explicit member of K_k in the root). $\square_{6.14}$

The rest of this section deals with $Q[I]$. Note that by 6.21(2) below in the interesting case the set of standard $p \in Q[I]$ is dense. For the rest of this section:

6.15 Notation. 1) Let Q^0 be the forcing of adding $\aleph_1$ Cohen reals $\langle r_i : i < \omega_1 \rangle$, $r_i \in {}^\omega\omega$. We usually work in $V_1 = V^{Q^0}$.

2) Let $\mathcal{A} = \{A_i : i < \alpha^*\}$ denote an infinite family of infinite subsets of ω (usually the members are pairwise almost disjoint).

3) Let $I = I_\mathcal{A}$ be the ideal of $\mathcal{P}(\omega)$, including all finite subsets of ω but $\omega \notin I$ and generated by $\mathcal{A} \cup \{[0, n) : n < \omega\}$. So $I_\mathcal{A}$ depends on the universe (the interesting case here is $\mathcal{A}$ a MAD family in V, of the form $\{A_i : i < \omega_1\}$, $Q[I_\mathcal{A}]$ means in $V_1 = V^{Q^0}$). If not said otherwise we assume $\emptyset \notin I_\mathcal{A}$.

6.16 Claim. Assume $\mathcal{A} \in V$ is a family of subsets of ω (not necessarily MAD), and we work in $V_1 = V^{Q^0}$, and $I = I_\mathcal{A}$ so $Q[I]$ is from V^{Q^0}:

1) If $p \in Q[I]$ and $\underset{\sim}{\tau}_n$ $(n < \omega)$ are $Q[I]$-names of ordinals *then* there is a pure standard extension q of p such that: $q \in Q[I]$, and letting $T^q = \{\mathbf{t}_n : n < \omega\}$, for every $n < \omega$ and $w \subseteq [\max \operatorname{int}(\mathbf{t}_n) + 1]$ let $q^n_w = (w, \{\mathbf{t}_\ell : n < \ell < \omega\})$, then ($q^n_w \in Q[I]$, of course, and) for every $k \leq n$ we have: q^n_w forces a value on $\underset{\sim}{\tau}_k$ iff some pure extension of q^n_w in $Q[I]$ forces a value on $\underset{\sim}{\tau}_\kappa$.

2) $Q[I]$ is proper, (moreover α-proper for every $\alpha < \omega_1$ (not used)).

3) $\Vdash_{Q[I]}$ "$\{n : (\exists p \in G_{Q[I]}) n \in w^p\}$ is an infinite subset of ω which is almost disjoint from every $A \in I$ (equivalently $A \in \mathcal{A}$)."

Proof. 1) Let λ be regular large enough, N a countable elementary submodel of $(H(\lambda), \in, V \cap H(\lambda))$ to which I, $\langle r_i : i < \omega_1 \rangle$, $Q[I]$, p and $\langle \underset{\sim}{\tau}_n : n < \omega \rangle$ belong and $N' = N \cap V \in V$ (remember we are working in V_1). Let $\delta = N \cap \omega_1$ (so $\delta \notin N$). So $N = N'[\langle r_i : i < \delta \rangle]$ belongs to $V[\langle r_i : i < \delta \rangle]$.

We define by induction on $n < \omega$, $q^n \in Q[I] \cap N$, $\mathbf{t}_n$ and $k_n < \omega$ such that:

a) each q^n is a pure extension of p.

b) $q^n \geq q^\ell$ for $\ell < n$ and if $w \subseteq k_n$, $m < n+1$ and some pure extension of (w, T^{q^n}) forces a value on $\underset{\sim}{\tau}_m$, then (w, T^{q^n}) does it.

c) $k_n > k_\ell$ and $k_n >\ \max \mathrm{int}(\mathbf{t}_\ell)$ for $\ell < n$.

d) every $\ell \in \mathrm{cnt}(q^{n+1})$ is $> k_n$ i.e. $\mathbf{t} \in T^{q^{n+1}} \Rightarrow \min[\mathrm{int}(\mathbf{t})] > k_n$.

e) $\mathbf{t}_n \geq \mathbf{t}'_n$ for some $\mathbf{t}'_n \in T^{q^n}$ and $\mathrm{lev}(\mathbf{t}_n) > n$ and $\min[\mathrm{int}(\mathbf{t}_n)]$ is $> k_n$.

There is no problem in doing this: in stage n, we first choose k_n, then q^n and at last $\mathbf{t}_n$. We want in the end to let $T^q = \{\mathbf{t}_n : n < \omega\}$ (and $w^q = w^p$). One point is missing. Why does $q = (w^p, T^q)$ belong to $Q[I]$ (not just to Q)? But we can use some function in $V[\langle r_i : i < \delta\rangle]$ to choose k_n, q^n and then let $\mathbf{t}_n$ be the $r_\delta(n)$-th member of T^{q^n} which satisfies the requirement (in some fixed well ordering from V of the hereditarily finite sets). As $\mathcal{A} \in V$ and $r_\delta \in {}^\omega\omega$ is Cohen generic over $V[\langle r_i : i < \delta\rangle]$, this should be clear.

2) Easy by part (1).

3) Use Definition 6.10 and Fact 6.7. $\square_{6.16}$

6.17 Claim. Assume $\mathcal{A} = \{A_i : i < \alpha^*\} \in V$ is a MAD family, and in V_1 we have that $\Vdash_Q$"$\{A_i : i < \alpha^*\}$ is a MAD family". In V_1, let I be the ideal generated by $\{A_i : i < \alpha^*\}$ and the finite subsets of ω. Then: $(w, \{\mathbf{t}_n : n < \omega\})$ is a [standard] condition in $Q'[I]$ *iff*

it is a [standard] condition in Q and there are finite (non empty) pairwise disjoint $u_\ell \subseteq \alpha^*$ (for $\ell < \omega$) such that for each ℓ, for every k for some $n < \omega$, for some $\mathbf{t}'_n$, $\mathbf{t}'_n \geq \mathbf{t}_n$, $\mathrm{lev}(\mathbf{t}'_n) \geq k$ and $\mathrm{int}(\mathbf{t}'_n) \subseteq \bigcup_{i \in u_\ell} A_i$ *iff*

as before but there are singletons u_ℓ as above.

6.17A Remark. Note: if $\mathcal{A} \in V$, by 6.17 the standard $q \in Q[I]$ are dense in $Q[I]$, but otherwise we do not know. In the proof it does not matter.

Proof. The third condition implies trivially the second. We shall prove [second $\Rightarrow$ first] and then [first$\Rightarrow$third]. Suppose there are u_ℓ ($\ell < \omega$) as in the second condition above and we shall prove the first one. So for each $\ell < \omega$ we can find $\langle \mathbf{t}'_n : n \in B_\ell\rangle$, $B_\ell \subseteq \omega$ is infinite, $\mathbf{t}'_n \geq \mathbf{t}_n$, $\mathrm{lev}(\mathbf{t}'_n) \geq |B_\ell \cap n|$ and

$\mathrm{int}(\mathbf{t}'_n) \subseteq \bigcup_{i\in u_\ell} A_i$. Wlog $\langle B_\ell : \ell < \omega\rangle$ are pairwise disjoint, so $p \le p' \stackrel{\mathrm{def}}{=} (w, \{\mathbf{t}'_n : n \in \bigcup_{\ell<\omega} B_\ell\})$ and $p' \in Q$, so it suffices to show $p' \in Q[I]$. Now every $B \in I$ is included in $\bigcup_{i\in u} A_i \cup \{0, \dots, n^* - 1\}$ for some finite $u \subseteq \omega_1$ and $n < \omega$. But for some ℓ, u_ℓ is disjoint from u, hence $B \cap (\bigcup_{i\in u_\ell} A_i)$ is finite. We know that for infinitely many $n \in B_\ell$, $\mathrm{int}(\mathbf{t}'_n) \subseteq \bigcup_{i\in u_\ell} A_i$ and the $\mathrm{int}(\mathbf{t}'_n)$ $(n < \omega)$ are pairwise disjoint, hence for the infinitely many $n < \omega$, $\mathrm{int}(\mathbf{t}_n) \cap B = \emptyset$, as required in the first condition.

Lastly assume the first condition and we shall prove the third one. Suppose $p = (w, \{\mathbf{t}_n : n < \omega\}) \in Q'[I]$, see Definition 6.10(2), w.l.o.g. $p \in Q[I]$. We choose by induction on m a finite $u_m \subseteq \alpha^*$, disjoint from $\bigcup_{\ell<m} u_\ell$ such that

$$B_m = \{n < \omega : \text{for some } \mathbf{t}'_n \ge \mathbf{t}_n \text{ we have } \mathrm{lev}(\mathbf{t}'_n) \ge \mathrm{lev}(\mathbf{t}_n)/2 - 1$$
$$\text{and } \mathrm{int}(\mathbf{t}_n) \subseteq \bigcup_{i\in u_m} A_i\}$$

are infinite and moreover u_m is a singleton.

Assume we have arrived to stage m. Let $B \stackrel{\mathrm{def}}{=} \{n : \mathrm{int}(\mathbf{t}_n)$ is disjoint to $\bigcup\{A_i : i \in \bigcup_{\ell<m} u_\ell\}\}$, so B is necessarily infinite (by the Definition of $Q[I]$), moreover $p^0 \stackrel{\mathrm{def}}{=} (w, \{\mathrm{half}(\mathbf{t}_n) : n \in B\})$ belongs to $Q[I]$ and is above p. Now clearly $Q[I] \subseteq Q$, hence $p^0 \in Q$. By an assumption of 6.17, we know $\mathcal{A} = \{A_i : i < \alpha^*\}$ is a MAD family even after forcing by Q, so there are $p^1 = (w', \{\mathbf{t}'_n : n < \omega\}) \in Q$, $p^0 \le p^1$ and $i_0 < \alpha^*$ such that

$(*)$ $\quad p^1 \Vdash$ "$\{n : (\exists q \in \underset{\sim}{G}_Q)[n \in w^q]\} \cap A_{i_0}$ is infinite".

Let n^* be $> \sup(A_{i_0} \cap \bigcup\{A_i : i \in \bigcup_{\ell<m} u_\ell\})$. By 6.7 (more exactly, as in the proof of 6.16(3)), without loss of generality, $\bigcup_{n<\omega} \mathrm{cnt}(\mathbf{t}'_n) \subseteq A_{i_0} \setminus n^*$ or $\bigcup_{n<\omega} \mathrm{cnt}(\mathbf{t}'_n) \cap A_{i_0} = \emptyset$, but the second possibility contradicts $(*)$ so the first holds.

But $p^1 \ge p^0$ (in Q) so for each $n < \omega$ for some $k < \omega$ and $j_{n,0} < \cdots < j_{n,k-1}$ from B, $\mathbf{t}'_n$ is built from $\mathrm{half}(\mathbf{t}_{j_{n,0}}), \dots, \mathrm{half}(\mathbf{t}_{j_{n,k-1}})$. So for some $y \in \mathbf{t}'_n$ we have $(\mathbf{t}'_n)^{[y]} \ge \mathrm{half}(\mathbf{t}_{j_{n,0}})$, hence clearly (or see 6.20 below) there is $\mathbf{t}''_{j_{n,0}} \ge \mathbf{t}_{j_{n,0}}$, such that $\mathrm{int}(\mathbf{t}''_{j_{n,0}}) \subseteq A_{i_0} \setminus n^*$ and $\mathrm{lev}(\mathbf{t}''_{j_{n,0}}) \ge \mathrm{lev}(\mathbf{t}_{j_{n,0}})/2$. Lastly let $u_m = \{i_0\}$ (i.e. all depend on m). $\square_{6.17}$

6.18 Claim. Let V_1, $\mathcal{A}$, $I = I_{\mathcal{A}}$ be as in 6.16 + 6.17 (so $\mathcal{A}$ is a MAD family in V, V_1 and V^Q). Assume we are given $k^* < \omega$, $(\emptyset, T) = (\emptyset, \{\mathbf{t}_n : n < \omega\}) \in Q[I]$, and a family W of finite subsets of $\mathrm{cnt}(T)$ such that
$(*)$ if $(\emptyset, T) \leq (\emptyset, T') \in Q[I]$ *then* there is $w \subseteq \mathrm{cnt}(T')$ such that $w \in W$.
Then there is $\mathbf{t} \in L_{k^*}$ appearing in some $(T', \emptyset) \geq (T, \emptyset)$ such that:

$$\mathbf{t}' \geq \mathbf{t} \Rightarrow (\exists w \in W)[w \subseteq \mathrm{int}(\mathbf{t}')].$$

Proof. Without loss of generality T is standard (by 6.16(1)) and W upward closed (check). Moreover we may assume that $\mathrm{lev}(\mathbf{t}_n) \geq 2k^*$ for each $n < \omega$.

We know, by 6.17 above, that there is $T^* = \{\mathbf{t}_n : n < \omega\}$, such that $(\emptyset, T^*) \in Q$ is standard, $(\emptyset, T) \leq (\emptyset, T^*)$ and for some sequence $\langle j_m : m < \omega \rangle$ of pairwise distinct ordinals $< \alpha^*$ and partition $\langle B_m : m < \omega \rangle$ of ω to infinite sets we have:

$$n \in B_m \Rightarrow \mathrm{int}(\mathbf{t}_n) \subseteq A_{j_m}.$$

For every finite $u \subseteq \omega$ define

$$\begin{aligned} \mathrm{nor}(u) = \ & \max\{m : \text{for every cover } \langle u_\ell : \ell < 2^m \rangle \text{ of } u \\ & \qquad (\text{i.e. } u_\ell \subseteq u \text{ and } \bigcup_{\ell < 2^m} u_\ell = u), \\ & \quad \text{for some } \ell < 2^m \text{ and for every } \mathbf{t}'_i \geq \mathrm{half}(\mathbf{t}_i) \\ & \quad \text{for } i \in u_\ell \text{ we have: } [\bigcup_{i \in u_\ell} \mathrm{int}(\mathbf{t}'_i)] \in W\}. \end{aligned}$$

If for some finite $u \subseteq \omega$, $\mathrm{nor}(u) \geq k^*$ we can finish. Why? Just as in the end of the proof of 6.14 we define $\mathbf{t} = (S, H)$ as follows: $S = \bigcup\{S^{\mathbf{t}_\ell} : \ell \in u\} \cup \{u\}$, u is the root, its set of (immediate) successor is $\{\mathrm{root}(S^{\mathbf{t}_\ell}) : \ell \in u\}$, the order is defined by: restricted to $S^{\mathbf{t}_\ell}$ is as in $\mathbf{t}_\ell$, for $x \in S^{\mathbf{t}_\ell}$ we let $H^{\mathbf{t}}_x = H_x^{\mathrm{half}(\mathbf{t}_\ell)}$ and $H^{\mathbf{t}}_u$ is defined by: for $v \subseteq u$ let $H^{\mathbf{t}}_u(\{\mathrm{root}(S^{\mathbf{t}_\ell}) : \ell \in v\}) = \mathrm{nor}(v)$. We know $H_u(\{\mathrm{root}(S^{\mathbf{t}_\ell}) : \ell \in u\}) \geq k^*$. This suffices as $\oplus_1$ below holds. Clearly by the definition of nor we have

$\oplus_0 \quad \mathbf{t} \geq \mathbf{t}' \Rightarrow (\exists w \in W)[w \subseteq \mathrm{int}(\mathbf{t}')]$

Now we have to prove

$\oplus_1$ if $v = v_1 \cup v_2$, $\mathrm{nor}(v) \geq m+1$ then: $\mathrm{nor}(v_1) \geq m$ or $\mathrm{nor}(v_2) \geq m$.

Proof of $\oplus_1$. If $\mathrm{nor}(v_1) \ngeq m$, then there is a cover $\langle v^1_\ell : \ell < 2^m \rangle$ of v_1 such that:

$(*)_1$ for every $\ell < 2^m$ for some $\mathbf{t}'_m \geq \mathrm{half}(\mathbf{t}_m)$ (for $m \in v^1_\ell$) we have $[\bigcup_{m \in v^1_\ell} \mathrm{int}(\mathbf{t}'_m))] \notin W$.

Similarly, if $\mathrm{nor}(v_2) \ngeq m$ then there is a cover $\langle v^2_\ell : \ell < 2^m \rangle$ of v_2 such that:

$(*)_2$ for every $\ell < 2^m$ for some $\mathbf{t}'_m \geq \mathrm{half}(\mathbf{t}_m)$ (for $m \in v^2_\ell$) we have $\bigcup\limits_{m \in v^2_\ell} \mathrm{int}(\mathbf{t}'_m) \notin W$

Define for $i < 2^{m+1}$:

$$v_i = \begin{cases} v^1_i & \textit{if } i < 2^m \\ v^2_{i-2^m} & \textit{if } i \in [2^m, 2^{m+1}). \end{cases}$$

So if the conclusion fails then $\langle v_i : i < 2^{m+1} \rangle$ exemplifies $\mathrm{nor}(v) \ngeq m+1$, a contradiction.

We can conclude from all this that, toward contradiction we can assume that

$\otimes$ $u \subseteq \omega$ finite $\Rightarrow \mathrm{nor}(u) \ngeq k^*$.

So

$\otimes_1$ for every $n = \{0, \ldots, n-1\}$, $\mathrm{nor}(n) \ngeq k^*$ so there is a cover $\langle v^n_\ell : \ell < 2^{k^*} \rangle$ of n such that:

$\oplus$ for every ℓ for some $\mathbf{t}'_i \geq \mathrm{half}(\mathbf{t}_i)$ (for $i \in v^n_\ell$) we have $[\bigcup_{i \in v^n_\ell} \mathrm{int}(\mathbf{t}'_i)] \notin W$.

By König's lemma there is a sequence $\langle v_\ell : \ell < 2^{k^*} \rangle$ of subsets of ω such that for every $m < \omega$ for some $n = n(m) > m$ we have $v_\ell \cap m = v^n_\ell \cap m$.

Now for some $\ell = \ell(*) < 2^{k^*}$, for infinitely many $m < \omega$ for infinitely many $n \in B_m$ we have $n \in v_\ell$ (on the B_m's, see beginning of the proof of 6.18), so by 6.17 we know that $(\emptyset, \{\mathbf{t}_i : i \in v_{\ell(*)}\}) \in Q[I]$ (and of course is $\geq (\emptyset, T)$). (Alternatively, for some $\ell < 2^{k^*}$, for every $A \in I_{\mathcal{A}}$, for infinitely many $n \in v_\ell$ we have $\mathrm{int}(\mathbf{t}_n) \subseteq \omega \setminus A$. If not then for each ℓ some $A_\ell \in I_{\mathcal{A}}$ fails it. So let $A = \bigcup\limits_{\ell < 2^{k^*}} A_\ell \in I_{\mathcal{A}}$ and we get contradiction to $(\emptyset, \{\mathbf{t}_i : i < \omega\}) \in Q[I]$.) Now for every k letting $n = n(k)$ be such that $v_{\ell(*)} \cap k = v^n_{\ell(*)} \cap k$, we apply $\oplus$. So there are $\mathbf{t}'_i \geq \mathrm{half}(\mathbf{t}_i)$ (for $i \in v^n_{\ell(*)}$)) such that $\bigcup_{i \in v^n_{\ell(*)}} \mathrm{int}(\mathbf{t}'_i) \notin W$, and by monotonicity $\bigcup_{i \in v_{\ell(*)} \cap k} \mathrm{int}(\mathbf{t}'_i) \notin W$. By König's lemma (as W is upward closed) there is $\langle \mathbf{t}^*_i : i \in v_{\ell(*)} \rangle$, $\mathbf{t}^*_i \geq \mathrm{half}(\mathbf{t}_i)$ such that for every n we have

$\bigcup_{i \in n \cap v_{\ell(*)}} \mathrm{int}(\mathbf{t}_i^*) \notin W$. So (again as in the proof of 6.14, see 6.20 below) choose $(S^\ell, H^\ell) \in L$ such that $\mathrm{int}(S^\ell) = \mathrm{int}(\mathbf{t}_\ell^*)$ and $\mathrm{lev}(S^\ell, H^\ell) \geq \mathrm{lev}(\mathbf{t}_\ell)/2$, i.e. $S^\ell = S^{\mathbf{t}_\ell^*}$, $H_x^\ell(v) = \min\{[\mathrm{lev}(\mathbf{t})/2], H_x^{\mathbf{t}_\ell}(v)\}$ (when $v \subseteq \mathrm{Suc}_{S^\ell}(x)$). So clearly $(\emptyset, T^*) \leq (\emptyset, \{(S^\ell, H^\ell) : \ell < \omega\}) \in Q[I]$ (see 6.10A(2)) and we apply $(*)$ from the assumption and we get a contradiction, so finishing the proof of 6.18. $\square_{6.18}$

6.19 Claim. Let $\mathcal{A}$, $I = I_{\mathcal{A}}$ be as in 6.18. Let q, $\underset{\sim}{\tau}_n$ be as in 6.16(1). Then for some pure standard extension $r \in Q[I]$ of q, letting $T^r = \{\mathbf{t}'_n : n < \omega\}$, (standard (see Definition 6.8(4)) so $\mathrm{lev}(\mathbf{t}'_n)$ strictly increasing, of course) the following holds:

$(*)$ *For every* $n < \omega$, $w \subseteq [\max(\mathrm{int}(\mathbf{t}'_{n-1})) + 1]$, and $\mathbf{t}''_n \geq \mathbf{t}'_n$ (so we ask only $\mathrm{lev}(\mathbf{t}''_n) \geq 0$) *there is* $w' \subseteq \mathrm{int}(\mathbf{t}''_n)$, such that the condition $(w \cup w',$ $\{\mathbf{t}_\ell : \ell > n\})$ forces a value on $\underset{\sim}{\tau}_m$ for $m \leq n$ (we let $\max \mathrm{int}(\mathbf{t}'_{-1})$ be $\max(w^q \cup \{-1\})$).

Proof. Like the proof of 6.16(1) but using as the induction step claim 6.18. $\square_{6.19}$

6.20 Fact. If $\mathbf{t}_1 \geq \mathrm{half}(\mathbf{t}_0)$, *then* for some $\mathbf{t}_2 \geq \mathbf{t}_0$ we have $\mathrm{int}(\mathbf{t}_2) = \mathrm{int}(\mathbf{t}_1)$, $\mathrm{lev}(\mathbf{t}_2) \geq \mathrm{lev}(\mathbf{t}_0)/2$.

Proof. Included in earlier proof: 6.14. $\square_{6.20}$

6.21 Conclusion. Let V_1, $\mathcal{A}$, $I = I_{\mathcal{A}}$ be as in 6.18.

1) If $p \in Q[I]$ and $\omega = \bigcup_{\ell<k} A_\ell$ where $k < \omega$ *then* for some p', $p \leq_{\mathrm{pr}} p' \in Q[I]$ and for some $\ell < k$ we have $\mathrm{cnt}(T^p) \subseteq A_\ell$.

2) The set of standard $p \in Q[I]$ is dense, in fact for any $p \in Q[I]$ there is a standard q, $p \leq q \in Q[I]$, $w^q = w^p$ and $T^q \subseteq T^p$.

Proof. 1) By repeated use w.l.o.g. $k = 2$. Let $p \in Q[I]$ and $T^p = \{\mathbf{t}_n : n < \omega\}$. For each n apply 6.7 to find $\mathbf{t}'_n \geq \mathbf{t}_n$ such that $\mathrm{int}(\mathbf{t}'_n) \subseteq A$ or $\mathrm{int}(\mathbf{t}'_n) \subseteq \omega \setminus A$ and $\mathrm{lev}(\mathbf{t}'_n) \geq \mathrm{lev}(\mathbf{t}_n) - 1$. Let $Y_0 = \{n : \mathrm{int}(\mathbf{t}'_n) \subseteq A\}$, $Y_1 = \omega \setminus Y_0$, so for some

$\ell \in \{0,1\}$ we have: for every $X \in I$ the set $\{n \in Y_\ell : \mathrm{int}(\mathbf{t}'_n) \cap X = 0\}$ is infinite. [Why? If X_ℓ contradict the demand for $\ell = \{0,1\}$ then $X_0 \cup X_1 \in I$ contradict $p \in Q[I]$ by Definition 6.10.] So $(w^p, \{\mathbf{t}'_n : n \in Y_\ell\}) \in Q[I]$ is above p, and it forces $\bigcup\{w^r : r \in \underset{\sim}{G}_{Q[I]}\} \setminus w^p$ is included in A_ℓ.

We give also an alternative proof, which can be applied for more general question. Let $p = (w, \{\mathbf{t}_n : n < \omega\}) \in Q[I]$. By the proof of [first condition $\Rightarrow$ third condition] in 6.17, there are pairwise distinct $j_m < \omega_1$ (for $m < \omega$) such that for each m the set

$$B_m \stackrel{\text{def}}{=} \{n < \omega : \text{ there is } \mathbf{t}'_n \geq \mathrm{half}(\mathbf{t}_n) \text{ such that } \mathrm{int}(\mathbf{t}'_n) \subseteq A_{j_m}\}$$

is infinite. So we can find $B'_m \subseteq B_m$ for $m < \omega$ such that: $\langle B'_m : m < \omega \rangle$ is a sequence of infinite pairwise disjoint sets. For each $m < \omega$, $n \in B_m$ choose $\mathbf{t}'_n \geq \mathrm{half}(\mathbf{t}_n)$ such that $\mathrm{int}(\mathbf{t}'_n) \subseteq A_{j_m}$. Let $\mathbf{t}''_n \geq \mathbf{t}_n$ be such that $\mathrm{int}(\mathbf{t}''_n) = \mathrm{int}(\mathbf{t}'_n)$ and $\mathrm{lev}(\mathbf{t}''_n) \geq \mathrm{lev}(\mathbf{t}_n)/2$.

If $\mathrm{lev}(\mathbf{t}''_n) > k$, let $\mathbf{t}^3_n \geq \mathbf{t}''_n$ be such that $\mathrm{lev}(\mathbf{t}^3_n) \geq \mathrm{lev}(\mathbf{t}''_n) - k$ (really $\geq \mathrm{lev}(\mathbf{t}''_n) - [1 + \log_2(k)]$ suffices) and for some $\ell = \ell(n)$, $\mathrm{int}(\mathbf{t}^3_n) \subseteq A_{\ell(n)}$. For each $m < \omega$, for some ℓ_m the set $B''_m = \{n \in B_m : \mathrm{lev}(\mathbf{t}^3_n) > k, \ell(n) = \ell_m\}$ is infinite and for some $\ell(*) < k$ the set $\{m < \omega : \ell_m = \ell(*)\}$ is infinite. Now $p \stackrel{\text{def}}{=} (w^p, \{\mathbf{t}^3_n :$ for some m we have $\ell_m = \ell(*)$, and $n \in B''_m\})$ is as required.

2) Left to the reader (or see 6.16(1)). $\square_{6.21}$

Now we pay a debt needed for the proof of 3.23(2).

6.22 Claim. Assume V_1, $\mathcal{A}$, $I = I_{\mathcal{A}}$ are as in 6.18. Then $Q[I]$ is almost ${}^\omega\omega$-bounding *or* for some $p \in Q$ we have $p \Vdash_Q$ "$\{A_i : i < \aleph^*\}$ is not a MAD."

Proof. Assume the second possibility fails. So let $p \in Q[I]$ and $\underset{\sim}{f}$ be a $Q[I]$-name of a function from ω to ω. Let $\underset{\sim}{\tau}_n = \underset{\sim}{f}(n)$, and apply 6.16(1) and get q as there. Next apply 6.19 to those q, $\underset{\sim}{\tau}_n$ and get r which satisfies $(*)$ from 6.19.

By 6.17, 6.21(2) and we can find $r_1 = (w^p, \{\mathbf{t}'_n : n < \omega\})$, a standard member of $Q[I]$ such that $r \leq r_1$ and for some pairwise distinct $j_m < \omega_1$ the sets

$B_m \stackrel{\text{def}}{=} \{n < \omega : \text{int}(\mathbf{t}'_n) \subseteq A_{j_m}\}$ are infinite. Clearly also r_1 satisfies $(*)$ of 6.19. Choose pairwise distinct $n(m,\ell)$ for $m < \ell < \omega$ such that $n(m,\ell) \in B_m \setminus \{0\}$ and $\min \text{int}(\mathbf{t}'_{n(m,\ell)}) > \ell$. Now we define a function $g : \omega \to \omega$ (in V_1) by

$$\begin{aligned} g(\ell) = \ \max\{\{\ell+1\} \cup \{k : & \text{ for some } m < \ell \text{ and } w \subseteq [0,\ \max \text{int}(\mathbf{t}'_{n(m,\ell)-1})] \\ & \text{and } w_1 \subseteq \text{int}(\mathbf{t}'_{n(m,\ell)}) \text{ we have} \\ & (w^p \cup w_1, \{\mathbf{t}'_n : n > n(m,\ell)\}) \Vdash_{Q[I]} \text{ ``}\underset{\sim}{f}(\ell) = k\text{''}\}\}. \end{aligned}$$

So $g \in ({}^\omega\omega)^{V_1}$, and let $A \subseteq \omega$ ($A \in V_1$) be infinite, and let $p_A = (w^p, \{\mathbf{t}'_{n(m,\ell)} : m < \ell \text{ and } \ell \in A\})$. Now clearly $r_1 \leq p_A \in Q$, p_A standard and even $p_A \in Q[I]$ because still for each $m < \omega$ the set $\{n : \mathbf{t}'_n \in T^{p_A} \text{ and } \text{int}(\mathbf{t}'_n) \subseteq A_{j_m}\}$ is infinite: it includes $\{n : n = n(m,\ell) \text{ for some } \ell \in A \setminus (m+1)\}$. Now one can easily finish the proof. $\square_{6.16}$

A trivial remark is

6.23 Fact. Cohen forcing and even the forcing for adding λ Cohen reals (by finite information) is almost ${}^\omega\omega$-bounding.

§7. On $\mathfrak{s} > \mathfrak{b} = \mathfrak{a}$

See background in §6.

7.1 Theorem. Assume $V \models CH$. Then for some forcing notion P^*, P^* is proper, satisfies the $\aleph_2$-c.c., is weakly bounding and:

(∗) In V^{P^*} we have $2^{\aleph_0} = \aleph_2$, there is an unbounded family of ${}^\omega\omega$ of power $\aleph_1$ (i.e. $\mathfrak{b} = \aleph_1$) and also a MAD family of power $\aleph_1$ i.e. $\mathfrak{a} = \aleph_1$, but there is no splitting family of power $\aleph_1$ i.e. $\mathfrak{s} > \aleph_1$ (so $\mathfrak{s} = \aleph_2$).

Proof. The forcing $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \omega_2\rangle$, $P^* = P_{\omega_2}$ are as in the proof of 3.23(1). So the only new point is the construction of a MAD of power $\aleph_1$. This will be done in V; the proof of its being MAD will be done directly rather than through a preservation theorem (though the proof is similar).

Let $\{\langle B^i_n : n < \omega\rangle : i < \aleph_1\}$ enumerate (in V) all sequences $\langle B_n : n < \omega\rangle$ of finite nonempty subsets of ω (remember CH holds in V). Next choose a MAD family $\langle A_\alpha : \alpha < \aleph_1\rangle$ such that

(**) for each infinite ordinal $\alpha < \omega_1$ and $i < \alpha$: *if* for every $k < \omega$, $\alpha_1, \ldots, \alpha_k < \alpha$ for every m for some (equivalently infinitely many) $n < \omega$, $\min(B^i_n) > m$ and $B^i_n \cap (A_{\alpha_1} \cup \ldots \cup A_{\alpha_k}) = \emptyset$

then

(a) for infinitely many $n < \omega$, $B^i_n \subseteq A_\alpha$

(b) for any $k < \omega$ and $\alpha_1, \ldots, \alpha_k \leq \alpha$ for infinitely many $n < \omega$ we have $B^i_n \cap (\bigcup\limits_{\ell=1}^{k} A_{\alpha_\ell}) = \emptyset$.

[How? let $A_n = \{k^2 + n : k \in (n, \omega)\}$, and then choose A_α for $\alpha \in [\omega, \omega_1)$ by induction on α as required in (**).]

Let λ be a regular large enough cardinal, $\alpha \leq \omega_2$. For a generic $G_\alpha \subseteq P_\alpha$, a model $N \prec (H(\lambda)[G_\alpha], \in)$ is called *good* if it is countable, G_α, $\langle P_j, \underset{\sim}{Q}_i : i < \alpha, j \leq \alpha\rangle$, $\langle A_i : i < \omega_1\rangle$, $\langle\langle B^i_n : n < \omega\rangle : i < \omega_1\rangle \in N$ and for every set $\{B_n : n < \omega\} \in N$ of finite nonempty subsets of ω, letting $\delta = N \cap \omega_1$ we have *if*

$\otimes_1$ $(\forall m, k < \omega)(\forall \alpha_1, \ldots, \alpha_k < \delta)(\exists^* n < \omega)[B_n \cap (A_{\alpha_1} \cup \ldots \cup A_{\alpha_k}) = \emptyset \;\&\; \min(B_n) > m]$

then $(\exists^* n)[B_n \subseteq A_\delta]$ (remember $\exists^* n$ stands for "for infinitely many").

Note that in the definition of goodness, we have that $\otimes_1$ is equivalent to

$\otimes_2$ $(\forall m, k < \omega)(\forall \alpha_1, \ldots, \alpha_k < \omega_1)(\exists^* n < \omega)[B_n \cap (A_{\alpha_1} \cdots \cup A_{\alpha_k}) = \emptyset \;\&\; \min(B_n) > m]$

(as $N \prec (H(\lambda)[G_\alpha], \in)$).

We shall prove by induction on $\alpha \leq \omega_2$, that:

$(\circledast)_\alpha$ for every $\beta < \alpha$, a countable $N \prec (H(\lambda), \in)$ to which $\langle P_j, \underset{\sim}{Q}_i : i < \alpha, j \leq \alpha\rangle$, and α, β belong and generic $G_\beta \subseteq P_\beta$ *if* $N[G_\beta] \cap \omega_1 = N \cap \omega_1$ (so $N[G_\beta] \cap V = N$), $N[G_\beta]$ is good (in $V[G_\beta]$ of course) and

$p \in N[G_\beta] \cap P_\alpha/G_\beta$ *then* there is $q \in P_\alpha/G_\beta$, $q \geq p$, $\mathrm{Dom}(q) \setminus \alpha = N \cap \beta \setminus \alpha$, q is $(N[G_\beta], P_\alpha/G_\beta)$-generic and: if $G_\alpha \subseteq P_\alpha$ is generic, $G_\beta \subseteq G_\alpha$, $q \in G_\alpha$, then $N[G_\alpha]$ is good.

This is proved by induction. The case $\alpha = \omega_2$, $\beta = 0$ gives the desired conclusion.

[Why? If not for some $p \in P^* = P_{\omega_2}$ and a P_{ω_2}-name $\underset{\sim}{B} = \{\underset{\sim}{k}_n : n < \omega\}$ we have

$$p \Vdash_{P_{\omega_2}} \text{“}\underset{\sim}{B} \text{ is an infinite subset of } \omega, \text{ moreover } \underset{\sim}{k}_n < \underset{\sim}{k}_{n+1} < \omega \text{ for } n < \omega, \text{ and } \underset{\sim}{B} \cap A_\alpha \text{ is finite for every } \alpha < \omega_1\text{”}.$$

Let $N \prec (H(\lambda), \in)$ be countable such that $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \omega_2 \rangle$, $P^* = P_{\omega_2}$, p, $\underset{\sim}{B}$, $\langle \underset{\sim}{k}_n : n < \omega \rangle$ belong to N, and let $\delta \overset{\text{def}}{=} N \cap \omega_1$. Clearly $N \cap \{\langle B^i_n : n < \omega \rangle : i < \omega_1\} = \{\langle B^i_n : n < \omega \rangle : i < \delta\}$, so by the choice of the A_α's (see (**) above), N is good (in $V = V^{P_0}$). Hence there is $q \in P_{\omega_2}$ such that $p \leq q$, q is (N, P_{ω_2})-generic and $q \Vdash_{P_{\omega_2}}$ “$N[\underset{\sim}{G}_{\omega_2}]$ is good”. Let $G \subseteq P_{\omega_2}$ be generic over V, $q \in G$, (hence $p \in G$) so $N[G]$ is good, $N[G] \cap \omega_1 = \delta$ and $\langle \{\underset{\sim}{k}_n[G]\} : n < \omega \rangle$ belongs to $N[G]$. Hence by the definition of good, $(\exists^* m)[\underset{\sim}{k}_m[G] \in A_\delta]$, but this means $A_\delta \cap \underset{\sim}{B}[G]$ is infinite, contradicting the choice of p (as $p \in G$).]

The case $\alpha = 0$ is trivial (saying nothing) and the case α limit is similar to the proof of 3.13. In the case α successor, by using the induction hypothesis we can assume $\alpha = \beta + 1$.

By renaming $V[G_\beta]$, $N[G_\beta]$ as V, N we see that it is enough to prove that for any good N and $p \in Q \cap N$ (remember $Q_\beta = Q^{V[G_\beta]}$) *there is* $q \geq p$ which is (N, Q)-generic and $q \Vdash_Q$ “$N[G]$ is good”.

Let $\delta = N \cap \omega_1$, and let $\delta = \{\gamma(\ell) : \ell < \omega\}$. Let $\{\underset{\sim}{\tau}_\ell : \ell < \omega\}$ be a list of all Q-names of ordinals which belong to N, and $\{\langle \underset{\sim}{B}^\ell_n : n < \omega \rangle : \ell < \omega\}$ be a list of all Q-names of ω-sequences of nonempty finite subsets of ω which belong to N, and which are forced to satisfy $\otimes_2$, each appearing infinitely often. For notational simplicity only, assume p is pure. We shall define by induction on $\ell < \omega$ pure $p_\ell = (\emptyset, T^{p_\ell}) = (\emptyset, \{\mathbf{t}^\ell_n : n < \omega\})$ such that:

a) $p_\ell \in N$, p_ℓ standard (so $\max \mathrm{int}(\mathbf{t}^\ell_n) < \min \mathrm{int}(\mathbf{t}^\ell_{n+1})$),

b) $p_0 = p$, $p_{\ell+1} \geq p_\ell$,

c) $\mathbf{t}^\ell_n = \mathbf{t}^{\ell+1}_n$ for $n \leq \ell$ and $\mathrm{lev}(\mathbf{t}^\ell_\ell) \geq \ell$,

d) for any finite $w \subseteq \omega$ and finite $T' \subseteq T^{p_{\ell+1}}$ we have $(w, T^{p_{\ell+1}} \setminus T') \Vdash_Q$ "$\underset{\sim}{\tau}_\ell \in C^*_\ell$" for some countable set of ordinals C^*_ℓ which belongs to N,

e) for every $w_0 \subseteq (\max[\mathrm{int}(\mathbf{t}^\ell_\ell)] + 1)$, $m < \ell$, and $\mathbf{t} \geq \mathbf{t}^{\ell+1}_{\ell+1}$ there is $w_1 \subseteq \mathrm{int}(\mathbf{t})$ such that the condition $(w_0 \cup w_1, \{\mathbf{t}^{\ell+1}_i : \ell+1 < i < \omega\})$ forces that

$$\text{“}(\exists j)[\min(\underset{\sim}{B}^m_j) > \ell \text{ and } \underset{\sim}{B}^m_j \subseteq A_\delta]\text{”}.$$

Below we shall let $p^m_\ell = (\emptyset, \{\mathbf{t}^{\ell,m}_n : n < \omega\})$. Let $p_0 = p$.

Suppose p_ℓ is defined. By 6.12 there is a pure $p^0_\ell \geq p_\ell$ in N such that $\mathbf{t}^{\ell,0}_i = \mathbf{t}^\ell_i$ for $i \leq \ell$, and for any finite $w \subseteq \omega$ and finite $T' \subseteq T^{p^0_\ell}$ we have $(w, T^{p^0_\ell} \setminus T') = p^0_\ell \Vdash$ "$\underset{\sim}{\tau}_\ell \in C^*_\ell$" for some countable set of ordinals C^*_ℓ from N [why? read $(*)$ of 6.12].

Given p^0_ℓ we define:

$\mathbb{B} = \{B : B \subseteq \omega$ is finite, $\min(B) > \ell$, and there is standard

$p^* = (\emptyset, \{\mathbf{t}^*_n : n < \omega\}) \geq p^0_\ell$ such that $\bigwedge\limits_{i \leq \ell} \mathbf{t}^*_i = \mathbf{t}^\ell_i$

(so $\mathrm{lev}(\mathbf{t}^*_{\ell+1}) \geq \ell + 1$) and:

for every $w_0 \subseteq \max \mathrm{int}(\mathbf{t}^\ell_\ell) + 1$ and $\mathbf{t} \geq \mathbf{t}^*_{\ell+1}$ and $m < \ell$, for some

$w_1 \subseteq \mathrm{int}(\mathbf{t})$, the condition $(w_0 \cup w_1, \{\mathbf{t}^*_i : i > \ell + 1 \text{ and } i < \omega\})$

forces " for some $j < \omega$ we have $\underset{\sim}{B}^m_j \subseteq B$"$\}$.

Clearly

(A) $\mathbb{B} \in N$

(B) $\mathbb{B}$ satisfies $(\otimes_1)$ from the definition of good.

[Why? Let $k < \omega$, $\alpha_1, \ldots, \alpha_k < \delta$. By the assumption, $\Vdash_Q$ "for each $m < \ell$, the sequence $\langle \underset{\sim}{B}^m_\ell : j < \omega \rangle$ satisfies $\otimes_2$" hence $\Vdash$ " for every $m < \ell$ for some $n = n(m)$ we have $\min(\underset{\sim}{B}^m_n) > \ell$ and $\underset{\sim}{B}^m_n \cap (A_{\alpha_1} \cup \ldots A_{\alpha_k}) = \emptyset$. Hence there is standard $q = (\emptyset, \{\mathbf{s}_{\ell+1}, \mathbf{s}_{\ell+2}, \ldots\}) \in Q$, $q \geq (\emptyset, \{\mathbf{t}^{\ell,0}_{\ell+1}, \mathbf{t}^{\ell,0}_{\ell+2}, \ldots\})$ such that $\mathrm{lev}(\mathbf{s}_{\ell+1}) \geq \ell + 1$ and:

$\oplus$ if $w_0 \subseteq \max(\mathrm{int}(\mathbf{t}^\ell_\ell)) + 1$, $m < \ell$, $\mathbf{t} \geq \mathbf{s}_{\ell+1}$ then for some $w_1 \subseteq \mathrm{int}(\mathbf{t})$ and $n^m_{w_0,w_1} < \omega$ and $C^m_{w_0,w_1}$ we have

(α) $(w_0 \cup w_1, \{\mathbf{s}_{\ell(1)+1}, \mathbf{s}_{\ell(1)+2}, \ldots\}) \Vdash$ "$\underset{\sim}{B}^m_{n^m_{w_0,w_1}} = C^m_{w_0,w_1}$".

(β) $C^m_{w_0,w_1} \subseteq \omega \backslash \ell$, and $C^m_{w_0,w_1}$ is nonempty finite disjoint to $A_{\alpha_0} \cup \ldots \cup A_{\alpha_k}$.

So necessarily $\bigcup\{C^m_{w_0,w_1} : w_0 \subseteq \max(\mathrm{int}(\mathbf{t}^\ell_\ell)) + 1$ and $\mathbf{t} \geq \mathbf{s}_{\ell+1}$, and $w_1 \subseteq \mathrm{int}(\mathbf{t})$ and $C^m_{w_0,w_1}$ is well defined and $m < \ell\} \in \mathbb{B}$ is as required finishing the proof of clause (B). We could have demand in $\oplus$ above for one w_1 to be O.K. for all $m < \ell$.]

(C) We can define $p_{\ell+1}$.

[Why? As $\mathbb{B} \in N$ satisfies $(\otimes_1)$ and N is good necessarily there is $B \in \mathbb{B}$, $B \subseteq A_\delta$. For this B there is p^* as in the definition of $\mathbb{B}$. Let $\mathbf{t}^{\ell+1}_n = \mathbf{t}^\ell_n$ for $n \leq \ell$, $\mathbf{t}^{\ell+1}_n = \mathbf{t}^*_n$ for $n > \ell$. So $p_{\ell+1} = (\emptyset, \{\mathbf{t}^{\ell+1}_n : n < \omega\})$ is defined.]

So we have defined $p_{\ell+1}$ satisfying (a)-(e). So we can define p_ℓ for $\ell < \omega$ and now $q \stackrel{\mathrm{def}}{=} (\emptyset, \{\mathbf{t}^n_n : n < \omega\})$ is as required. $\square_{7.1}$

§8. On $\mathfrak{h} < \mathfrak{s} = \mathfrak{b}$

See background in §6. We first recall well known definitions.

8.1 Definition. 1) Let $\mathfrak{h}$ be the minimal cardinal λ such that there is a tree T with λ levels (not normal!) and $A_t \in [\omega]^{\aleph_0}$ for $t \in T$ such that $[t < s \Rightarrow A_s \subseteq_{ae} A_t]$ and $(\forall B \in [\omega]^{\aleph_0})(\exists t \in T)[A_t \subseteq_{ae} B]$ and if $t, s \in T$ are $<_T$-incomparable then $A_s \cap A_t$ is finite. See Balcar, Pelant and Simon [BPS] on it (and in particular why it exists which was for long an open problem).

2) Let $Q^d = \{(n, f) : n < \omega, f \in {}^\omega\omega\}$ with the order defined by

$(n_1, f_1) \leq (n_2, f_2)$ if and only if

$$n_1 \leq n_2, f_1 \restriction n_1 = f_2 \restriction n_1 \text{ and } f_1 \leq f_2 (\text{ i.e. } \bigwedge_\ell f_1(\ell) \leq f_2(\ell)).$$

This forcing adds a dominating real and it satisfies c.c.c. This is called Hechler forcing or dominating real forcing.

8.2 Theorem. Assume $V \models CH$.
For some proper forcing P of power $\aleph_2$ satisfying the $\aleph_2$-c.c., in V^P, $\mathfrak{h} = \aleph_1$, $\mathfrak{b} = \mathfrak{s} = \aleph_2$ (and $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$).

Proof. We shall use the direct limit P of the CS iteration $\langle P_i, \underset{\sim}{Q}_i : i < \omega_2 \rangle$ where:

A) letting $i = (\omega_1)^3\gamma + j, j < (\omega_1)^3$, if $j \neq \omega_1, \omega_1 + 1$ then $\underset{\sim}{Q}_i$ is Cohen forcing; if $j = \omega_1$ then $\underset{\sim}{Q}_i$ is Q from Definition 6.8 (in V^{P_j}) and if $j = \omega_1 + 1$ then $\underset{\sim}{Q}_i$ is Q^d (see Definition 8.1(2), also other nicely definable forcing notions are O.K.).

B) We use the presentation of countable support defined in III, proof of Theorem 4.1, i.e. using only hereditarily countable names. We let $\underset{\sim}{r}_i$ be the generic real of $\underset{\sim}{Q}_i$.

Clearly $|P| = \aleph_2$, P satisfies the $\aleph_2$-c.c. and is proper (see III §3, §4), hence forcing by P preserves cardinals. Clearly in V^P, $\mathfrak{s} \geq \aleph_2$ (because for unboundedly many $i < \aleph_2$, $\underset{\sim}{Q}_i = Q$ (from Definition 6.6, and 6.11(3)) and $\mathfrak{b} \geq \aleph_2$ (because for unboundedly many $i < \aleph_2$, $\underset{\sim}{Q}_i = Q^d$) and $2^{\aleph_0} = \aleph_2$. Hence in V^P we have $\mathfrak{s} = \mathfrak{b} = \aleph_2$ (so $\mathfrak{d} = \aleph_2$) and always $\mathfrak{h} \geq \aleph_1$. So the only point left is $V^P \models$ "$\mathfrak{h} \leq \aleph_1$".

We define by induction on $i < \omega_2$ (an ordinal $\alpha(i)$ and) $P_{\alpha(i)}$-names $\underset{\sim}{\eta}_i, \underset{\sim}{A}_i$ such that

(a) $\alpha(i) = (\omega_1)^3(i+1)$,

(b) $\underset{\sim}{\eta}_i \in \bigcup_{\beta < \omega_1} {}^{\beta+1}(\omega_2) \setminus \{\underset{\sim}{\eta}_j : j < i\}$ and for every successor $\beta < \ell g(\underset{\sim}{\eta}_i)$ we have $\underset{\sim}{\eta}_i \restriction \beta \in \{\underset{\sim}{\eta}_j : j < i\}$) (i.e. those things are forced),

(c) $\underset{\sim}{\eta}_j \lhd \underset{\sim}{\eta}_i \Rightarrow \underset{\sim}{A}_i \subseteq_{ae} \underset{\sim}{A}_j$ (for $j < i$) and $\underset{\sim}{A}_i$ is an infinite subset of ω,

(d) if $\underset{\sim}{A} \subseteq \omega$ is infinite and $\underset{\sim}{A} \in V^{P_j}$ then for some $i < j + \omega_1$, $\underset{\sim}{A} \subseteq \underset{\sim}{A}_j$,

(e) $\underset{\sim}{A}_i$ includes no infinite set from $V^{P_{\alpha(j)}}$ when $j < i$, and moreover is a subset of the generic real of $Q_{\omega_1^3 i+3}$,

(f) if $\underset{\sim}{\eta}_i$, $\underset{\sim}{\eta}_j$ are $\triangleleft$-incomparable then $\underset{\sim}{A}_i \cap \underset{\sim}{A}_j$ is finite (i.e. this is forced).

There is no problem to do this if you know the known way to build trees exemplifying the definition of $\mathfrak{h}$ (by Balcar, Pelant and Simon [BPS]), provided that no ω_1-branch has an intersection. I.e. for no $\eta \in {}^{\omega_1}(\omega_2)$ and $B \in [\omega]^{\aleph_0}$ (in $V^{P_{\omega_2}}$) do we have $B \subseteq_{ae} A_{i_\alpha}$ where $\eta \restriction (\alpha+1) = \eta_{i_\alpha}$ for $\alpha < \omega_1$; by clause (e) above necessarily i_α is strictly increasing. Let $i(*) = \bigcup_{\gamma<\omega_1} i_\gamma$ and $\alpha(*) = \bigcup_{\gamma<\omega_1} \alpha(i_\gamma)$, in $V^{P_{\alpha(*)}}$ there is no intersection by clause (e) (even in the case $\eta \notin V^{P_{\alpha(*)}}$). So it is enough to prove this for a fixed $i(*)$ hence also $\alpha(*)$.

We can look, in $V^{P_{\alpha(*)}}$, at the iteration $\bar{Q}' = \langle P'_\beta, \underset{\sim}{Q}_\gamma : \alpha(*) \le \gamma < \omega_2,$ $\alpha(*) \le \beta \le \omega_2 \rangle$, where $P'_\beta \overset{\text{def}}{=} P_\beta / P_{\alpha(*)}$. Let $G_1 \subseteq P_{\alpha(*)}$ be generic, $V_1 = V[G_1]$. Note that every element of P'_{ω_2} can be represented by a countable function from ordinals ($< \omega_2$) to hereditarily countable sets (built from ordinals $< \omega_2$). The set of elements of P'_{ω_2} as well as its partial order are definable from ordinal parameters only (all this in $V[G_1]$). Suppose $p \in P'_{\omega_2}$ forces $\underset{\sim}{B}$ (a P'_{ω_2}-name of a subset of ω) and $\underset{\sim}{i}_\gamma$ (for $\gamma < \omega_1$) to be as above (so with limit $i(*)$). W.l.o.g. for each $n < \omega$ there is an antichain $\langle q_{n,\ell} : \ell < \omega \rangle$ which is predense above p, such that $q_{n,\ell} \Vdash$ "$n \in \underset{\sim}{B}$ *iff* $\mathbf{t}_{n,\ell}$", $\mathbf{t}_{n,\ell}$ a truth value. So for some $j(*) < \alpha(*)$ we have p, $\langle \langle q_{n,\ell} : \ell < \omega \rangle : n < \omega \rangle \in V[G_1 \cap P_{j(*)}]$.

There is p_1, $p \le p_1 \in P'_{\omega_2}$ such that $p_1 \Vdash$ "$\underset{\sim}{i}_\gamma = i$" for some γ, i such that $j(*) < (\omega_1)^3 i < \alpha(*)$ so $p_1 \Vdash$ "$\underset{\sim}{B} \subseteq \underset{\sim}{r}_{\omega_1^3 i+3}$" where $\underset{\sim}{r}_{\omega_1^3 i+3}$ is the generic real that the set $G_1 \cap Q_{\omega_1^3 i+3}$ gives (see the end of clause (e)). Now using automorphisms of the forcing $P_{\alpha(*)}/P_{j(*)}$ we see that there is p_2, $p \le p_2 \in P'_{\omega_2}$ such that $p_2 \Vdash$ "$\underset{\sim}{B}$ is almost disjoint from $\underset{\sim}{r}_{\omega_1^3 i+3}$". From this we can conclude that $p \Vdash$ "$\bigcup_{\gamma<\omega_1} \underset{\sim}{\eta}_{i_\gamma} \notin V[G_1]$" (otherwise some $p_0 \ge p$ forces a particular value and repeat the argument above for p_0). Hence it suffices to prove by induction on $\beta \in [\alpha(*), \omega_2]$ that forcing with P'_β adds no new ω_1-branches to the tree $T \in V_1$ where $T = \{\underset{\sim}{\eta}_i[G_1] : i < i(*)\}$, ordered by $\triangleleft$, (i.e. all are on $V^{P_{\alpha(*)}}$). Let $\eta_i = \underset{\sim}{\eta}_i[G_1]$ for $i < i(*)$.

We prove by induction on $\beta \in [\alpha(*), \omega_2]$ that

$(*)^1_\beta$ P'_β adds no new ω_1-branch to $T \in V_1$.

So assume $p_0 \in P'_\beta$ is such that $p_0 \Vdash$ "$\langle \underset{\sim}{\nu}_\gamma : \gamma < \omega_1 \rangle$ is a new ω_1-branch of $\{\eta_i : i < i(*)\} \in V_1$".

In V_1 choose a sequence $\langle N_m : m < \omega \rangle$ of countable elementary submodels of $(H(\chi), \in, <^*_\chi)$ such that β, $\bar{Q}'$, P, $\underset{\sim}{B} \in N_0$, $N_m \in N_{m+1}$, $N_\omega = \bigcup_{m<\omega} N_m$. Let $\delta_m = N_m \cap \omega_1$, and let

$$A_m = \{\bar{\nu} = \langle \nu_\gamma : \gamma < \delta_m \rangle : \bar{\nu} \in N_{m+1} \text{ and for every } \gamma < \delta_m, \bar{\nu} \restriction \gamma \in N_m\}.$$

So $\langle A_m : m < \omega \rangle \in V_1$, and we can list $A_m = \{\bar{\nu}^{m,\ell} : \ell < \omega\}$, $\langle \langle \bar{\nu}^{m,\ell} : \ell < \omega \rangle : m < \omega \rangle \in V_1$. The real $\underset{\sim}{r}_{i(*)}$ is a Cohen real over V_1 (as $\langle i_\gamma : \gamma < \omega_1 \rangle$ is strictly increasing with limit $i(*)$), and we can interpret $Q_{i(*)}$ as ${}^{\omega>}\omega$, so let $\underset{\sim}{r}_{i(*)} = \langle \underset{\sim}{\ell}_m : m < \omega \rangle$.

Clearly for proving $(*)^1_\beta$ it is enough to find q such that:

$(*)_2$ $q \in P'_\beta$, $p_0 \leq q$, q is (N_m, P'_β)-generic for each $m < \omega$ and $q \Vdash_{P'_\beta}$ "$\langle \underset{\sim}{\nu}_\gamma : \gamma < \delta_m \rangle \neq \langle \nu_\gamma^{m,\underset{\sim}{\ell}_m} : \gamma < \delta_m \rangle$" for each m.

The proof splits to cases, the first four cases give $(*)^1_\beta$ directly, the last three do it through $(*)_2$.

Case 1: For $\beta = \omega_2$ no new branches appear (by $\aleph_2$-c.c.).

Case 2: For $\beta = i(*)$ trivial.

Case 3: For $\beta = \alpha + 1$, Q_α Cohen: use "Q_α is the union of $\aleph_0$ directed sets" (and such forcing notions do not add a new ω_1-branch to any old tree).

Case 4: For $\beta = \alpha + 1$, $Q_\alpha = Q^d$: similarly, as

$$Q^d = \bigcup_n \{\{(n, f) : f \in {}^\omega\omega \ \& \ f \restriction n = \eta\} : n < \omega, \eta \in {}^n\omega\}.$$

Case 5: For $\beta = \alpha + 1$, $Q_\alpha = Q$: so for some γ we have $\alpha = \omega_1^3\gamma + \omega_1$, shortly we shall work in the universe $V^{P'_{\omega_1^3\gamma}}$. Let $q' \in P'_\alpha$ be (N_m, P'_α)-generic for each $m < \omega$, $p_0 \restriction \alpha \leq q'$ (such q' exists as all those forcing notions are ω-proper and ω-properness is preserved by CS iteration by V §3). Let $G'_\alpha \subseteq P'_\alpha$ be generic over V_1 such that $q' \in G'_\alpha$, and we work in $V_2 = V_1[G'_\alpha]$. Let $N'_m = N_m[G'_\alpha]$, $w = w^{p_0(\alpha)}$ (a finite subset of ω, actually it is $w^{p_0(\alpha)[G'_\alpha]}$).

We choose by induction on $m < \omega$, $q_m = (w, T_m) \in Q_\alpha$, $T_m = \langle \mathbf{t}^m_n : n < \omega \rangle$ such that:

(a) $q_m \in N'_m \cap Q_\alpha$, $q_m \leq q_{m+1}$, $[n < m \Rightarrow \mathbf{t}^m_n = \mathbf{t}^{m+1}_n]$, $p_0(\alpha) \leq q_0$,

(b) q_{m+1} is (N'_m, Q_α)-generic

(c) $q_{m+1} \Vdash$ "$\langle \underset{\sim}{\nu}_\gamma : \gamma < \delta_m \rangle \neq \langle \nu^{m,\ell_m}_\gamma : \gamma < \delta_m \rangle$".

This clearly suffices and for the induction step, clauses (a), (b) are possible by the proof of "Q_α is proper" (in §6), and reflecting on the proof there also clause (c) [in more details given q_m, let $\langle \underset{\sim}{\tau}_{m,k} : k < \omega \rangle$ list the Q-names of ordinals which belong to N_m, now we choose by induction on $k < \omega$, $q_{m,k} = (w, T_{m,k})$ and $\gamma_{m,k}$, $T_{m,k} = \{\mathbf{t}^{m,k}_n : n < \omega\}$ standard, $\{q_{m,k}, \gamma_{m,k}\} \in N_m$, $q_m = q_{m,0}$, $q_{m,k} \leq q_{m,k+1}$, $[n \leq m + k \Rightarrow \mathbf{t}^{m,k}_n = \mathbf{t}^{m,k}_{n+1}]$, and for every $w_0 \subseteq \max[\mathrm{int}(\mathbf{t}^{m,k}_{m+k})] + 1$ and $\mathbf{t} \geq \mathbf{t}^{m,k}_{m+k+1}$, for some $w_1 \subseteq \mathrm{int}(\mathbf{t})$ we have $(w_0 \cup w_1, \{\mathbf{t}^{m,k}_{m+k+2}, \mathbf{t}^{m,k}_{m+k+3}, \ldots\})$ force $\underset{\sim}{\nu}_{\gamma_{m,k}} = \rho$, $\rho \neq \nu^{m,\ell_m}_{\gamma_{m,k}}$ and forces a value to $\underset{\sim}{\tau}_{i,k}$; for the induction step get a candidate for all $\gamma < \omega_1$ and use Δ-system (and "the branch is new" i.e. not from $V^{P'_\alpha}$).]

Case 6: $\beta > \alpha(*)$ is a limit ordinal; $\mathrm{cf}(\beta) = \aleph_0$

Quite straightforward as in the proof of the preservation of ω-properness (of course we could work in V rather than in V_1 and use the induction hypothesis). Choose $\langle \beta_n : n < \omega \rangle$ such that $\beta = \bigcup\limits_{n<\omega} \beta_n$, $i(*) = \beta_0$, $\beta_n < \beta_{n+1} < \beta$, and $\beta_n \in N_0$. We choose by induction on n, q_n, p_n such that:

(a) $q_n \in P'_{\beta_n}$, $\mathrm{Dom}(q_n) = (\bigcup\limits_{k<\omega} N_k) \cap [\alpha(*), \beta_n)$

(b) q_n is (N_i, P_{β_n})-generic for each $i \leq \omega$

(c) p_n is a P_{β_n}-name of a member of $P_\beta \cap N_n$

(d) $p_n \restriction \beta_n \leq q_n$, $q_{n+1} \restriction \beta_n = q_n$

(e) $p_n \leq p_{n+1}$

(f) p_{n+1} is (N_n, P_β)-generic (i.e. forced to be)

(g) $q_n \cup (p_n \restriction [\beta_n, \beta)) \Vdash_{P_\beta}$ "$\langle \underset{\sim}{\nu}_\gamma : \gamma < \delta_0 \rangle \neq \langle \nu^{n,\ell_n}_\gamma : \gamma < \delta_0 \rangle$".

The induction should be clear and $q \overset{\mathrm{def}}{=} \bigcup\limits_n q_n = \bigcup\limits_n (q_{n+1} \restriction [\beta_n, \beta_{n+1}))$ is as required.

Case 7: $\beta > \alpha(*)$ a limit ordinal, $\mathrm{cf}(\beta) > \aleph_0$

Like case 6, but $\beta_n = \sup(N_n \cap \beta)$. $\square_{8.2}$

Concluding Remark. The proof of "no new ω_1-branch" has little to do with the specific problem. More on definable forcing notions see [Sh:630].