Chapter VII
Trees and Large Cardinals in L

In this chapter we concentrate on the notion of a κ-tree in the case where κ is an inaccessible cardinal. In this case, assuming $V = L$, both the notion of a κ-Souslin tree and of a κ-Kurepa tree turn out to be closely related to large cardinal properties. Thus this chapter extends both Chapter IV, where we studied κ^+-trees, and (parts of) Chapter V, where we dealt with large cardinals.

1. Weakly Compact Cardinals and κ-Souslin Trees

The notion of a weakly compact cardinal has already been introduced in V.1, and we refer the reader back there for basic definitions. In particular, V.1.3 gives several equivalent definitions of weak compactness, and V.1.5 proves the result, relevant to us here, that if κ is a weakly compact cardinal, then $[\kappa$ is weakly compact$]^L$.

Assuming $V = L$, we shall prove that if κ is an inaccessible cardinal, then κ is weakly compact iff there is no κ-Souslin tree. This extends V.1.3 (viii), which says that, in ZFC, an inaccessible cardinal κ is weakly compact iff there is no κ-Aronszajn tree. We shall also show that under $V = L$, V.1.3 (ii) may be extended.

We shall require the following characterisation of weak compactness, which is really just a $V = L$ analogue of Π_1^1-indescribability (V.1.3 (iv)).

1.1 Lemma. *Assume $V = L$. Let κ be an inaccessible cardinal. Then κ is weakly compact iff, whenever $\varphi(\mathring{U}, \mathring{A}_1, \dots, \mathring{A}_n)$ is a sentence of the language $\mathscr{L}(U, A_1, \dots, A_n)$, if $A_1, \dots, A_n \subseteq J_\kappa$ are such that*

$$(\forall U \subseteq J_\kappa)\, [\langle J_\kappa, \in, U, A_1, \dots, A_n \rangle \models \varphi],$$

then for some $\alpha < \kappa$,

$$(\forall U \subseteq J_\alpha)\, [\langle J_\alpha, \in, U, A_1 \cap J_\alpha, \dots, A_n \cap J_\alpha \rangle \models \varphi]. \quad \square$$

There are various ways of proving 1.1. One way is to make minor modifications to the proof that Π_1^1-indescribability characterises weak compactness in ZFC (V.1.3 (iv)). Another way is to prove that under the assumption $V = L$, the property in 1.1 is actually equivalent to the Π_1^1-indescribability condition, by noting that if λ is inaccessible, then $J_\lambda = V_\lambda$. (This requires a lemma that the $\alpha < \kappa$ of 1.1 can always be assumed to be an inaccessible cardinal. The proof of this fact involves adding a conjunct to the sentence φ which ensures this.) In any event, the proof of 1.1 is of no direct relevance to our work here, being essentially a part of large cardinal theory itself, rather than constructibility theory. So we do not give a full proof.

Now, by V.1.3(viii), if κ is a weakly compact cardinal, then there is no κ-Aronszajn tree, so certainly there can be no κ-Souslin tree. We shall prove that if $V = L$, then if κ is not weakly compact, there is a κ-Souslin tree. As usual when dealing with trees, we are assuming that κ is regular here. In fact, since we know from IV.2.4 that (if $V = L$) there is a κ-Souslin tree whenever κ is a successor cardinal, we need only consider the case where κ is inaccessible. Our construction of a κ-Souslin tree closely resembles that of IV.2.4. Indeed, since $\diamondsuit_\kappa(E)$ is valid for any stationary set $E \subseteq \kappa$ (assuming $V = L$), by examining the proof of IV.2.4 we see that it is sufficient, in order to show that there is a κ-Souslin tree for inaccessible, non-weakly compact κ, to prove the following combinatorial result:

1.2 Theorem. *Assume $V = L$. Let κ be an inaccessible cardinal which is not weakly compact. Then there is a stationary set $E \subseteq \kappa$ and a sequence $(C_\alpha \mid \alpha < \kappa \wedge \lim(\alpha))$ such that*:

(i) $\alpha \in E \to \mathrm{cf}(\alpha) = \omega$;
(ii) *C_α is a club subset of α*;
(iii) *if $\bar\alpha < \alpha$ is a limit point of C_α, then $\bar\alpha \notin E$ and $C_{\bar\alpha} = \bar\alpha \cap C_\alpha$.* □

By means of a slightly different argument, depending on VI.6.1′ rather than VI.6.1, it is possible to prove the following more general form of 1.2.

1.2′ Theorem. *Assume $V = L$. Let κ be an inaccessible cardinal which is not weakly compact. Let $A \subseteq \kappa$ be a stationary set of limit ordinals. Then there is a stationary set $E \subseteq A$ and a sequence $(C_\alpha \mid \alpha < \kappa \wedge \lim(\alpha))$ such that*:

(i) *C_α is a club subset of α*;
(ii) *if $\bar\alpha < \alpha$ is a limit point of C_α, then $\bar\alpha \notin E$ and $C_{\bar\alpha} = \bar\alpha \cap C_\alpha$.* □

(See Exercise 4.)

Before we turn to the proof of 1.2, we obtain some consequences of this result.

1.3 Theorem. *Assume $V = L$. Let κ be an inaccessible cardinal. Then the following are equivalent*:

(i) *κ is weakly compact*;
(ii) *if $E \subseteq \kappa$ is stationary in κ, then for some regular cardinal $\lambda < \kappa$, $E \cap \lambda$ is stationary in λ*;
(iii) *there is no κ-Souslin tree*;
(iv) *for all n, λ such that $1 < n < \omega$ and $1 < \lambda \leqslant \kappa$, the partition property*

$$\kappa \to [\kappa]^n_\lambda$$

(see below) is valid;

(v) *for some λ such that $1 < \lambda \leqslant \kappa$, the partition property*

$$\kappa \rightarrow [\kappa]^2_\lambda$$

(see below) is valid.

Proof. (i) $\rightarrow$ (ii). This is a simple application of Π^1_1-indescribability, and is left to the reader. $V = L$ is not required for this implication.

(ii) $\rightarrow$ (i). This follows from 1.2. If κ is not weakly compact, then the set $E \subseteq \kappa$ of 1.2 is stationary in κ, but if $\lambda < \kappa$ is regular, the set of limit points of C_λ is a club subset of λ which is disjoint from $E \cap \lambda$, so $E \cap \lambda$ is not stationary in λ.

(i) $\rightarrow$ (iii). This is a consequence of V.1.3 (viii) (there are no κ-Aronszajn trees). This part does not require $V = L$.

(iii) $\rightarrow$ (i). If κ is not weakly compact, then, using 1.2 we may repeat the argument of IV.2.4.

(i) $\rightarrow$ (iv). Condition (iv) involves a new partition relation. We write

$$\kappa \rightarrow [\mu]^n_\lambda$$

iff, whenever $f: [\kappa]^n \rightarrow \lambda$, there is a set $X \subseteq \kappa$, $|X| = \mu$, such that $f''[X]^n \neq \lambda$. Provided that $\lambda > 2$, this would seem to be much weaker than the condition

$$\kappa \rightarrow (\kappa)^n_\lambda,$$

which requires that the set X satisfy $|f''[X]^n| = 1$. An indeed, it is known that the two partition relations are not provably equivalent in ZFC. But as the theorem shows, in L these two relations are equivalent.

Since $\kappa \rightarrow (\kappa)^n_\lambda$ is a consequence of weak compactness (V.1.3(ii)), the implication (i) $\rightarrow$ (iv) is provable in ZFC.

(iv) $\rightarrow$ (v). This is trivial, since (v) is a special case of (iv).

(v) $\rightarrow$ (i). It suffices to prove $\neg$ (iii) $\rightarrow \neg$ (v). So let $\mathbf{T} = \langle \kappa, \leqslant_\mathbf{T} \rangle$ be a κ-Souslin tree. By discarding levels of $\mathbf{T}$ we may assume that for every $x \in \mathbf{T}$ the set $S(x)$ of all immediate successors of x in $\mathbf{T}$ has cardinality at least $|x|$. Let f_x be a map from $S(x)$ onto $x (= \{y \mid y < x\})$ for each $x \in \mathbf{T}$. Define $f: [\kappa]^2 \rightarrow \kappa$ as follows. If x, y are incomparable in $\mathbf{T}$, let $f(\{x, y\}) = 0$. Suppose $x, y \in \mathbf{T}$ are such that $x <_\mathbf{T} y$. Let $\bar{y}$ be the unique predecessor of y in $S(x)$. Let $f(\{x, y\}) = f_x(\bar{y})$. We show that f witnesses $\kappa \nrightarrow [\kappa]^2_\kappa$.

Assume that $X \in [\kappa]^\kappa$ and $\alpha < \kappa$ are given. For each $x \in X - (\alpha + 1)$, let $y_x \in S(x)$ be such that $f_x(y_x) = \alpha$. Since $\mathbf{T}$ is κ-Souslin, there must be $x, x' \in X - (\alpha + 1)$ such that $y_x <_\mathbf{T} y_{x'}$. Then by definition, $f(\{x, x'\}) = \alpha$. The proof is complete. $\square$

We turn now to the proof of 1.2. We assume $V = L$ from now on. We fix κ an inaccessible cardinal which is not weakly compact. By 1.1 there is a sentence φ of $\mathscr{L}(\mathring{B}, \mathring{D})$ and a set $B \subseteq \kappa$ such that

(a) $$(\forall D \subseteq \kappa)\,[\langle J_\kappa, \in, B, D\rangle \models \varphi];$$

(b) $$(\forall \alpha < \kappa)\,(\exists D \subseteq \alpha)\,[\langle J_\alpha, \in, B \cap \alpha, D\rangle \models \neg\, \varphi].$$

(We have made some simplifications here. In 1.1 we allowed any finite number of predicate letters in φ. But by using pairing functions we can always replace a finite number of predicates by a single predicate. Also, we have only considered predicates on ordinals in the above. But since there is a uniformly J_α-definable map from α onto J_α for all ordinals closed under the Gödel Pairing Function (see VI.3.19), and since we can always add a conjunct to φ to ensure that α is closed under the Gödel function, this also causes no loss of generality.)

Our proof of 1.2 depends heavily upon the proof of the global $\Box$ principle in VI.6. We begin by recalling the definition of the class E of VI.6.

E is the class of all limit ordinals α such that for some ordinal $\beta > \alpha$:

(i) α is regular over J_β; and
(ii) there is a $p \in J_\beta$ such that whenever $p \in X \prec J_\beta$ and $X \cap \alpha$ is transitive, then $X = J_\beta$.

We define $\bar{E} \subseteq \kappa$ to be the set of all limit cardinals (note: *cardinals*) $\alpha < \kappa$ such that $\alpha \in E$ and for some $\beta > \alpha$ satisfying (i) and (ii) above, it is the case that:

(iii) $B \cap \alpha \in J_\beta$;
(iv) if $D \in \mathscr{P}(\alpha) \cap J_\beta$, then $\langle J_\alpha, \in, B \cap \alpha, D\rangle \models \varphi$.

Since $\bar{E} \subseteq E$, by VI.6.4, $\alpha \in \bar{E}$ implies $\mathrm{cf}(\alpha) = \omega$.

By VI.6.3 we know that $E \cap \kappa$ is stationary in κ. By modifying the proof of VI.6.3 slightly, we prove:

1.4 Lemma. *$\bar{E}$ is stationary in κ.*

Proof. Let $C \subseteq \kappa$ be club. We prove that $\bar{E} \cap C \neq \emptyset$. Since the set of all limit cardinals $\alpha < \kappa$ is club in κ, we may assume that all members of C are limit cardinals. Much as in VI.6.3, let N be the smallest $N \prec J_{\kappa^+}$ such that $(B, C) \in N$ and $N \cap \kappa$ is transitive. Let $\alpha = N \cap \kappa$. Let $\pi\colon J_\beta \cong N$. Then $\pi \restriction \alpha = \mathrm{id} \restriction \alpha$ and $\pi(\alpha) = \kappa$. Moreover, $\pi(B \cap \alpha) = B$ and $\pi(C \cap \alpha) = C$.

Exactly as in VI.6.3, we may prove that α, β are as in conditions (i) and (ii) above, with $p = (B \cap \alpha, C \cap \alpha)$. Moreover, we know that $B \cap \alpha \in J_\beta$, so (iii) holds. Finally, by choice of φ and absoluteness,

$$\models_{J_{\kappa^+}} (\forall D \subseteq \kappa)\,[\langle J_\kappa, \in, B, D\rangle \models \varphi].$$

Applying π^{-1},

$$\models_{J_\beta} (\forall D \subseteq \alpha)\,[\langle J_\alpha, \in, B \cap \alpha, D\rangle \models \varphi].$$

So by absoluteness,

$$(\forall D \in \mathscr{P}(\alpha) \cap J_\beta)\,[\langle J_\alpha, \in, B \cap \alpha, D\rangle \models \varphi].$$

But $\alpha \in C$ (as in VI.6.3), so α is a limit cardinal. Thus $\alpha \in \bar{E}$, and so $\alpha \in \bar{E} \cap C$, and we are done. $\square$

We shall let S, $(C_\alpha \mid \alpha \in S)$ be as in VI.6. So, in particular, $(C_\alpha \mid \alpha \in S)$ is a $\square(E)$-sequence. We define a sequence $(\bar{C}_\alpha \mid \alpha < \kappa \wedge \lim(\alpha))$ to satisfy 1.2 for the stationary set $\bar{E} \subseteq \kappa$. That is, we shall define the sets $\bar{C}_\alpha$ so that $\bar{C}_\alpha$ is a club subset of α and whenever $\bar{\alpha} < \alpha$ is a limit point of $\bar{C}_\alpha$, then $\bar{\alpha} \notin \bar{E}$ and $\bar{C}_{\bar{\alpha}} = \bar{\alpha} \cap \bar{C}_\alpha$. There are several cases to consider. First a trivial case: set $\bar{C}_\omega = \omega$. From now on we shall assume $\alpha > \omega$. S, C_α

Case 1. α is not a limit cardinal.

In this case, let τ be the largest limit cardinal less than α, and set $\bar{C}_\alpha = \alpha - (\tau + 1)$. Since $\bar{E}$ consists only of limit cardinals, no limit point of $\bar{C}_\alpha$ can be in $\bar{E}$. Moreover, if $\bar{\alpha} < \alpha$ is a limit point of $\bar{C}_\alpha$, then $\tau < \bar{\alpha} < \alpha$, so $\bar{\alpha}$ falls under Case 1 as well, and $\bar{C}_{\bar{\alpha}} = \bar{\alpha} - (\tau + 1) = \bar{\alpha} \cap \bar{C}_\alpha$. There is nothing further to check in this case.

In order to describe the next case we require some preliminary notions.

Let U be the set of all limit cardinals $\alpha < \kappa$ such that for some $\beta > \alpha$: U

(i) α is regular over J_β;
(ii) $B \cap \alpha \in J_\beta$;
(iii) there is a $D \in \mathscr{P}(\alpha) \cap J_\beta$ such that $\langle J_\alpha, \in, B \cap \alpha, D\rangle \models \neg\varphi$.

We shall say that any β as above *testifies* that $\alpha \in U$.

1.5 Lemma. $U \cap \bar{E} = \emptyset$.

Proof. Let $\alpha \in \bar{E}$ and let $\beta > \alpha$ satisfy the definition for $\alpha \in \bar{E}$. Thus, in particular,

$$(\forall D \in \mathscr{P}(\alpha) \cap J_\beta)\,[\langle J_\alpha, \in, B \cap \alpha, D\rangle \models \varphi].$$

Now suppose that $\alpha \in U$, and let $\beta' > \alpha$ testify this fact. Thus, in particular,

$$(\exists D \in \mathscr{P}(\alpha) \cap J_{\beta'})\,[J_\alpha, \in, B \cap \alpha, D\rangle \models \neg\varphi].$$

Hence $\beta < \beta'$. But by VI.6.4, α is Σ_1-singular over $J_{\beta+1}$. Hence α is not regular over $J_{\beta'}$. Contradiction, since β' testifies $\alpha \in U$. Thus $\alpha \notin U$, and the lemma is proved. $\square$

Now let W be the set of all $\alpha \in U$ such that if $\beta > \alpha$ is the least to testify $\alpha \in U$, then whenever $p \in J_\beta$ there is an $X \prec J_\beta$ such that $p \in X$ and $X \cap \alpha \in \alpha$. W

1.6 Lemma. *$U - W \subseteq E$. Moreover, if $\alpha \in U - W$ and $\beta > \alpha$ is the least to testify $\alpha \in U$, then β satisfies the definition for $\alpha \in E$.*

Proof. Let α, β be as above. Since $\alpha \notin W$ there is a $p \in J_\beta$ such that whenever $X \prec J_\beta$ is such that $p \in X$ and $X \cap \alpha$ is transitive, then $X \cap \alpha = \alpha$. Let p be in fact the $<_J$-least such element of J_β. Since $\alpha \in U$, let $D \in \mathscr{P}(\alpha) \cap J_\beta$ be the $<_J$-least subset of α such that $\langle J_\alpha, \in, B \cap \alpha, D\rangle \models \neg \varphi$. Let $q = (p, \alpha, B \cap \alpha, D)$. We prove the lemma by showing that if $q \in X \prec J_\beta$ and $X \cap \alpha$ is transitive, then $X = J_\beta$. It suffices to prove this for the smallest X for which $q \in X \prec J_\beta$ and $X \cap \alpha$ is transitive.

Let $\pi: X \cong J_{\bar\beta}$, $\bar\beta \leqslant \beta$. Since $p \in X$, we have $X \cap \alpha = \alpha$, so $\pi \restriction \alpha = \mathrm{id} \restriction \alpha$, $\pi(\alpha) = \alpha$, $\pi(B \cap \alpha) = B \cap \alpha$, $\pi(D) = D$. Moreover, since α is regular over J_β, α is regular over $J_{\bar\beta}$. Thus $\bar\beta$ testifies that $\alpha \in U$. So by the minimality of β, we have $\bar\beta = \beta$.

Suppose now that $\pi(p) \in Y \prec J_\beta$ and $Y \cap \alpha$ is transitive. Let $\bar Y = \pi^{-1}{}'' Y$. Then $\bar Y \cap \alpha = Y \cap \alpha$, so, as $\pi^{-1}: J_\beta \prec J_\beta$, we have $p \in \bar Y \prec J_\beta$ and $\bar Y \cap \alpha$ is transitive. Thus by choice of p, $\bar Y \cap \alpha = \alpha$. Thus $Y \cap \alpha = \alpha$. But Y was arbitrary here. Hence $\pi(p)$ has the same property as p. So as p was chosen $<_J$-minimally and $\pi(p) \leqslant_J p$ (because π is a collapsing map) we have $\pi(p) = p$. It follows at once that $\pi(q) = q$.

Now by choice of X, every element of X is definable from parameters in $\alpha \cup \{q\}$ in J_β. (Because the set of all elements of J_β which are so definable is an elementary submodel of J_β containing q which is transitive on α, and X is the smallest such.) But we have $X \prec J_\beta$, $\pi: X \cong J_\beta$, $\pi \restriction \alpha = \mathrm{id} \restriction \alpha$, $\pi(q) = q$. Hence $\pi = \mathrm{id} \restriction X$. Thus $X = J_\beta$, and we are done. $\square$

Case 2. $\alpha \in W$.

Let $\beta > \alpha$ be the least to testify $\alpha \in U$, and let $D \in \mathscr{P}(\alpha) \cap J_\beta$ be $<_J$-least such that

$$\langle J_\alpha, \in, B \cap \alpha, D\rangle \models \neg \varphi.$$

Since $\alpha \in W$, we can define submodels $X_\nu \prec J_\beta$, $\nu < \theta$ (some θ), as follows:

$$X_0 = \text{the smallest } X \prec J_\beta \text{ such that } (\alpha, B \cap \alpha, D) \in X \text{ and } X \cap \alpha \text{ is transitive;}$$

$$X_{\nu+1} = \text{the smallest } X \prec J_\beta \text{ such that } (\alpha, B \cap \alpha, D, \alpha_\nu) \in X \text{ and } X \cap \alpha \text{ is transitive;}$$

$$X_\lambda = \bigcup_{\nu<\lambda} X_\nu, \quad \text{if } \lim(\lambda) \text{ and } \sup_{\nu<\lambda}\alpha_\nu < \alpha \quad \text{(otherwise undefined),}$$

where for each ν we set

$$\alpha_\nu = X_\nu \cap \alpha.$$

Since $\alpha \in W$, the definition proceeds until a limit ordinal θ is reached for which $\sup_{\nu<\theta}\alpha_\nu = \alpha$. Thus the set

$$\bar C_\alpha = \{\alpha_\nu \mid \nu < \theta\}$$

is a club subset of α.

1.7 Lemma. *Let $\alpha \in W$. Let $\bar{\alpha} < \alpha$ be a limit point of $\bar{C}_\alpha$. Then $\bar{\alpha} \in W$, $\bar{\alpha} \in \bar{E}$, and $\bar{C}_{\bar{\alpha}} = \bar{\alpha} \cap \bar{C}_\alpha$.*

Proof. Let $\bar{\alpha} = \alpha_\lambda$, $\lim(\lambda)$, and let $\pi: J_{\bar{\beta}} \cong X_\lambda$. Thus, $\pi \restriction \bar{\alpha} = \mathrm{id} \restriction \bar{\alpha}$, $\pi(\bar{\alpha}) = \alpha$, $\pi(B \cap \bar{\alpha}) = B \cap \alpha$, $\pi(D \cap \bar{\alpha}) = D \cap \alpha$. So, as $\pi: J_{\bar{\beta}} \prec J_\beta$, it is immediate that $\bar{\beta}$ testifies that $\bar{\alpha} \in U$, and moreover that $\bar{\beta}$ is the least such ordinal. In particular, by 1.5, we have $\bar{\alpha} \notin E$.

Let $\bar{p} \in J_{\bar{\beta}}$. Then $p = \pi(\bar{p}) \in X_\lambda$, so $p \in X_\nu$ for some $\nu < \lambda$. Let $X = \pi^{-1}{}'' X_\nu$. Then $\bar{p} \in X \prec J_{\bar{\beta}}$ and $X \cap \bar{\alpha} = \alpha_\nu < \bar{\alpha}$. Thus $\bar{\alpha} \in W$.

Define $\bar{D}, (\bar{X}_\nu \mid \nu < \bar{\theta}), (\bar{\alpha}_\nu \mid \nu < \bar{\theta})$ from $\bar{\alpha}, \bar{\beta}$ just as $D, (X_\nu \mid \nu < \theta), (\alpha_\nu \mid \nu < \theta)$ were defined from α, β. Thus, in particular, $\bar{C}_{\bar{\alpha}} = \{\bar{\alpha}_\nu \mid \nu < \bar{\theta}\}$. It is easily seen that $\bar{\theta} = \lambda$ and $X_\nu = \pi'' \bar{X}_\nu$ for all $\nu < \bar{\theta}$. Hence $\bar{\alpha}_\nu = \alpha_\nu$ for all $\nu < \bar{\theta}$, and we have $\bar{C}_{\bar{\alpha}} = \bar{\alpha} \cap \bar{C}_\alpha$. The lemma is proved. □

That completes the discussion in Case 2. Notice that this case includes all regular $\alpha > \omega$, since if α is regular, then $\beta = \alpha^+$ testifies $\alpha \in U$, and $\alpha \in W$ holds by regularity. From now on we assume that $\alpha > \omega$ does not fall under either of Cases 1 or 2. Hence α is a singular limit cardinal. We now make use of the sequences C_α, $\alpha \in S$, from VI.6.

Let C'_α be the set of all limit cardinals $\lambda < \alpha$ which are limit points of C_α. Then C'_α is closed in α, and if $\mathrm{cf}(\alpha) > \omega$, C'_α is also unbounded in α. C'_α

Case 3. C'_α is bounded in α.

Then we must have $\mathrm{cf}(\alpha) = \omega$. Let $\bar{C}_\alpha$ be any ω-sequence cofinal in α. Since $\bar{C}_\alpha$ has no limit points, there is nothing to check in this case.

Now, if $\alpha \in E$, then in the definition of C_α in VI.6, α falls under either Case 1 ($\alpha < \omega_1$) or else Case 4 ($n(\alpha) = 1$ and $\mathrm{succ}(\beta(\alpha))$), so C_α is an ω-sequence cofinal in α. Hence $C'_\alpha = \emptyset$ for all $\alpha \in E$. Thus Cases 1 through 3 above include all $\alpha \in E$. So by 1.6, Cases 1 through 3 include all $\alpha \in U - W$. So if we assume from now on that $\alpha > \omega$ does not fall under any of cases 1 through 3, then $\alpha \notin U$ and C'_α is unbounded in α. We shall take $\bar{C}_\alpha$ to be a certain club subset of C_α.

In the definition of C_α in VI.6, in Case 1 ($\alpha < \omega_1$), Case 2 ($\alpha \notin Q$), and Case 3 ($\alpha \in Q$ and $\sup(Q \cap \alpha) < \alpha$), α is not a cardinal, and hence falls under our present Case 1 above. And in Case 4 of VI.6 ($n(\alpha) = 1$ and $\mathrm{succ}(\beta(\alpha))$) we have $C'_\alpha = \emptyset$. All of these possibilities are covered by our present Cases 1 through 3. Since we are assuming now that α does not fall under any of these three cases, it follows that in the definition of C_α in VI.6, α falls under Case 5. In particular, by VI.6.17, if $\bar{\alpha}$ is a limit point of C_α (*a fortiori*: of $\bar{C}_\alpha$, when it has been defined), then $\bar{\alpha} \notin E$, so $\bar{\alpha} \notin \bar{E}$, and hence we need only concern ourselves with the proof that $\bar{C}_{\bar{\alpha}} = \bar{\alpha} \cap \bar{C}_\alpha$.

Let $\beta = \beta(\alpha)$, $n = n(\alpha)$ be as in VI.6. Let $(\alpha_\nu \mid \nu < \theta)$ be the monotone enumeration of C'_α, and set $\beta_\nu = \beta(\alpha_\nu)$. If $\bar{\alpha} = \alpha_\nu$, then our β_ν is just the $\bar{\beta}$ of VI.6 and we have (VI.6.12) $n(\alpha_\nu) = n$; moreover there is a map $\tilde{\pi}: J_{\bar{\beta}} \prec_{n-1} J_\beta$ such that $\tilde{\pi} \restriction \bar{\alpha} = \mathrm{id} \restriction \bar{\alpha}$, and, in case $\bar{\alpha} < \bar{\beta}$, such that $\tilde{\pi}(\bar{\alpha}) \geqslant \alpha$. (See just prior to VI.6.10.) Let π_ν denote this embedding. Thus for each $\nu < \theta$ we have an embedding $\pi_\nu: J_{\beta_\nu} \prec_{n-1} J_\beta$ such that $\pi_\nu \restriction \alpha_\nu = \mathrm{id} \restriction \alpha_\nu$, and in case $\alpha_\nu < \beta_\nu$, $\pi_\nu(\alpha_\nu) \geqslant \alpha$. β, n, θ α_ν, β_ν π_ν

1.8 Lemma. *Let $\bar{\alpha} < \alpha$ be a limit point of C'_α. Then $\bar{\alpha}$ does not fall under either of Cases 1 and 3 above.*

Proof. Since $\bar{\alpha}$ is a limit cardinal, it cannot fall under Case 1. Since $C'_{\bar{\alpha}} = \bar{\alpha} \cap C'_\alpha$, $\bar{\alpha}$ is a limit point of $C'_{\bar{\alpha}}$, so $\bar{\alpha}$ cannot fall under Case 3. $\square$

Case 4. $\alpha = \beta$.

Set $\bar{C}_\alpha = C'_\alpha$ in this case. Suppose that $\bar{\alpha} < \alpha$ is a limit point of $\bar{C}_\alpha$. Note that by 1.8, $\bar{\alpha}$ cannot fall under either of Cases 1 or 3. For some limit ordinal $\lambda < \theta$, we have $\bar{\alpha} = \alpha_\lambda$. If $\alpha_\lambda < \beta_\lambda$, then we would have $\pi_\lambda(\alpha_\lambda) \geqslant \alpha$, which is impossible since $\alpha = \beta \notin J_\beta \supseteq \operatorname{ran}(\pi_\lambda)$. (Since α is a cardinal, $\omega\alpha = \alpha$, of course, so $\omega\beta = \beta$.) Thus $\alpha_\lambda = \beta_\lambda$, i.e. $\beta(\bar{\alpha}) = \bar{\alpha}$. It follows that $\bar{\alpha}$ does not fall under Case 2 above, because any $\bar{\beta} > \bar{\alpha}$ which would testify $\bar{\alpha} \in U$ would need to be less than $\beta(\bar{\alpha})$ in order for $\bar{\alpha}$ to be regular over $J_{\bar{\beta}}$. Hence as $\alpha_\lambda = \beta_\lambda$, $\bar{\alpha}$ falls under Case 4. Thus

$$\bar{C}_{\bar{\alpha}} = C'_{\bar{\alpha}} = \bar{\alpha} \cap C'_\alpha = \bar{\alpha} \cap \bar{C}_\alpha,$$

and we are done in this case.

We assume now that α does not fall under any of Cases 1 through 4. For
$\tau_{\nu\tau}$ $\nu \leqslant \tau < \theta$, set $\pi_{\nu\tau} = \pi_\tau^{-1} \circ \pi_\nu$. Thus $\pi_{\nu\tau}\colon J_{\beta_\nu} \prec_{n-1} J_{\beta_\tau}$ and $\pi_{\nu\tau} \restriction \alpha_\nu = \mathrm{id} \restriction \alpha_\nu$ for all $\nu \leqslant \tau < \theta$. Clearly, $\langle (J_{\beta_\nu})_{\nu<\theta}, (\pi_{\nu\tau})_{\nu\leqslant\tau<\theta}\rangle$ is a directed Σ_{n-1}-elementary system. (See V.5 for the relevant definitions.) What is its direct limit? We claim that it is $\langle J_\beta, (\pi_\nu)_{\nu<\theta}\rangle$. Clearly, what we must prove is that $J_\beta = \bigcup_{\nu<\theta} \operatorname{ran}(\pi_\nu)$. We do this below.

ϱ_ν Let $\varrho_\nu = \varrho(\alpha_\nu) = \varrho_{\beta_\nu}^{n-1}$ $(\nu < \theta)$. By VI.6.7(iii),

(1) $$\bigcup_{\nu<\theta} \pi_\nu'' J_{\varrho_\nu} = J_\varrho.$$

Suppose now that $\nu < \theta$, and let $\bar{\beta} = \beta_\nu$, $\pi = \pi_\nu$. By VI.5.6 we know that

$$(1)' \quad \begin{array}{l} (\pi \restriction J_{\varrho_{\bar{\beta}}^{n-2}})\colon \langle J_{\varrho_{\bar{\beta}}^{n-2}}, A_{\bar{\beta}}^{n-2}\rangle \prec_1 \langle J_{\varrho_\beta^{n-2}}, A_\beta^{n-2}\rangle, \\ \pi(p_{\bar{\beta}}^{n-1}) = p_\beta^{n-1}; \end{array}$$

. . . .

. . . .

$$(n-1)' \quad \begin{array}{l} (\pi \restriction J_{\varrho_{\bar{\beta}}^1})\colon \langle J_{\varrho_{\bar{\beta}}^1}, A_{\bar{\beta}}^1\rangle \prec_{n-2} \langle J_{\varrho_\beta^1}, A_\beta^1\rangle, \\ \pi(p_{\bar{\beta}}^2) = p_\beta^2; \end{array}$$

$$(n)' \quad \begin{array}{l} \pi\colon J_{\bar{\beta}} \prec_{n-1} J_\beta, \\ \pi(p_{\bar{\beta}}^1) = p_\beta^1. \end{array}$$

Now, by definition of p_β^{n-1}, every element of $J_{\varrho_\beta^{n-2}}$ is Σ_1-definable from elements of $J_\varrho \cup \{p_\beta^{n-1}\}$ in $\langle J_{\varrho_\beta^{n-2}}, A_\beta^{n-2}\rangle$. So by (1),

(2) $$\bigcup_{\nu<\theta} \pi_\nu'' J_{\varrho_{\beta_\nu}^{n-2}} = J_{\varrho_\beta^{n-2}}.$$

Repeating the same argument, using (2) in place of (1) and (2)′ in place of (1)′ now yields

(3) $$\bigcup_{\nu<\theta} \pi_\nu'' J_{\varrho_{\beta_\nu}^{n-3}} = J_{\varrho_\beta^{n-3}}.$$

Continuing in this fashion, we obtain, eventually,

$$(n-1)\ \bigcup_{\nu<\theta} \pi_\nu{}''J_{\varrho^1_{\beta_\nu}} = J_{\varrho^1_\beta}.$$

$$(n)\ \bigcup_{\nu<\theta} \pi_\nu{}''J_{\beta_\nu} = J_\beta.$$

This last equality is the one we require.

If $\lambda < \theta$ is a limit ordinal now and we set $\bar{\alpha} = \alpha_\lambda$, $\bar{\beta} = \beta_\lambda$, and if we define $\bar{\beta}_\nu, \bar{\alpha}_\nu, \bar{\pi}_\nu, \bar{\pi}_{\nu\tau}$, for $\nu \leqslant \tau < \bar{\theta}$, from $\bar{\alpha}, \bar{\beta}$ as $\beta_\nu, \alpha_\nu, \pi_\nu, \pi_{\nu\tau}$, for $\nu \leqslant \tau < \theta$, were defined from α, β, then (clearly) $\bar{\theta} = \lambda$ and for $\nu \leqslant \tau < \bar{\theta}$, $\bar{\beta}_\nu = \beta_\nu$, $\bar{\alpha}_n = \alpha_\nu$, $\bar{\pi}_\nu = \pi_{\nu\lambda}$, $\bar{\pi}_{\nu\tau} = \pi_{\nu\tau}$. We utilise these observations below. $\lambda, \bar{\alpha}, \bar{\beta}$ $\bar{\beta}_\nu, \bar{\alpha}_\nu, \bar{\pi}_\nu,$ $\bar{\pi}_{\nu\tau}, \bar{\theta}$

Case 5. $B \cap \alpha \in J_\beta$.

Since $J_\beta = \bigcup_{\nu<\theta} \pi_\nu{}''J_{\beta_\nu}$, we may pick ν_α to be the least $\nu < \theta$ such that α, $B \cap \alpha \in \pi_\nu{}''J_{\beta_\nu}$. Set

$$\bar{C}_\alpha = \{\alpha_\nu \mid \nu_\alpha \leqslant \nu < \theta\}.$$

Let $\bar{\alpha} < \alpha$ be a limit point of $\bar{C}_\alpha$. Thus $\bar{\alpha} = \alpha_\lambda$ for some limit ordinal λ, $\nu_\alpha < \lambda < \theta$. By 1.8, $\bar{\alpha}$ cannot fall under either of Cases 1 and 3. Moreover, $\bar{\alpha}$ cannot fall under Case 4, since $\alpha \in \pi_\lambda{}''J_{\beta_\nu}$, which implies that $\alpha_\lambda \in J_{\beta_\nu}$ and $\pi_\lambda(\alpha_\lambda) = \alpha$. (Recall that $\pi_\lambda \restriction \alpha_\lambda = \mathrm{id} \restriction \alpha_\lambda$.) We show that $\bar{\alpha}$ also cannot fall under Case 2. Indeed, not only do we have $\bar{\alpha} \notin W$, but the stronger condition $\bar{\alpha} \notin U$. For suppose that $\bar{\beta} > \bar{\alpha}$ were to testify that $\bar{\alpha} \in U$. Since $\bar{\alpha}$ must be regular over $J_{\bar{\beta}}$, we have $\bar{\beta} < \beta_\lambda$. Now, $B \cap \bar{\alpha} \in J_{\bar{\beta}}$, so as $\lambda > \nu_\alpha$, we must have $\pi_\lambda(B \cap \bar{\alpha}) = B \cap \alpha$. Thus $B \cap \alpha \in J_{\pi_\lambda(\bar{\beta})}$. Again, we can pick $\bar{D} \in \mathscr{P}(\bar{\alpha}) \cap J_{\bar{\beta}}$ so that $\langle J_{\bar{\alpha}}, \in, B \cap \bar{\alpha}, \bar{D}\rangle \models \neg\varphi$. Let $D = \pi_\lambda(\bar{D})$. Since $\pi_\lambda\colon J_{\beta_\lambda} \prec_0 J_\beta$, we have $D \in \mathscr{P}(\alpha) \cap J_{\pi_\lambda(\bar{\beta})}$ and $\langle J_\alpha, \in, B \cap \alpha, D\rangle \models \neg\varphi$. Thus as $\pi_\lambda(\bar{\beta}) < \beta$, $\pi_\lambda(\bar{\beta})$ testifies that $\alpha \in U$. But α falls under Case 5, so $\alpha \notin U$. Contradiction! Hence $\bar{\alpha}$ does not fall under any of Cases 1 through 4. But $\lambda > \nu_\alpha$, so $B \cap \bar{\alpha} \in J_{\beta_\lambda} = J_{\beta(\bar{\alpha})}$. Hence $\bar{\alpha}$ falls under Case 5. But it is clear from the remarks we made just prior to Case 5, together with the facts that $\pi_\lambda(\bar{\alpha}) = \alpha$ and $\pi_\lambda(B \cap \bar{\alpha}) = B \cap \alpha$ (which are valid because $\pi_\lambda \restriction \bar{\alpha} = \mathrm{id} \restriction \bar{\alpha}$ and α, $B \cap \alpha \in \pi_\lambda{}''J_{\beta_\lambda}$), that $\nu_{\bar{\alpha}} = \nu_\alpha$. Hence

$$\bar{C}_{\bar{\alpha}} = \{\alpha_\nu \mid \nu_{\bar{\alpha}} \leqslant \nu < \bar{\theta}\} = \{\alpha_\nu \mid \nu_\alpha \leqslant \nu < \lambda\} = \bar{\alpha} \cap \bar{C}_\alpha.$$

That completes the proof in this case.

Case 6. Otherwise.

In particular, in this case we have $B \cap \alpha \notin J_\beta$. Suppose that $\nu < \theta$ were such that $B \cap \alpha_\nu \in J_{\beta_\nu}$. Then there must be a $\tau > \nu$ such that $\pi_{\nu\tau}(B \cap \alpha_\nu) \neq B \cap \alpha_\tau$, since otherwise we would have

$$B \cap \alpha = \bigcup_{\nu<\tau<\theta} \pi_{\nu\tau}(B \cap \alpha_\nu) = \pi_\nu(B \cap \alpha_\nu) \in J_\beta.$$

So we can define a normal sequence $(\nu(\iota) \mid \iota < \bar{\theta})$, for some $\bar{\theta} \leqslant \theta$, as follows.

$$\begin{aligned} \nu(0) &= 0; \\ \nu(\iota + 1) &= \text{the least } \nu > \nu(\iota) \text{ such that} \\ &\qquad B \cap \alpha_{\nu(\iota)} \in J_{\beta_{\nu(\iota)}} \to \pi_{\nu(\iota),\nu}(B \cap \alpha_{\nu(\iota)}) \neq B \cap \alpha_\nu; \\ \nu(\lambda) &= \sup_{\iota < \lambda} \nu(\iota), \qquad \text{if this is less than } \theta \text{ (otherwise undefined),} \\ &\qquad\qquad\qquad\quad \text{for } \lim(\lambda). \end{aligned}$$

$\bar{\theta}$ The definition proceeds until an ordinal $\bar{\theta}$ is reached for which $\sup_{\iota < \bar{\theta}} \nu(\iota) = \theta$. (Clearly, $\lim(\bar{\theta})$.) Set

$$\bar{C}_\alpha = \{\alpha_{\nu(\iota)} \mid \iota < \bar{\theta}\}.$$

Let $\bar{\alpha} < \alpha$ be a limit point of $\bar{C}_\alpha$. Thus $\bar{\alpha} = \alpha_{\nu(\lambda)}$ for some limit ordinal $\lambda < \bar{\theta}$. As in Case 5, 1.8 implies that $\bar{\alpha}$ cannot fall under Cases 1 and 3, and since $\bar{\alpha} = \alpha_{\nu(\lambda)} < \beta_{\nu(\lambda)} = \beta(\bar{\alpha})$, $\bar{\alpha}$ cannot fall under Case 4. We show that $\bar{\alpha}$ cannot fall under Case 2. In fact, as in Case 5 we show that $\bar{\alpha} \notin U$. Suppose, on the contrary, that $\bar{\alpha} \in U$. Thus, in particular, $B \cap \bar{\alpha} \in J_{\beta_{\nu(\lambda)}}$. (Clearly, the least $\bar{\beta} > \bar{\alpha}$ which testifies $\bar{\alpha} \in U$ has to be less than $\beta(\bar{\alpha}) = \beta_{\nu(\lambda)}$.) But (as we proved earlier for β)

$$J_{\beta_{\nu(\lambda)}} = \bigcup_{\iota < \lambda} \pi_{\nu(\iota),\nu(\lambda)}{}'' J_{\beta_{\nu(\iota)}},$$

so for some $\iota < \lambda$, $B \cap \bar{\alpha} \in \pi_{\nu(\iota),\nu(\lambda)}{}'' J_{\beta_{\nu(\iota)}}$. Thus $B \cap \bar{\alpha} = \pi_{\nu(\iota),\nu(\lambda)}(B \cap \alpha_{\nu(\iota)})$. But this implies that $\pi_{\nu(\iota),\nu(\iota+1)}(B \cap \alpha_{\nu(\iota)}) = B \cap \alpha_{\nu(\iota+1)}$, contrary to the choice of $\nu(\iota + 1)$. Hence $\bar{\alpha}$ does not fall under any of Cases 1 through 4. But the above argument shows that $\bar{\alpha}$ does not fall under Case 5 either. Thus $\bar{\alpha}$ falls under Case 6, and we have

$$\bar{C}_{\bar{\alpha}} = \{\alpha_{\nu(\iota)} \mid \iota < \lambda\} = \bar{\alpha} \cap \bar{C}_\alpha.$$

The proof of 1.2 is complete.

2. *Ineffable Cardinals and κ-Kurepa Trees*

Ineffability is a large cardinal property which strengthens weak compactness. By definition, an uncountable, regular cardinal κ is said to be weakly compact iff, whenever $f\colon [\kappa]^2 \to 2$, there is an *unbounded* set $X \subseteq \kappa$ such that $|f''[X]^2| = 1$. We say that an uncountable, regular cardinal κ is *ineffable* iff, whenever $f\colon [\kappa]^2 \to 2$, there is a *stationary* set $X \subseteq \kappa$ such that $|f''[X]^2| = 1$.

Clearly, all ineffable cardinals are weakly compact. The converse is not true, and indeed, as we shall show presently, ineffability is a much stronger notion than weak compactness. It should be said that the notion of ineffability is a rather specialised one, not covered in many of the standard texts dealing with large cardinals. (For instance, it is not covered in *Drake* (1974) or *Jech* (1978).) Consequently we give here a few of the basic results concerning ineffable cardinals.

2.1 Theorem. *Let $\kappa > \omega$ be regular. Then κ is ineffable iff, whenever $(A_\alpha \mid \alpha < \kappa)$ is such that $A_\alpha \subseteq \alpha$ for all $\alpha < \kappa$, there is a set $A \subseteq \kappa$ such that the set $\{\alpha \in \kappa \mid A \cap \alpha\}$ is stationary in κ.*

Proof. ($\rightarrow$) Let $(A_\alpha \mid \alpha < \kappa)$ be given, $A_\alpha \subseteq \alpha$ for all $\alpha < \kappa$. For each $\alpha < \kappa$, let $f_\alpha : \alpha \rightarrow 2$ be the characteristic function of A_α. If we can find a function $f: \kappa \rightarrow 2$ such that $\{\alpha \in \kappa \mid f \upharpoonright \alpha = f_\alpha\}$ is stationary, then $A = f^{-1}{}''\{1\}$ will be as required.

Let $\dashv$ be the lexicographic ordering on the set $\{f_\alpha \mid \alpha < \kappa\}$. Define a function $h: [\kappa]^2 \rightarrow 2$ by

$$h\{\{\alpha, \beta\}\} = 0 \quad \text{iff } f_\alpha \dashv f_\beta \quad (\alpha < \beta < \kappa).$$

By assumption there is a stationary set $X \subseteq \kappa$ such that $|h''[X]^2| = 1$. Suppose, for definiteness, that $h''[X]^2 = \{0\}$. (The other case is similar.) Thus

$$\alpha, \beta \in X \quad \text{and} \quad \alpha < \beta \quad \text{implies } f_\alpha \dashv f_\beta.$$

For each $\nu < \kappa$, let α_ν be the least member of X such that $\alpha_\nu \geqslant \nu$ and

$$(\forall \beta \in X)(\beta \geqslant \alpha_\nu \rightarrow f_\beta \upharpoonright \nu = f_{\alpha_\nu} \upharpoonright \nu).$$

By choice of X, this definition is always possible. Let

$$C = \{\gamma \in \kappa \mid (\forall \nu)(\nu < \gamma \rightarrow \alpha_\nu < \gamma\}.$$

Clearly, C is a club subset of κ. Thus the set

$$Y = X \cap C \cap \{\nu \in \kappa \mid \lim(\nu)\}$$

is stationary in κ. Now, if $\lim(\nu)$, α_ν is the first member of X not less than $\sup_{\eta < \nu} \alpha_\eta$. So, if $\nu \in Y$, we will have $\alpha_\nu = \nu$. Hence

$$\alpha \in Y \quad \text{implies } (\forall \beta \in Y)(\beta \geqslant \alpha \rightarrow f_\beta \upharpoonright \alpha = f_\alpha).$$

Define $f: \kappa \rightarrow 2$ by

$$f = \bigcup_{\alpha \in Y} f_\alpha.$$

Since $Y \subseteq \{\alpha \in \kappa \mid f \upharpoonright \alpha = f_\alpha\}$, we are done.

($\leftarrow$) Let $f: [\kappa]^2 \rightarrow 2$ be given. For $\alpha < \kappa$, define $f_\alpha : \alpha \rightarrow 2$ by

$$f_\alpha(\nu) = f(\{\nu, \alpha\}) \quad (\nu < \alpha).$$

By assumption there is a function $\bar{f}: \kappa \to 2$ such that the set

$$X = \{\alpha \in \kappa \mid f_\alpha = \bar{f} \upharpoonright \alpha\}$$

is stationary in κ. (Consider the sets $A_\alpha \subseteq \alpha$ for which f_α is the characteristic function.) Now, $\bar{f}$ is regressive on $X - 2$, so by Fodor's Theorem (III.3.1) there is a stationary set $Y \subseteq X$ and an integer $i \in 2$ such that

$$\alpha \in Y \to \bar{f}(\alpha) = i.$$

For $\nu, \alpha \in Y$, $\nu < \alpha$, we have

$$f(\{\nu, \alpha\}) = f_\alpha(\nu) = (\bar{f} \upharpoonright \alpha)(\nu) = \bar{f}(\nu) = i.$$

Hence $|f''[Y]^2| = 1$. $\square$

Strengthening the notion of Π_1^1-indescribability, which we have already noted as being equivalent to weak compactness (V.1.3), is that of Π_2^1-indescribability. An inaccessible cardinal κ is said to be Π_2^1-*indescribable* if, whenever $\varphi(\mathring{X}, \mathring{Y}, \mathring{U}_1, \ldots, \mathring{U}_n)$ is a sentence of $\mathscr{L}(X, Y, U_1, \ldots, U_n)$ and $U_1, \ldots, U_n \subseteq V_\kappa$ are such that

$$(\forall X \subseteq V_\kappa)(\exists Y \subseteq V_\kappa)[\langle V_\kappa, \in, X, Y, U_1, \ldots, U_n \rangle \models \varphi(\mathring{X}, \mathring{Y}, \mathring{U}_1, \ldots, \mathring{U}_n)],$$

then for some $\alpha < \kappa$,

$$(\forall X \subseteq V_\alpha)(\exists Y \subseteq V_\alpha)[\langle V_\alpha, \in, X, Y, U_1 \cap V_\alpha, \ldots, U_n \cap V_\alpha \rangle \models \varphi(\mathring{X}, \mathring{Y}, \mathring{U}_1, \ldots, \mathring{U}_n)].$$

Clearly, if κ is Π_2^1-indescribable, it must be Π_1^1-indescribable, i.e. weakly compact. The converse is not true. Indeed, we have:

2.2 Theorem. *If κ is Π_2^1-indescribable, then the set*

$$\{\lambda \in \kappa \mid \lambda \text{ is weakly compact}\}$$

is unbounded in κ.

Proof. (Sketch) There is a sentence $\varphi(\mathring{X}, \mathring{Y})$ of $\mathscr{L}(X, Y)$ such that an ordinal α is weakly compact iff

$$(\forall X \subseteq V_\alpha)(\exists Y \subseteq V_\alpha)[\langle V_\alpha, \in, X, Y \rangle \models \varphi(\mathring{X}, \mathring{Y})].$$

(Simply consider the defining property $\alpha \to (\alpha)_2^2$.) Given $\gamma < \kappa$ now, apply Π_2^1-indescribability for the structure $\langle V_\kappa, \in, X, Y, \{\gamma\}\rangle$ and the sentence $\varphi(\mathring{X}, \mathring{Y}) \wedge \exists x(x \in \mathring{U})$. $\square$

2.3 Theorem. *If κ is ineffable, then κ is Π^1_2-indescribable.*

Proof. Let $\varphi(\mathring{X}, \mathring{Y}, \mathring{U}_1, \ldots, \mathring{U}_n)$ be a sentence of $\mathscr{L}(X, Y, U_1, \ldots, U_n)$, and let $U_1, \ldots, U_n \subseteq V$ be such that

$$(\forall X \subseteq V_\kappa)(\exists Y \subseteq V_\kappa)[\langle V_\kappa, \in, X, Y, U_1, \ldots, U_n\rangle \models \varphi].$$

Let

$$C = \{\lambda \in \kappa \mid |V_\lambda| = \lambda\}.$$

Clearly, C is a club subset of κ. We claim that for some $\lambda \in C$,

$$(\forall X \subseteq V_\lambda)(\exists Y \subseteq V_\lambda)[\langle V_\lambda, \in, X, Y, U_1 \cap V_\lambda, \ldots, U_n \cap V_\lambda\rangle \models \varphi],$$

thereby proving the theorem.

Suppose not. Then for each $\lambda \in C$ we can pick a set $X_\lambda \subseteq V_\lambda$ such that for all $Y \subseteq V_\lambda$,

$$\langle V_\lambda, \in, X, Y, U_1 \cap V_\lambda, \ldots, U_n \cap V_\lambda\rangle \models \neg\varphi.$$

Since $|V_\kappa| = \kappa$ and $|V_\lambda| = \lambda$ for all $\lambda \in C$, we may apply ineffability using 2.1 to conclude that there is a set $X \subseteq V_\kappa$ such that the set

$$A = \{\lambda \in C \mid X_\lambda = X \cap V_\lambda\}$$

is stationary in κ.

By assumption, we can find a set $Y \subseteq V_\kappa$ such that

$$\langle V_\kappa, \in, X, Y, U_1, \ldots, U_n\rangle \models \varphi.$$

Let

$$E = \{\lambda \in \kappa \mid \langle V_\lambda, \in, X \cap V_\lambda, Y \cap V_\lambda, U_1 \cap V_\lambda, \ldots, U_n \cap V_\lambda\rangle \prec \langle V_\kappa, \in, X, Y, U_1, \ldots, U_n\rangle\}.$$

Clearly, E is club in κ. Hence we can find a $\lambda \in E \cap A$. But then we have

$$\langle V_\lambda, \in, X_\lambda, Y \cap V_\lambda, U_1 \cap V_\lambda, \ldots, U_n \cap V_\lambda\rangle \models \varphi,$$

contrary to the choice of X_λ.

The theorem is proved. $\square$

We shall show presently that if $V = L$, then ineffability is closely related to the Kurepa Hypothesis. Indeed, as we shall see, it plays the same role for Kurepa trees as does weak compactness for Souslin trees. But first it is of interest (though of no use to us here) to present the following result, which, it should be emphasised, is a theorem of ZFC.

2.4 Theorem. *If κ is an ineffable cardinal, then $\diamondsuit_\kappa$ holds.*

Proof. For each $\alpha < \kappa$, let (S_α, C_α) be, if possible, any pair of subsets of α such that C_α is club in α and $(\forall \gamma \in C_\alpha)(\gamma \cap S_\alpha \neq S_\gamma)$. In case no such pair exists, define

$S_\alpha = C_\alpha = \emptyset$. This defines $((S_\alpha, C_\alpha) \mid \alpha < \kappa)$ by recursion. We show that $(S_\alpha \mid \alpha < \kappa)$ is a $\diamondsuit_\kappa$-sequence.

Let $S \subseteq \kappa$. Suppose that the set

$$\{\alpha \in \kappa \mid S \cap \alpha = S_\alpha\}$$

were not stationary in κ. Then we could find a club set $C \subseteq \kappa$ such that

$$(\forall \alpha \in C)(S \cap \alpha \neq S_\alpha).$$

By the ineffability of κ and 2.1, together with some simple coding device, we can find sets $\bar{S}, \bar{C} \subseteq \kappa$ such that the set

$$A = \{\alpha \in \kappa \mid \bar{S} \cap \alpha = S_\alpha \wedge \bar{C} \cap \alpha = C_\alpha\}$$

is stationary in κ. Pick $\alpha, \beta \in A \cap C$, $\alpha < \beta$. Then

$$(*) \qquad S_\beta \cap \alpha = \bar{S} \cap \alpha = S_\alpha, \quad \text{and}$$

$$(**) \qquad C_\beta \cap \alpha = \bar{C} \cap \alpha = C_\alpha.$$

Since C_α is club in α is club in β, using $(**)$ we have

$$\alpha = \sup(C_\alpha) = \sup(\alpha \cap C_\beta) \in C_\beta.$$

Thus by choice of (S_β, C_β) we must have $\alpha \cap S_\beta \neq S_\alpha$. But this contradicts $(*)$. Thus the set $\{\alpha \in \kappa \mid S \cap \alpha = S_\alpha\}$ is stationary in κ, and the theorem is proved. $\square$

We turn now to the study of ineffable cardinals in L. As was the case with weakly compact cardinals (V.1.5), we can prove that ineffability relativises to L.

2.5 Lemma. *If κ is ineffable, then $[\kappa$ is ineffable$]^L$.*

Proof. We make use of 2.1. In L, let $(A_\alpha \mid \alpha < \kappa)$ be such that $A_\alpha \subseteq \alpha$ for all $\alpha < \kappa$. By absoluteness, this set is such a sequence in V, so by ineffability using 2.1, there is a set $A \subseteq \kappa$ such that

$$X = \{\alpha \in \kappa \mid A_\alpha = A \cap \alpha\}$$

is stationary in κ. Now, for each $\alpha \in X$, $A \cap \alpha = A_\alpha \in L$. Hence as X is cofinal in κ, $A \cap \gamma \in L$ for all $\gamma < \kappa$. But κ is weakly compact, so by V.1.4 this implies that $A \in L$. Hence $X \in L$ as well. But, by absoluteness, in L, X is stationary and $X = \{\alpha \in \kappa \mid A_\alpha = A \cap \alpha\}$. Thus by 2.1 applied inside L, we conclude that κ is ineffable in the sense of L. $\square$

We shall prove that if $V = L$, an inaccessible cardinal κ will be ineffable iff there is no κ-Kurepa tree. But what exactly do we mean by a "κ-Kurepa tree" for inaccessible κ? For if κ is inaccessible, the κ-tree consisting of all binary sequences of lengths less than κ, ordered by inclusion, has 2^κ many κ-branches, and we surely do not want such a trivial example to be a "Kurepa tree". The only reason this tree is a κ-tree at all is because the inaccessibility of κ keeps the cardinality of each level less than κ. A more interesting notion is supplied by the following considerations.

A κ-tree $\mathbf{T}$ is said to be *slim* if $|T_\alpha| \leqslant |\alpha|$ for all infinite α. By a *κ-Kurepa tree* we shall mean a slim κ-tree with at least κ^+ many κ-branches. In the case where κ is a successor cardinal, this is at variance with the definition of IV.1, but the distinction is clearly unimportant in this case, as it is the *cofinal* behaviour of trees that is of interest to us. Let us agree to adopt the new definition for all κ from now on. Likewise for the definition of a "κ-Kurepa family", given below.

The restriction that our trees be slim could also be applied to the notion of a κ-Souslin tree. In fact it is easily seen that the κ-Souslin trees constructed (in L) in 1.3 and in IV.2.4 are slim. Consequently there would have been no loss if we had required *all* of our κ-trees to be slim.

By a *κ-Kurepa family* we shall mean a family, $\mathscr{F}$, of subsets of κ such that $|\mathscr{F}| \geqslant \kappa^+$ but for all infinite $\alpha < \kappa$, $|\{x \cap \alpha \mid x \in \mathscr{F}\}| \leqslant |\alpha|$. The same argument as in III.2.1 shows that the existence of a (slim) κ-Kurepa tree is equivalent to the existence of a κ-Kurepa family.

The following result is a theorem of ZFC.

2.6 Theorem. *If κ is ineffable, then there is no κ-Kurepa tree.*

Proof. Let $\mathscr{F} \subseteq \mathscr{P}(\kappa)$ be such that $|\{x \cap \alpha \mid x \in \mathscr{F}\}| \leqslant |\alpha|$ for all infinite $\alpha < \kappa$. Assuming that κ is ineffable, we show that $|\mathscr{F}| \leqslant \kappa$, so that $\mathscr{F}$ cannot be a κ-Kurepa family.

For each $\alpha \geqslant \omega$, let $(f_\nu^\alpha \mid \nu < \alpha)$ enumerate $\{x \cap \alpha \mid x \in \mathscr{F}\}$. Set

$$R_\alpha = \{(\tau, \nu) \mid \tau \in f_\nu^\alpha\}.$$

Thus $R_\alpha \subseteq \alpha \times \alpha$. By ineffability (using 2.1 and a simple coding device) there is a set $R \subseteq \kappa \times \kappa$ such that the set

$$E = \{\alpha \in \kappa \mid R \cap (\alpha \times \alpha) = R_\alpha\}$$

is stationary in κ. For each $\nu < \kappa$, set $f_\nu = R''\{\nu\}$. We shall prove that $\mathscr{F} \subseteq \{f_\nu \mid \nu < \kappa\}$.

Let $f \in \mathscr{F}$, and suppose that $f \neq f_\nu$ for all $\nu < \kappa$. Since κ is regular we can find a club set $C \subseteq \kappa$ such that

$$\alpha \in C \rightarrow (\forall \nu < \alpha)(f \cap \alpha \neq f_\nu \cap \alpha).$$

Pick $\alpha \in C \cap E$. Then for $\nu < \alpha$,

$$f \cap \alpha \neq f_\nu \cap \alpha = \alpha \cap R''\{\nu\} = \alpha \cap R''_\alpha\{\nu\} = \{\tau \mid \tau \in f_\nu^\alpha\} = f_\nu^\alpha.$$

Hence
$$f \cap \alpha \notin \{f_\nu^\alpha \mid \nu < \alpha\} = \{x \cap \alpha \mid x \in \mathscr{F}\}.$$

Since $f \in \mathscr{F}$, this is absurd. This contradiction proves our result. $\square$

Using $V = L$, we now prove the converse to the above theorem.

2.7 Theorem. *Assume $V = L$. Let κ be an uncountable regular cardinal which is not ineffable. Then there is a κ-Kurepa tree.*

Proof. The proof is very similar to that of IV.3.3 (the construction of a κ^+-Kurepa tree). As there, it is more convenient to construct a κ-Kurepa family.

By 2.1, let $(A_\alpha \mid \alpha < \kappa)$ be the $<_J$-least sequence such that $A_\alpha \subseteq \alpha$ for all α, and whenever $A \subseteq \kappa$, the set $\{\alpha \in \kappa \mid A \cap \alpha = A_\alpha\}$ is not stationary in κ. Notice that $(A_\alpha \mid \alpha < \kappa)$ is a definable element of J_{κ^+}.

For each $\alpha < \kappa$, let M_α be the smallest $M \prec J_\kappa$ such that $(\alpha + 1) \cup \{(A_\nu \mid \nu \leqslant \alpha)\} \subseteq M$, and let $\sigma_\alpha \colon M_\alpha \cong J_{f(\alpha)}$. Notice that for infinite α, $|f(\alpha)| = |\alpha|$. It is clear that the function $f \colon \kappa \to \kappa$ so defined is a definable element of J_{κ^+}.

Let
$$\mathscr{F} = \{x \subseteq \kappa \mid (\forall \alpha < \kappa)(x \cap \alpha \in J_{f(\alpha)})\}.$$

If we can show that $|\mathscr{F}| \geqslant \kappa^+$, then $\mathscr{F}$ will be a κ-Kurepa family, and we shall be done. We assume $|\mathscr{F}| \leqslant \kappa$ and derive a contradiction.

Let $X = (x_\nu \mid \nu < \kappa)$ be the $<_J$-least enumeration of $\mathscr{F}$. Notice that both $\mathscr{F}$ are X are definable elements of J_{κ^+}.

By recursion, define submodels $N_\nu \prec J_{\kappa^+}$, for $\nu < \kappa$, as follows.

$$N_0 = \text{the smallest } N \prec J_{\kappa^+} \text{ such that } N \cap \kappa \in \kappa;$$
$$N_{\nu+1} = \text{the smallest } N \prec J_{\kappa^+} \text{ such that } N_\nu \cup \{N_\nu\} \subseteq N \text{ and } N \cap \kappa \in \kappa;$$
$$N_\delta = \bigcup_{\nu < \delta} N_\nu, \quad \text{if } \lim(\delta).$$

Set
$$\alpha_\nu = N_\nu \cap \kappa.$$

Then $(\alpha_\nu \mid \nu < \kappa)$ is a normal sequence in κ. Set

$$x = \{\alpha_\nu \mid \nu < \kappa \wedge \alpha_\nu \notin x_\nu\}.$$

Then $x \subseteq \kappa$ and $x \neq x_\nu$ for all $\nu < \kappa$, so $x \notin \mathscr{F}$. We obtain our contradiction by showing that $x \cap \alpha \in J_{f(\alpha)}$ for all $\alpha < \kappa$. We argue much as in IV.3.3.

Let $\alpha < \kappa$ be given. Let η be the largest limit ordinal such that $\alpha_\eta \leqslant \alpha$. Since $x \cap \alpha$ differs from $x \cap \alpha_\eta$ by at most a finite set, in order to show that $x \cap \alpha \in J_{f(\alpha)}$ it suffices to show that $x \cap \alpha_\eta \in J_{f(\alpha_\eta)}$. (The function f is clearly non-decreasing.)

Since $x \cap \alpha_\eta = \{\alpha_\nu \mid \nu < \eta \wedge \alpha_\nu \notin x_\nu\}$, it is in fact enough to show that $(\alpha_\nu \mid \nu < \eta)$ and $(x_\nu \cap \alpha_\eta \mid \nu < \eta)$ are elements of $J_{f(\alpha_\eta)}$.

Let $\pi \colon N_\eta \cong J_\beta$. Then $\pi \restriction \alpha_\eta = \mathrm{id} \restriction \alpha_\eta$, $\pi(\kappa) = \alpha_\eta$, and $\pi(x) = (x_\nu \cap \alpha_\eta \mid \nu < \alpha_\eta)$. In particular, $(x_\nu \cap \alpha_\eta \mid \nu < \eta) \in J_\beta$. And by an argument just as in IV.3.3, we see that $(\alpha_\nu \mid \nu < \eta)$ is ZF^--definable from J_β. It thus suffices to show that $\beta < f(\alpha_\eta)$.

Suppose, on the contrary, that $f(\alpha_\eta) \leqslant \beta$. Since $\alpha_\eta + 1 \subseteq M_{\alpha_\eta}$, we have $\sigma_{\alpha_\eta}((A_\nu \mid \nu \leqslant \alpha_\eta)) = (A_\nu \mid \nu \leqslant \alpha_\eta)$, so $(A_\nu \mid \nu \leqslant \alpha_\eta) \in J_{f(\alpha_\eta)} \subseteq J_\beta$. Let

$$E = \{\gamma \in \alpha_\eta \mid A_\gamma = \gamma \cap A_{\alpha_\eta}\}.$$

Then $E \in J_\beta$. Suppose that

$$\models_{J_\beta} \text{“}E \text{ is stationary in } \alpha_\eta\text{”}.$$

Setting $\tilde{E} = \pi^{-1}(E)$ and applying $\pi^{-1}\colon J_\beta \prec J_{\kappa^+}$, we get

$$\models_{J_{\kappa^+}} \text{“}\tilde{E} \text{ is stationary in } \kappa\text{”}.$$

Hence $\tilde{E}$ really is stationary in κ (by absoluteness). But $(A_\nu \mid \nu < \kappa) \in N_\eta$ (by definability), so $\pi^{-1}((A_\nu \mid \nu < \alpha_\eta)) = (A_\nu \mid \nu < \kappa)$. Hence, setting $\tilde{A} = \pi^{-1}(A_{\alpha_\eta})$, we have $A \subseteq \kappa$ and

$$\models_{J_{\kappa^+}} \text{“}\tilde{E} = \{\gamma \in \kappa \mid A_\gamma = \gamma \cap \tilde{A}\}\text{”}.$$

This is contrary to the choice of $(A_\gamma \mid \gamma < \kappa)$, because the above sentence is absolute. Hence,

$$\models_{J_\beta} \text{“}E \text{ is not stationary in } \alpha_\eta\text{”}.$$

Thus for some $C \in J_\beta$ we have

$$\models_{J_\beta} \text{“}C \text{ is a club subset of } \alpha_\eta \text{ and } (\forall \gamma \in C)(A_\gamma \neq \gamma \cap A_{\alpha_\eta})\text{”}.$$

Setting $\tilde{C} = \pi^{-1}(C)$ we get, applying $\pi^{-1}\colon J_\beta \prec J_{\kappa^+}$,

$$\models_{J_{\kappa^+}} \text{“}\tilde{C} \text{ is a club subset of } \kappa \text{ and } (\forall \gamma \in \tilde{C})(A_\gamma \neq \gamma \cap \tilde{A})\text{”}.$$

Since $\pi^{-1} \upharpoonright \alpha_\eta = \mathrm{id} \upharpoonright \alpha_\eta$, we have $\tilde{C} \cap \alpha_\eta = C$. Hence as C is unbounded in α_η and $\tilde{C}$ is closed in κ (by absoluteness), $\alpha_\eta \in \tilde{C}$. Thus $A_{\alpha_\eta} \neq \alpha_\eta \cap \tilde{A}$. But $\tilde{A} = \pi^{-1}(A_{\alpha_\eta})$, so in fact we do have $\tilde{A} \cap \alpha_\eta = A_{\alpha_\eta}$, because $\pi^{-1} \upharpoonright \alpha_\eta = \mathrm{id} \upharpoonright \alpha_\eta$. Contradiction! The proof is complete. $\square$

3. *Generalised Kurepa Families and the Principles* $\diamondsuit^+_{\kappa,\lambda}$

The following natural generalisation of the notion of a κ-Kurepa family was put forward by C. C. Chang. Let κ, λ denote uncountable cardinals, with κ regular[8]

8 The principle $\mathrm{KH}(\kappa, \kappa)$ is of some interest in the case where κ is singular. This is considered in Exercise 3.

and $\lambda \leqslant \kappa$. We define

$$\mathscr{P}_\lambda(\kappa) = \{x \subseteq \kappa \mid \omega \leqslant |x| < \lambda\}.$$

The (κ, λ)-*Kurepa Hypothesis*, $\mathrm{KH}(\kappa, \lambda)$, is the assertion that there is a family $\mathscr{F} \subseteq \mathscr{P}(\kappa)$ such that $|\mathscr{F}| \geqslant \kappa^+$ and for every $x \subseteq \mathscr{P}_\lambda(\kappa)$,

$$|\{f \cap x \mid f \in \mathscr{F}\}| \leqslant |x|.$$

Clearly, $\mathrm{KH}(\kappa, \kappa)$ implies the existence of a κ-Kurepa family. Hence by 2.6, we have

3.1 Theorem. *If κ is ineffable, then* $\mathrm{KH}(\kappa, \kappa)$ fails. □

We shall prove that if $V = L$, the converse to 3.1 holds, a result which strengthens 2.7. We shall also prove that $V = L$ implies that $\mathrm{KH}(\kappa, \lambda)$ holds for all uncountable regular κ and all uncountable $\lambda < \kappa$. We do this by introducing a two cardinal version of the combinatorial principle $\diamondsuit_\kappa^+$.

We assume throughout that κ, λ are as stated at the outset of this section.

$\diamondsuit_{\kappa,\lambda}^+$ asserts the existence of a function $(S_x \mid x \in \mathscr{P}_\lambda(\kappa))$ such that:

(i) $S_x \subseteq \mathscr{P}(\bigcup x)$;
(ii) $|S_x| \leqslant |x|$;
(iii) if $X \subseteq \kappa$, then there is an unbounded set $B \subseteq \kappa$ with the property that whenever $x \in \mathscr{P}_\lambda(\kappa)$ has no largest element and is such that $B \cap x$ is cofinal in x, then $X \cap \alpha, B \cap \alpha \in S_x$, where $\alpha = \bigcup x$.

3.2 Theorem. *$\diamondsuit_{\kappa,\lambda}^+$ implies* $\mathrm{KH}(\kappa, \lambda)$.

Proof. Recall that H_κ is a model of ZF^-. Fix some set of skolem functions for H_κ. Let $(S_x \mid x \in \mathscr{P}_\lambda(\kappa))$ satisfy $\diamondsuit_{\kappa,\lambda}^+$, and for each $x \in \mathscr{P}_\lambda(\kappa)$, let M_x be the smallest (with respect to the chosen skolem functions) $M \prec H_\kappa$ such that $x \cup \{x\} \subseteq M$ and $(\forall \alpha \leqslant \bigcup x)(S_{x \cap \alpha} \subseteq M)$. Notice that $|M_x| = |x|$. Set

$$\mathscr{F} = \{f \subseteq \kappa \mid (\forall x \in \mathscr{P}_\lambda(\kappa))(f \cap x \in M_x)\}.$$

In order to prove $\mathrm{KH}(\kappa, \lambda)$, it clearly suffices to show that $|\mathscr{F}| \geqslant \kappa^+$, since in that case $\mathscr{F}$ will satisfy $\mathrm{KH}(\kappa, \lambda)$. We shall assume that $|\mathscr{F}| \leqslant \kappa$ and derive a contradiction. Notice that $\kappa \in \mathscr{F}$, so $\mathscr{F} \neq \emptyset$.

Let $(f_\nu \mid \nu < \kappa)$ enumerate all unbounded members of $\mathscr{F}$. (This enumeration need not be one-one.) For each $\nu < \kappa$, let C_ν be the set of all limit points of f_ν. Let X be the diagonal intersection of the sequence $(C_\nu \mid \nu < \kappa)$, i.e.

$$X = \{\alpha \in \kappa \mid (\forall \nu < \alpha)(\alpha \in C_\nu)\}.$$

Each set C_ν is club in κ, so X is club in κ. For each $\alpha \in X$, α is a limit ordinal and for any $\nu < \alpha$, $f_\nu \cap \alpha$ is unbounded in α.

By $\diamondsuit^{+}_{\kappa,\lambda}$, let $B \subseteq \kappa$ be unbounded and such that whenever $x \in \mathscr{P}_\lambda(\kappa)$ is such that $\alpha = \bigcup x$ is a limit point of $B \cap x$, then $X \cap \alpha, B \cap \alpha \in S_x$. Let $(\alpha_\nu \mid \nu < \kappa)$ be the monotone enumeration of the set

$$\{\alpha \in X \mid \alpha \text{ is a limit point of } B\}.$$

For $\nu < \kappa$, set

$$\beta_\nu = \min(B - \alpha_\nu).$$

Notice that

$$\alpha_\nu \leqslant \beta_\nu < \alpha_{\nu+1}.$$

Set

$$f = \{\beta_\nu \mid \nu < \kappa\}.$$

Then f in an unbounded subset of κ. Since $f_\nu \cap \alpha_{\nu+1}$ is unbounded in $\alpha_{\nu+1}$, but $f \cap \alpha_{\nu+1} \subseteq \beta_\nu + 1 < \alpha_{\nu+1}$ for each $\nu < \kappa$, we have $f \neq f_\nu$ for all $\nu < \kappa$. We obtain our contradiction by showing that $f \in \mathscr{F}$.

Let $x \in \mathscr{P}_\lambda(\kappa)$. We prove that $f \cap x \in M_x$. Let β be the greatest limit point of $f \cap x$. Then

$$f \cap x = (f \cap x \cap \beta) \cup (f \cap x - \beta),$$

where $f \cap x - \beta$ is finite. Being a finite subset of x, $f \cap x - \beta$ must be an element of M_x, since $x \subseteq M_x$ and $M_x \models \mathrm{ZF}^-$. So in order to show that $f \cap x \in M_x$, it suffices to show that $f \cap x \cap \beta \in M_x$.

Now, β is a limit point of $f \cap x$. But $f \subseteq B$. Thus β is a limit point of $B \cap (x \cap \beta)$. Hence $X \cap \beta$, $B \cap \beta \in S_{x \cap \beta} \subseteq M_x$. But clearly, $f \cap \beta$ is ZF^--definable from $X \cap \beta$, $B \cap \beta$ in exactly the same way that f was defined from X and B. Hence $f \cap \beta \in M_x$. Thus $f \cap \beta \cap x \in M_x$, and we are done. □

3.3 Theorem. *Assume $V = L$. If $\lambda < \kappa$, then $\diamondsuit^{+}_{\kappa,\lambda}$ is valid.*

Proof. For each $x \in \mathscr{P}_\lambda(\kappa)$, let M_x be the smallest $M \prec J_\kappa$ such that M_x
$x \cup \{x\} \cup \{\lambda\} \subseteq M$, and set $S_x = \mathscr{P}(\bigcup x) \cap M_x$. We prove that $(S_x \mid x \in \mathscr{P}_\lambda(\kappa))$ S_x
satisfies $\diamondsuit^{+}_{\kappa,\lambda}$.

Suppose otherwise, and let $X \subseteq \kappa$ be the $<_J$-least set such that there is no X
unbounded set $B \subseteq \kappa$ as in $\diamondsuit^{+}_{\kappa,\lambda}$. Note that both $(S_x \mid x \in \mathscr{P}_\lambda(\kappa))$ and X are definable from λ in J_{κ^+}.

By recursion on $\nu < \kappa$, define a chain of submodels

$$N_0 \prec N_1 \prec \ldots \prec N_\nu \prec \ldots \prec J_{\kappa^+}$$ N_ν

as follows.

$$N_0 = \text{the smallest } N \prec J_{\kappa^+} \text{ such that } \lambda \in N \cap \kappa \in \kappa;$$

$$N_{\nu+1} = \text{the smallest } N \prec J_{\kappa^+} \text{ such that } N_\nu \cup \{N_\nu\} \subseteq N \text{ and } N \cap \kappa \in \kappa;$$

$$N_\delta = \bigcup_{\nu<\delta} N_\nu, \quad \text{if } \lim(\delta).$$

It is easily seen that this causes no difficulties. In particular, $|N_\nu| < \kappa$ for all $\nu < \kappa$. Moreover

$$\nu < \tau < \kappa \rightarrow N_\nu \prec N_\tau \prec J_{\kappa^+}.$$

For each $\nu < \kappa$, set

α_ν
$$\alpha_\nu = N_\nu \cap \kappa.$$

Clearly, $(\alpha_\nu \mid \nu < \kappa)$ is a normal sequence in κ.

For each $\nu < \kappa$, let

$\sigma_\nu, \beta(\nu)$
$$\sigma_\nu \colon N_\nu \cong J_{\beta(\nu)}.$$

Clearly,

$$\sigma_\nu \upharpoonright \alpha_\nu = \mathrm{id} \upharpoonright \alpha_\nu, \qquad \sigma_\nu(\kappa) = \alpha_\nu, \qquad \sigma_\nu(X) = X \cap \alpha_\nu.$$

Set

B
$$B = \{\beta(\nu) \mid \nu < \kappa\}.$$

B is an unbounded subset of κ. We shall obtain the desired contradiction by showing that B satisfies the requirements of $\diamondsuit^+_{\kappa,\lambda}$ for X.

x Fix x an arbitrary element of $\mathscr{P}_\lambda(\kappa)$ such that $\alpha = \bigcup x$ is a limit point of $B \cap x$. We shall show that $X \cap \alpha, B \cap \alpha \in M_x$, thereby completing the proof.

For each $\nu < \kappa$, we have $N_\nu \in N_{\nu+1} \prec J_{\kappa^+}$, and hence σ_ν, $\beta(\nu) \in N_{\nu+1}$. But $|N_\nu| < \kappa$. Thus $\beta(\nu) \in N_{\nu+1} \cap \kappa = \alpha_{\nu+1}$. Also, since $\sigma_{\nu+1}(\kappa) = \alpha_{\nu+1}$ we have $\alpha_{\nu+1} < \beta(\nu+1)$. Thus for all $\nu < \kappa$ we have

(1) $$\beta(\nu) < \alpha_{\nu+1} < \beta(\nu+1).$$

But α is a limit point of $B = \{\beta(\nu) \mid \nu < \kappa\}$. Thus we must have $\alpha = \alpha_\eta$ for some

η limit ordinal $\eta < \kappa$.

Now, as we remarked earlier, X is J_{κ^+}-definable from λ. But

$$\sigma_\eta^{-1} \colon J_{\beta(\eta)} \prec J_{\kappa^+}, \qquad \sigma_\eta^{-1}(\lambda) = \lambda, \qquad \sigma_\eta^{-1}(X \cap \alpha_\eta) = X.$$

Thus $X \cap \alpha_\eta$ is $J_{\beta(\eta)}$-definable from λ.

Similarly, $B \cap \alpha_\eta$ is ZF^--definable from $J_{\beta(\eta)}$ and λ in exactly the same way that B was defined from J_{κ^+} and λ. (This uses the fact that $\sigma_\eta^{-1} \upharpoonright \alpha_\eta = \mathrm{id} \upharpoonright \alpha_\eta$.)

Since $\lambda \in M_x$ and $M_x \models \mathrm{ZF}^-$, it follows that in order to prove that $X \cap \alpha$, $B \cap \alpha \in M_x$ it is sufficient to show that $\beta(\eta) \in M_x$. This will take some time, and requires some considerable extra machinery before we can even motivate the argument.

To avoid confusion between ordinals and sequences of ordinals, from now on

$\vec{\beta}, \vec{\beta} \upharpoonright \tau$ we shall use $\vec{\beta}$ to denote $(\beta(\nu) \mid \nu < \kappa)$, and for any $\tau < \kappa$ we shall write $\vec{\beta} \upharpoonright \tau$ for $(\beta(\nu) \mid \nu < \tau)$.

For $\nu < \mu < \kappa$, set

$\sigma_{\nu\mu}$
$$\sigma_{\nu\mu} = \sigma_\mu \circ \sigma_\nu^{-1}.$$

Thus

$$\sigma_{\nu\mu}\colon J_{\beta(\nu)} \prec J_{\beta(\mu)}.$$

Note that $\langle (J_{\beta(\nu)})_{\nu<\kappa}, (\sigma_{\nu\mu})_{\nu<\mu<\kappa}\rangle$ is a directed elementary system. We write $\vec{\sigma}$ for $(\sigma_{\nu\mu} \mid \nu < \mu < \kappa)$, and for any $\tau < \kappa$ we write $\vec{\sigma} \upharpoonright \tau$ for $(\sigma_{\nu\mu} \mid \nu < \mu < \tau)$. $\vec{\sigma}$ $\vec{\sigma} \upharpoonright \tau$

The following result is central to our entire argument.

(2) If $\gamma \in \eta \cap M_x$, then $\alpha_\gamma, \beta(\gamma), \vec{\beta} \upharpoonright (\gamma + 1), \vec{\sigma} \upharpoonright (\gamma + 1) \in M_x$.

To prove (2), let $\gamma \in \eta \cap M_x$. Since $\alpha = \alpha_\eta$ is a limit point of $B \cap x$ we can find a $\tau < \eta$ such that $\tau > \gamma$ and $\beta(\tau) \in x \subseteq M_x$. γ

Define a sequence $(N'_\nu \mid \nu < \theta)$, for some θ, as follows.

$$N'_0 = \text{the smallest } N \prec J_{\beta(\tau)} \text{ such that } \lambda \in N \cap \alpha_\tau \in \alpha_\tau;$$

$$N'_{\nu+1} = \text{the smallest } N \prec J_{\beta(\tau)} \text{ such that } N'_\nu \cup \{N'_\nu\} \subseteq N \text{ and } N \cap \alpha_\tau \in \alpha_\tau;$$

$$N'_\delta = \bigcup_{\nu<\delta} N'_\nu, \quad \text{if } \lim(\delta).$$

The definition will break down at some stage θ when $\sup(\bigcup_{\nu<\theta} N'_\nu \cap \alpha_\tau) = \alpha_\tau$.

We have

$$(*) \qquad \sigma_\tau^{-1}\colon J_{\beta(\tau)} \prec J_{\kappa^+}, \quad \sigma_\tau^{-1}(\lambda) = \lambda, \quad \sigma_\tau^{-1}(\alpha_\tau) = \kappa, \quad \sigma_\tau^{-1} \upharpoonright \alpha_\tau = \mathrm{id} \upharpoonright \alpha_\tau.$$

So by induction on ν we see that

$$\nu < \tau \rightarrow N'_\nu \text{ is defined and } \sigma_\tau^{-1}{}'' N'_\nu = N_\nu.$$

It follows that $\theta = \tau$, of course, since $\sup_{\nu<\tau}(N'_\nu \cap \alpha_\tau) = \sup_{\nu<\tau}\alpha_\nu = \alpha_\tau$. For each $\nu < \tau$, let

$$\sigma'_\nu\colon N'_\nu \cong J_{\beta'(\nu)}.$$

Since $N'_\nu \cong N_\nu$ (by $\sigma_\tau^{-1} \upharpoonright N'_\nu$). We have $\beta'(\nu) = \beta(\nu)$ for all $\nu < \tau$. Thus $(\beta'(\nu) \mid \nu < \tau) = \vec{\beta} \upharpoonright \tau$. This shows that $\vec{\beta} \upharpoonright \tau$ is ZF^--definable from $\beta(\tau), \lambda, \alpha_\tau$. Now, $\beta(\tau), \lambda \in M_x$. And by $(*)$ above,

$$\alpha_\tau = [\text{the largest cardinal}]^{J_{\beta(\tau)}},$$

so $\alpha_\tau \in M_x$ as well. Thus $\vec{\beta} \upharpoonright \tau \in M_x$. Since $\gamma \in M_x$ and $\gamma < \tau$ it follows that $\beta(\gamma) = (\vec{\beta} \upharpoonright \tau)(\gamma) \in M_x$ and $\vec{\beta} \upharpoonright (\gamma + 1) = (\vec{\beta} \upharpoonright \tau) \upharpoonright (\gamma + 1) \in M_x$. Also, $\alpha_\gamma = [\text{the largest cardinal}]^{J_{\beta(\gamma)}} \in M_x$. It remains to prove that $\vec{\sigma} \upharpoonright (\gamma + 1) \in M_x$.

Now, in the definition of $(N_\nu \mid \nu < \kappa)$, if we replace J_{κ^+} by N_μ and κ by α_μ, we will obtain the sequence $(N_\nu \mid \nu < \mu)$, as is easily seen. So, as $\sigma_\mu\colon N_\mu \cong J_{\beta(\mu)}$ and $\sigma_\mu \upharpoonright \alpha_\mu = \mathrm{id} \upharpoonright \alpha_\mu$, the same definition with parameters $J_{\beta(\mu)}$ and α_μ will produce the sequence $(\sigma''_\mu N_\nu \mid \nu < \mu)$. But it is easily seen that $\sigma_{\nu\mu}^{-1}$ is the collapsing isomorphism for $\sigma''_\mu N_\nu$. Since $\alpha_\mu = [\text{the largest cardinal}]^{J_{\beta(\mu)}}$ for all μ, this shows that $(\sigma_{\nu\mu} \mid \nu < \mu < \tau)$ is ZF^--definable from $\vec{\beta} \upharpoonright \tau$. But $\vec{\beta} \upharpoonright \tau \in M_x$. Thus $\vec{\sigma} \upharpoonright \tau \in M_x$ and it follows at once that $\vec{\sigma} \upharpoonright (\gamma + 1) \in M_x$. So (2) is proved.

Two further results follow easily from the above.

(3) For $\tau < \eta$, $\tau \in M_x$ iff $\beta(\tau) \in M_x$.

By (2), if $\tau \in M_x$, then $\beta(\tau) \in M_x$. To prove the converse, assume that $\beta(\tau) \in M_x$. Thus $\alpha_\tau = [\text{the largest cardinal}]^{J_{\beta(\tau)}} \in M_x$. Now, from λ, $\beta(\tau)$, α_τ we may define the sequence $(N'_\nu \mid \nu < \theta)$ as in the proof of (2) above. As we observed then, we must have $\theta = \tau$. So this defines τ from λ, $\beta(\tau)$, α_τ in a ZF$^-$ fashion. So as λ, $\beta(\tau)$, $\alpha_\tau \in M_x$, we conclude that $\tau \in M_x$, and (3) is proved.

(4) $\sup(\eta \cap M_x) = \eta$.

Clearly, $\sup(\eta \cap M_x) \leqslant \eta$. To prove the opposite inequality, let $\nu < \eta$. Then $\alpha_\nu < \alpha_\eta$, so as $\alpha_\eta = \alpha = \sup(B \cap x \cap \alpha)$, we can find a $\tau < \eta$ such that $\alpha_\nu < \beta(\tau) \in x \subseteq M_x$. By (1), $\nu < \tau$. By (3), $\tau \in M_x$. So $\nu \leqslant \sup(\eta \cap M_x)$, and (4) follows at once.

Now let

π, δ $$\pi: M_x \cong J_\delta$$

and set

η^* $$\eta^* = \pi''(\eta \cap M_x).$$

By virtue of (2) we may define

$\vec{\beta}^*$ $$\vec{\beta}^* = \bigcup_{\gamma \in \eta \cap M_x} \pi(\vec{\beta} \restriction (\gamma + 1)),$$

$\vec{\sigma}^*$ $$\vec{\sigma}^* = \bigcup_{\gamma \in \eta \cap M_x} \pi(\vec{\sigma} \restriction (\gamma + 1)).$$

Since π is a collapsing isomorphism, the following are easily checked:

(5) η^* is an ordinal.

$\beta^*(\nu)$ (6) $\vec{\beta}^*$ is an η^*-sequence of ordinals, say $\vec{\beta}^* = (\beta^*(\nu) \mid \nu < \eta^*)$.

$\sigma^*_{\nu\mu}$ (7) $\vec{\sigma}^*$ is a system of maps of the form $\vec{\sigma}^* = (\sigma^*_{\nu\mu} \mid \nu < \mu < \eta^*)$.

(8) $\beta^*(\nu) = \pi(\beta(\pi^{-1}(\nu)))$ for all $\nu < \eta^*$.

(9) $\sigma^*_{\nu\mu} = \pi(\sigma_{\pi^{-1}(\nu), \pi^{-1}(\mu)})$ for all $\nu < \mu < \eta^*$.

We know that $\langle (J_{\beta(\nu)})_{\nu<\eta}, (\sigma_{\nu\mu})_{\nu<\mu<\eta} \rangle$ is a directed elementary system with direct limit $\langle J_{\beta(\eta)}, (\sigma_{\nu\eta})_{\nu<\eta} \rangle$. Using (8) and (9) it is easily checked that $\langle (J_{\beta^*(\nu)})_{\nu<\eta^*}, (\sigma^*_{\nu\mu})_{\nu<\mu<\eta^*} \rangle$ is a directed elementary system. Let $\langle \langle U, E \rangle, (\sigma^*_\nu)_{\nu<\eta^*} \rangle$ be a direct limit of this system. We may define an embedding

h $$h: \langle U, E \rangle \prec \langle J_{\beta(\eta)}, \in \rangle$$

by letting ν range over η^* in the following commutative diagram:

$$\begin{tikzcd} J_{\beta(\pi^{-1}(\nu))} \arrow[r, "\sigma_{\pi^{-1}(\nu),\eta}"] & J_{\beta(\eta)} \\ J_{\beta^*(\nu)} \arrow[u, "\pi^{-1}"] \arrow[r, "\sigma^*_\nu"] & \langle U, E\rangle \arrow[u, "h"'] \end{tikzcd}$$

Thus $\langle U, E\rangle$ is well-founded, and we may take $\langle U, E\rangle$ to be of the form $\langle J_{\beta^*}, \in\rangle$ for some unique ordinal β^*. β^*

If $\nu < \eta^*$, then

$$(\sigma_{\pi^{-1}(\nu)})^{-1} \circ \pi^{-1}: J_{\beta^*(\nu)} \prec J_{\kappa^+},$$

so there is an $\alpha^*_\nu < \beta^*(\nu)$ such that

$$\alpha^*_\nu = [\text{the largest cardinal}]^{J_{\beta^*(\nu)}}.$$

Also,

$$\sigma_\eta^{-1} \circ h: J_{\beta^*} \prec J_{\kappa^+},$$

so there is an $\alpha^* < \beta^*$ such that

$$\alpha^* = [\text{the largest cardinal}]^{J_{\beta^*}}.$$ α^*

The following result is immediate:

(10) $\quad \alpha^*_\nu = \pi(\alpha_{\pi^{-1}(\nu)})$ and $\sigma^*_\nu(\alpha^*_\nu) = \alpha^*$ for all $\nu < \eta^*$, and $h(\alpha^*) = \alpha_\eta$.

Moreover, as we show next:

(11) $\quad \sigma^*_\nu \restriction \alpha^*_\nu = \mathrm{id} \restriction \alpha^*_\nu \quad$ for all $\nu < \eta^*$.

Since $\langle J_{\beta^*}, (\sigma^*_\nu)_{\nu<\eta^*}\rangle$ is the transitive direct limit of $\langle (J_{\beta^*(\nu)})_{\nu<\eta^*}, (\sigma^*_{\nu\mu})_{\nu<\mu<\eta^*}\rangle$, it suffices to prove that $\sigma^*_{\nu\mu} \restriction \alpha^*_\nu = \mathrm{id} \restriction \alpha^*_\nu$ for all $\nu < \mu < \eta^*$. But this follows easily from (9) and the properties of the system $\vec{\sigma}$.

(12) $\quad \alpha^* = \sup_{\nu<\eta^*} \alpha^*_\nu$.

Since $\sigma^*_\nu(\alpha^*_\nu) = \alpha^*$ for all $\nu < \eta^*$, we have $\sup_{\nu<\eta^*}\alpha^*_\nu \leqslant \alpha^*$. To prove the opposite inequality, suppose $\gamma < \alpha^*$. Pick $\nu < \eta^*$ so that $\gamma = \sigma^*_\nu(\bar{\gamma})$ for some $\bar{\gamma}$. Since $\sigma^*_\nu(\alpha^*_\nu) = \alpha^*$, we have $\bar{\gamma} < \alpha^*_\nu$. So by (11), $\gamma = \sigma^*_\nu(\bar{\gamma}) = \bar{\gamma}$. Thus $\gamma < \alpha^*_\nu$. This proves that $\alpha^* \leqslant \sup_{\nu<\eta^*}\alpha^*_\nu$, and completes the proof of (12).

We are now able to indicate the purpose of the above considerations. It is easily seen that $\vec{\beta}^*$ and $\vec{\sigma}^*$ are ZF$^-$-definable from β^*, α^*, and λ in the same way that $\vec{\beta}$ and $\vec{\sigma}$ were defined from κ^+, κ, and λ. (See, in particular, the proof of (2) above and the definitions of $\vec{\beta}^*$, $\vec{\sigma}^*$, β^*, α^*.) Since $\alpha^* = [\text{the largest cardinal}]^{J_{\beta^*}}$, this means that $\vec{\beta}^*$ and $\vec{\sigma}^*$ are ZF$^-$-definable from β^*, λ.

Assume for the time being that $\beta^* \in J_\delta$. Since $J_\delta \models \text{Zf}^-$ (because $M_x \models \text{ZF}^-$), it follows that $\vec{\beta}^*, \vec{\sigma}^* \in J_\delta$. Thus $\pi^{-1}(\vec{\beta}^*)$ and $\pi^{-1}(\vec{\sigma}^*)$ are defined. Since $\pi\colon M_x \cong J_\delta$ is a collapsing isomorphism and $\eta^* = \pi''(\eta \cap M_x)$, it is a routine consequence of (6), (7), (8), (9) and the definition of $\vec{\beta}^*$ and $\vec{\sigma}^*$ that

$$\vec{\beta} \restriction \eta = \pi^{-1}(\vec{\beta}^*) \restriction \eta \quad \text{ and } \vec{\sigma} \restriction \eta = \pi^{-1}(\vec{\sigma}^*) \restriction \eta.$$

Suppose first that $\pi^{-1}(\vec{\beta}^*) = \vec{\beta} \restriction \eta$. Since J_{β^*} is the unique transitive limit of the system $\langle (J_{\beta^*(\nu)})_{\nu<\eta^*}, (\sigma^*_{\nu\mu})_{\nu<\mu<\eta^*}\rangle$, it follows that $J_{\pi^{-1}(\beta^*)}$ is the unique transitive limit of the system $\langle (J_{\beta(\nu)})_{\nu<\eta}, (\sigma_{\nu\mu})_{\nu<\mu<\eta}\rangle$. Thus $\pi^{-1}(\beta^*) = \beta(\eta)$. Hence $\beta(\eta) \in \text{ran}(\pi^{-1}) = M_x$, and we are done.

Otherwise, $\pi^{-1}(\vec{\beta}^*)$ is a proper end-extension of $\vec{\beta} \restriction \eta$. Thus the directed elementary system determined by $\pi^{-1}(\vec{\beta}^*)$, $\pi^{-1}(\vec{\sigma}^*)$ is an end-extension of $\langle (J_{\beta(\nu)})_{\nu<\eta}, (\sigma_{\nu\mu})_{\nu<\mu<\eta}\rangle$. So $J_{\pi^{-1}(\beta^*)(\eta)}$ is the transitive direct limit of $\langle (J_{\beta(\nu)})_{\nu<\eta}, (\sigma_{\nu\mu})_{\nu<\mu<\eta}\rangle$, which means that $\pi^{-1}(\vec{\beta}^*)(\eta) = \beta(\eta)$. It follows that $\beta(\eta)$ is ZF$^-$-definable from $\pi^{-1}(\vec{\beta}^*)$ and α as the unique element γ of $\text{ran}(\pi^{-1}(\vec{\beta}^*))$ such that $\alpha = [\text{the largest cardinal}]^{J_\gamma}$. (By (1), each $\beta(\nu)$ has a unique α_ν associated with it, so the same will be true for the members of $\pi^{-1}(\vec{\beta}^*)$. Since $\pi^{-1}(\vec{\beta}^*)(\eta) = \beta(\eta)$, the relevant "$\alpha_\nu$" here is $\alpha_\eta = \alpha$.) But $\alpha = \bigcup x$ and $x \in M_x$, so $\alpha \in M_x$. Also, $\pi^{-1}(\vec{\beta}^*) \in \text{ran}(\pi^{-1}) = M_x$. Hence $\beta(\eta) \in M_x$, as required.

So we see that the proof boils down to showing that (as was assumed for the above discussion) $\beta^* \in J_\delta$. As a first step we prove:

(13) $\quad \mathscr{P}(\alpha_\eta) \cap M_x \nsubseteq J_{\beta(\eta)}$.

We know that $x \in \mathscr{P}(\alpha_\eta) \cap M_x$, so it suffices to show that $x \notin J_{\beta(\eta)}$. Well, we have $|x| < \lambda < \alpha_0 < \alpha_\eta$. Since λ is a cardinal, $|x|^{J_{\beta(\eta)}} < \lambda$. But $\sup(x) = \alpha_\eta$. Hence $\models_{J_{\beta(\eta)}}$ "α_η is singular". But this is a contradiction, since $\sigma_\eta^{-1}\colon J_{\beta(\eta)} \prec J_{\kappa^+}$ and $\sigma_\eta^{-1}(\alpha_\eta) = \kappa$. This proves (13). (Incidentally, this is the only point were we need the fact that $\lambda < \kappa$.)

We complete the proof by showing that if $\delta \leqslant \beta^*$, then, contrary to the above, $\mathscr{P}(\alpha_\eta) \cap M_x \subseteq J_{\beta(\eta)}$. First two results which do not require this assumption.

(14) $\quad$ If $z \in \mathscr{P}(\alpha_\eta) \cap M_x$, $\quad$ then $\pi(z) \in \mathscr{P}(\alpha^*) \cap J_\delta$.

Since $\pi(z) = \{\pi(\xi) \mid \xi \in z \cap M_x\}$, in order to prove (14) it suffices to show that if $\xi \in z \cap M_x$, then $\pi(\xi) \in \alpha^*$. Suppose $\xi \in z \cap M_x$. Then $\xi < \alpha_\eta$. Now, $\alpha_\eta = \sup_{\nu<\eta} \alpha_\nu$, so by (4) we can find a $\nu \in \eta \cap M_x$ such that $\xi < \alpha_\nu$. By (10), $\pi(\xi) < \pi(\alpha_\nu) = \alpha^*_{\pi(\nu)}$. But $\pi(\nu) < \eta^*$. So by (12), $\pi(\xi) < \alpha^*$. This proves (14).

(15) $\quad$ If $z \in J_{\beta^*}$ and z is a bounded subset of α^*, then $h(z) = \pi^{-1}(z)$.

Since z is a bounded subset of α^*, (12) tells us that we can pick $\nu < \eta^*$ sufficiently large so that $z \subseteq \alpha^*_\nu$. Since $z \in J_{\beta^*}$, we can assume that ν is chosen here so that $z = \sigma^*_\nu(\bar{z})$ for some $\bar{z} \subseteq \alpha^*_\nu$. By (11) and (10), $\sigma^*_\nu \restriction \alpha^*_\nu = \text{id} \restriction \alpha^*_\nu$ and $\sigma^*_\nu(\alpha^*_\nu) = \alpha^*$, so

$z = z \cap \alpha^*_\nu = \sigma^*_\nu{}''\bar{z} = \bar{z}$. Thus $z \in J_{\beta^*(\nu)}$ and $\sigma^*_\nu(z) = z$. Since $z \in J_{\beta^*(\nu)}$, $\pi^{-1}(z)$ is defined. We have $\pi^{-1}(z) \subseteq \pi^{-1}(\alpha^*_\nu)$, so by (10), $\pi^{-1}(z) \subseteq \alpha_{\bar{\nu}}$, where $\bar{\nu} = \pi^{-1}(\nu)$. Now, $\sigma_{\bar{\nu}\eta} \restriction \alpha_{\bar{\nu}} = \mathrm{id} \restriction \alpha_{\bar{\nu}}$, so $\sigma_{\bar{\nu}\eta}{}''\pi^{-1}(z) = \pi^{-1}(z)$. But by choosing ν large enough below η^* we may assume that z is a bounded subset of α^*_ν, and hence that $\pi^{-1}(z)$ is a bounded subset of $\alpha_{\bar{\nu}}$. Thus $\sigma_{\bar{\nu}\eta}(\pi^{-1}(z)) = \sigma_{\bar{\nu}\eta}{}''\pi^{-1}(z) = \pi^{-1}(z)$. By definition of h now, we have

$$h(z) = \sigma_{\bar{\nu}\eta} \circ \pi^{-1} \circ \sigma^{*\,-1}_\nu(z) = \sigma_{\bar{\nu}\eta} \circ \pi^{-1}(z) = \pi^{-1}(z),$$

which proves (15).

To complete the proof of the theorem we now have:

(16) If $\delta \leqslant \beta^*$, then $\mathscr{P}(\alpha_\eta) \cap M_x \subseteq J_{\beta(\eta)}$.

Let $\bar{z} \in \mathscr{P}(\alpha_\eta) \cap M_x$. Let $z = \pi(\bar{z})$. By (14), $z \in \mathscr{P}(\alpha^*) \cap J_\delta$. Since $\delta \leqslant \beta^*$, $z \in J_{\beta^*}$. Thus $h(z) \in J_{\beta(\eta)}$. It suffices, therefore, to prove that $h(z) = \bar{z}$.

Now, $z \subseteq \alpha^*$, so, using (10), $h(z) \subseteq h(\alpha^*) = \alpha_\eta$. But by (4), $\alpha_\eta = \sup_{\nu \in \eta \cap M_x} \alpha_\nu$. Thus

$$h(z) = \bigcup_{\nu \in \eta \cap M_x} [h(z) \cap \alpha_\nu].$$

Likewise

$$\bar{z} = \bigcup_{\nu \in \eta \cap M_x} [\bar{z} \cap \alpha_\nu].$$

So we have

$$\begin{aligned}
h(z) &= \bigcup_{\nu \in \eta \cap M_x} [h(z) \cap \alpha_\nu] \\
&= \bigcup_{\nu < \eta^*} [h(z) \cap \alpha_{\pi^{-1}(\nu)}] && \text{(by definition of } \eta^*) \\
&= \bigcup_{\nu < \eta^*} [h(z) \cap \pi^{-1}(\alpha^*_\nu)] && \text{(by (10))} \\
&= \bigcup_{\nu < \eta^*} [h(z) \cap h(\alpha^*_\nu)] && \text{(by (15) applied to } \alpha^*_\nu) \\
&= \bigcup_{\nu < \eta^*} [h(z \cap \alpha^*_\nu)] && \text{(since } h \text{ is an isomorphism)} \\
&= \bigcup_{\nu < \eta^*} [\pi^{-1}(z \cap \alpha^*_\nu)] && \text{(by (15) applied to } z \cap \alpha^*_\nu) \\
&= \bigcup_{\nu < \eta^*} [\pi^{-1}(z) \cap \pi^{-1}(\alpha^*_\nu)] && \text{(since } \pi^{-1} \text{ is an isomorphism)} \\
&= \bigcup_{\nu < \eta^*} [\bar{z} \cap \alpha_{\pi^{-1}(\nu)}] && \text{(since } \pi(\bar{z}) = z \text{ and by (10), respectively)} \\
&= \bigcup_{\nu \in \eta \cap M_x} [\bar{z} \cap \alpha_\nu] && \text{(by definition of } \eta^*) \\
&= \bar{z}.
\end{aligned}$$

We are done. □

By 3.2 and 3.3, if $V = L$, then $\mathrm{KH}(\kappa, \lambda)$ is valid whenever $\lambda < \kappa$. By virtue of 3.1 and 3.2, our next result shows that if $V = L$, then $\mathrm{KH}(\kappa, \kappa)$ iff κ is not ineffable.

3.4 Theorem. *Assume $V = L$. Then $\diamondsuit^+_{\kappa,\kappa}$ holds iff κ is not ineffable.*

Proof. If κ is ineffable, then by 3.1, $\neg\, \mathrm{KH}(\kappa, \kappa)$, so by 3.2, $\neg\, \diamondsuit^+_{\kappa,\kappa}$.

Conversely, suppose κ is not ineffable. We prove $\diamondsuit^+_{\kappa,\kappa}$ by means of an argument very similar to that used in 3.3 above. Because of this similarity, we simply describe the changes that must be made to the proof in the present case. The idea is to modify the definition of the models M_x so that an analogue of (13) may be proved, since this is the only point in the proof of 3.3 where we made use of the fact that $\lambda < \kappa$. (At all other points where λ was mentioned, we may now simply omit all mention of λ, and everything proceeds as before.)

Let $(A_\alpha \mid \alpha < \kappa)$ be the $<_J$-least sequence such that $A_\alpha \subseteq \alpha$ for all $\alpha < \kappa$, but for any $A \subseteq \kappa$, the set $\{\alpha \in \kappa \mid A_\alpha = \alpha \cap A\}$ is not stationary in κ. (Such a sequence exists by 2.1.) For each $x \in \mathscr{P}_\kappa(\kappa)$, let M_x be the smallest $M \prec J_\kappa$ such that $x \cup \{x\} \cup \{A_{\cup x}\} \subseteq M$. Now define S_x, $x \in \mathscr{P}_\kappa(\kappa)$, as before, and proceed exactly as in 3.3 except for the verification of (13). At this point we argue as follows.

We wish to prove that $\mathscr{P}(\alpha_\eta) \cap M_x \nsubseteq J_{\beta(\eta)}$. We assume otherwise and derive a contradiction. By definition, we have $A_{\alpha_\eta} \in M_x$, so by our assumption, $A_{\alpha_\eta} \in J_{\beta(\eta)}$.

Now, $(A_\gamma \mid \gamma < \kappa)$ is J_{κ^+}-definable, so $(A_\gamma \mid \gamma < \kappa) \in N_0 \subseteq N_\eta$, so $(A_\gamma \mid \gamma < \eta) = \sigma_\eta((A_\gamma \mid \gamma < \kappa)) \in J_{\beta(\eta)}$. Thus

$$X = \{\gamma \in \alpha_\eta \mid A_\gamma = \gamma \cap A_{\alpha_\eta}\} \in J_{\beta(\eta)}.$$

Suppose that

$$\models_{J_{\beta(\eta)}} \text{“}X \text{ is stationary in } \alpha_\eta\text{”}.$$

Set $\bar{X} = \sigma_\eta^{-1}(X)$. Since $\sigma_\eta^{-1}\colon J_{\beta(\eta)} \prec J_{\kappa^+}$, we have

$$\models_{J_{\kappa^+}} \text{“}\bar{X} \text{ is stationary in } \kappa\text{”}.$$

Thus $\bar{X}$ really is stationary in κ. (Because $\mathscr{P}(\kappa) \subseteq J_{\kappa^+}$.)

Again, by absoluteness,

$$\models_{J_{\beta(\eta)}} \text{“}X = \{\gamma \in \alpha_\eta \mid A_\gamma = \gamma \cap A_{\alpha_\eta}\}\text{”}.$$

So if we set $A = \sigma_\eta^{-1}(A_{\alpha_\eta})$, we have, since $\sigma_\eta^{-1}\colon J_{\beta(\eta)} \prec J_{\kappa^+}$,

$$\models_{J_{\kappa^+}} \text{“}\bar{X} = \{\gamma \in \kappa \mid A_\gamma = \gamma \cap A\}\text{”}.$$

Thus it really is the case that

$$\bar{X} = \{\gamma \in \kappa \mid A_\gamma = \gamma \cap A\}.$$

But we assumed that no A exists for which such a set $\bar{X}$ is stationary. This contradiction proves that

$$\vDash_{J_{\beta(\eta)}} \text{“}X \text{ is not stationary in } \alpha_\eta\text{”}.$$

So there is a set $C \in J_{\beta(\eta)}$ such that

$$\vDash_{J_{\beta(\eta)}} \text{“}C \text{ is club in } \alpha_\eta \text{ and } (\forall \gamma \in C)(A_\gamma \neq \gamma \cap A_{\alpha_\eta})\text{”}.$$

Let $\bar{C} = \sigma_\eta^{-1}(C)$. By $\sigma_\eta^{-1}\colon J_{\beta(\eta)} \prec J_{\kappa^+}$, we get

$$\vDash_{J_{\kappa^+}} \text{“}\bar{C} \text{ is club in } \kappa \text{ and } (\forall \gamma \in \bar{C})(A_\gamma \neq \gamma \in A)\text{”}$$

By absoluteness, $\bar{C}$ is thus a club subset of κ such that $(\forall \gamma \in \bar{C})(A_\gamma \neq \gamma \cap A)$.

Now, $\bar{C} \cap \alpha_\eta = C$ (because $\sigma_\eta^{-1} \restriction \alpha_\eta = \mathrm{id} \restriction \alpha_\eta$ and $\sigma_\eta^{-1}(\alpha_\eta) = \kappa$) and, by absoluteness from $J_{\beta(\eta)}$, C is club in α_η, so as $\bar{C}$ is closed in κ, we have $\alpha_\eta \in \bar{C}$. Thus $A_{\alpha_\eta} \neq \alpha_\eta \cap A$. But $\alpha_\eta \cap A = \sigma_\eta(A) = A_{\alpha_\eta}$ (by the two properties of σ_η^{-1} just mentioned), so we have a contradiction. The proof is complete. □

Exercises

1. Weakly Compact Cardinals and Set mappings

A *set mapping* is (for our purposes) a function $f\colon [\kappa]^n \to \kappa$ (for some $n \in \omega$) such that $f(\sigma) \notin \sigma$ for all $\sigma \in [\kappa]^n$. A set $X \subseteq \kappa$ is said to be *free* for such a set mapping if $f''[X]^n \cap X = \emptyset$. We write $(\kappa, n) \to \lambda$ if every set mapping $f\colon [\kappa]^n \to \kappa$ has a free set of cardinality λ.

1 A. Prove that if κ is weakly compact, then $(\kappa, n) \to \kappa$ for all $n \in \omega$.

1 B. Prove that if $V = L$, then κ is weakly compact iff κ is uncountable and regular and either $(\kappa, 2) \to \kappa$ or else $(\kappa, n) \to \kappa$ for all $n \in \omega$.

2. Weakly Compact Cardinals and Colourings of Graphs

A *graph* is a structure $\mathscr{G} = \langle G, E\rangle$, where G is a non-empty set, called the set of *vertices* of $\mathscr{G}$, and E is a set of pairs from G, called the set of *edges* of $\mathscr{G}$. If $\{x, y\} \in E$, we say that x and y are *joined* in $\mathscr{G}$. A *subgraph* of $\mathscr{G}$ is a substructure of $\mathscr{G}$ in the usual sense. If $H \subseteq G$, $\mathscr{G} \restriction H$ denotes the subgraph of $\mathscr{G}$ with domain H. We say $\mathscr{G} \restriction H$ is *small* if $|H| < |G|$.

Let $\mathscr{G} = \langle G, E\rangle$ be a graph, μ a cardinal. A mapping $h\colon G \to \mu$ is called a *μ-colouring* if $h(x) \neq h(y)$ whenever x and y are joined in $\mathscr{G}$. The least μ for which $\mathscr{G}$ has a μ-colouring is called the *chromatic number* of $\mathscr{G}$.

A basic question of graph theory is: how is the chromatic number of a graph $\mathscr{G}$ effected by the chromatic number of its small subgraphs? By $P(\kappa)$, let us mean the following assertion: if $\mathscr{G}$ is a graph of cardinality κ, all of whose small subgraphs have countable chromatic number, then $\mathscr{G}$ has countable chromatic number.

2A. Prove that if κ is weakly compact, then $P(\kappa)$ holds.

We shall prove that if $V = L$, then for uncountable regular κ, the converse to the above result is valid. Assume $V = L$ from now on. Let κ be an uncountable regular cardinal, not weakly compact. Let $E \subseteq \kappa$ be stationary and such that $E \cap \lambda$ is not stationary in λ for all limit ordinals $\lambda < \kappa$, with $\mathrm{cf}(\alpha) = \omega$ for all $\alpha \in E$. Assume that $\beta + \omega < \alpha$ whenever $\beta < \alpha \in E$. Let $(B_n^\alpha \mid n < \omega)$ be a partition of α, for each $\alpha \in E$, such that whenever $(B_n \mid n < \omega)$ is a partition of κ, the set

$$\{\alpha \in E \mid \mathrm{cf}(\alpha) = \omega \wedge (\forall n \in \omega)(B_n \cap \alpha = B_n^\alpha)\}$$

is stationary in κ. For $\alpha \in E$, let A_α be a cofinal ω-sequence in α, chosen so that

$$\forall n [B_n^\alpha \text{ unbounded in } \alpha \rightarrow A_\alpha \cap B_n^\alpha \neq \emptyset] .$$

Let $\mathscr{G}$ be the graph with domain κ, in which two points $v < \alpha$ are joined iff $\alpha \in E$ and $v \in A_\alpha$.

2B. Prove that $\mathscr{G}$ has chromatic number at least ω_1.

2C. Prove that, for any $\lambda < \kappa$, there is an enumeration $(x_v \mid v < \theta)$ of λ such that the set of all $\eta < v$ for which x_η is joined to x_v is finite for all $v < \theta$, and use this to deduce that $\mathscr{G} \restriction \lambda$ has countable chromatic number.

3. $\mathrm{KH}(\kappa, \kappa)$ *for Singular* κ

3A. Assume GCH. Prove that if κ is singular and $\mathrm{cf}(\kappa) > \omega$, then whenever $\mathscr{F} \subseteq \mathscr{P}(\kappa)$ is such that the set $\{\lambda \in \kappa \mid |\{f \cap \lambda \mid f \in \mathscr{F}\}| \leqslant \lambda\}$ is stationary in κ, then $|\mathscr{F}| \leqslant \kappa$. (Hint: Work on a club subset of κ of order-type $\mathrm{cf}(\kappa)$ and use Fodor's Theorem.)

By the above, assuming GCH, $\mathrm{KH}(\kappa, \kappa)$ fails for all singular cardinals κ of uncountable cofinality. The following exercises show that if $V = L$, $\mathrm{KH}(\kappa, \kappa)$ is valid for all singular cardinals κ of cofinality ω. We fix κ a singular cardinal of cofinality ω from now on.

3B. Assume GCH. Show that if $\mathscr{F}$ is a set of ω-sequences cofinal in κ, then for any uncountable set $X \subseteq \kappa$.

$$|\{f \cap X \mid f \in \mathscr{F}\}| \leqslant |X|.$$

By virtue of the above, in order to prove $\mathrm{KH}(\kappa, \kappa)$ assuming $V = L$, it suffices to construct (from $V = L$) a family $\mathscr{F}$ of κ^+ many ω-sequences cofinal in κ such that $|\{f \cap X \mid f \in \mathscr{F}\}| \leqslant \omega$ for all countable sets $X \subseteq \kappa$. Assume $V = L$ from now on. For each $x \in \mathscr{P}_\kappa(\kappa)$, let M_x be the smallest M such that $x \cup \{x\} \subseteq M \prec J_{\kappa^+}$. Let

$$\mathscr{F} = \{f \subseteq \kappa \mid \mathrm{otp}(f) = \omega \;\&\; \sup(f) = \kappa \;\&\; (\forall x \in \mathscr{P}_{\omega_1}(\kappa))[f \cap x \in M_x]\}.$$

The aim is to prove that $|\mathscr{F}| = \kappa^+$, which at once establishes $\mathrm{KH}(\kappa, \kappa)$, of course.

Let $(\kappa_n \mid n < \omega)$ be the $<_J$-least ω-sequence of cardinals cofinal in κ such that $\kappa_0 > \omega$. For each $n < \omega$, let N_n be the smallest $N \prec J_{\kappa^{++}}$ such that $\kappa_n \subseteq N$, and let $N = \bigcup_{n<\omega} N_n$.

3C. Prove that $N_0 \prec N_1 \prec \ldots \prec N_n \prec \ldots \prec N \prec J_{\kappa^{++}}$.

3D. Prove that $\kappa \subseteq N$ and that $N \cap \kappa^+ \in \kappa^+$.

Let

$$j: N \cong J_\varrho,$$

and for each $n < \omega$, set

$$\bar{N}_n = j'' N_n.$$

3E. Prove that $\bar{N}_0 \prec \bar{N}_1 \prec \ldots \prec \bar{N}_n \prec \ldots \prec J_\varrho$.

Let

$$j_n: \bar{N}_n \cong J_{\varrho_n},$$

and set

$$R = \{\varrho_n \mid n < \omega\}.$$

3F. Show that $R \notin N$. (Hint: We know that $(\kappa_n \mid n < \omega) \in N$. Then if $R \in N$, we get $\langle (J_{\varrho_n})_{n<\omega}, (j_n \circ j_m^{-1})_{m<n<\omega} \rangle \in N$. Thus $J_\varrho \in N$, a contadiction.)

Now assume, by way of contadiction, that $|\mathscr{F}| \leqslant \kappa$.

3G. Show that (under the above assumption) $\mathscr{F} \in J_\varrho$ and $\mathscr{F} \subseteq J_\varrho$.

3H. Obtain a contradiction with 3F by proving that $R \in \mathscr{F}$. (This is the challenging part of the exercise, and you are on your own from now on. Good luck!)

4. More on Weak Compactness in L

Prove Theorem 1.2′. (See Exercise VI.4.)