AXIOM SCHEMATA OF STRONG INFINITY IN AXIOMATIC SET THEORY

AZRIEL LÉVY

1. Introduction. There are, in general, two main approaches to the introduction of strong infinity assertions to the Zermelo-Fraenkel set theory. The arithmetical approach starts with the regular ordinal numbers, continues with the weakly inaccessible numbers and goes on to the ρ_0-numbers of Mahlo [4], etc. The model-theoretic approach, with which we shall be concerned, introduces the strongly inaccessible numbers and leads to Tarski's axioms of [14] and [15]. As we shall see, even in the model-theoretic approach we can use methods for expressing strong assertions of infinity which are mainly arithmetical. Therefore we shall introduce strong axiom schemata of infinity by following Mahlo [4,5,6,]. Using the ideas of Montague in [7] we shall give those axiom schemata a purely model-theoretic form. Also the axiom schemata of replacement in conjunction with the axiom of infinity will be given a similar form, and thus the new axiom schemata will be seen to be natural continuations of the axiom schema of replacement and infinity.

A provisional notion of a standard model, introduced in § 2, will be basic for our discussion. However, in § 5 it is shown that this definition cannot serve as a general definition for the notion of a standard model.

2. Standard models of set theories. For the forthcoming discussion we need the notion of a standard model of a set theory. A general principle which distinguishes between standard and non-standard models of set theory is not yet known. Nevertheless, a notion of a standard model for various set theories will be given here, but this will serve only as an ad-hoc principle and we shall see later that its general application is not justified.

The Zermelo-Fraenkel set theory is generally formalized in the simple applied first-order functional calculus, since this is the most natural language for a set theory. In that formulation the Zermelo-Fraenkel set theory has an infinite number of axioms. From that formulation one passes directly to a formulation of the Zermelo-Fraenkel set theory by a finite number of axioms in the non-simple applied first-order functional calculus (we shall denote functional variables with $p, p_1, p_2, \cdots$). The axioms of extensionality, pairing, sum-set, power-set and infinity are as in [2]. The changed axioms are

The axiom of subsets $(x)(\exists y)(z)(z \in y \equiv : z \in x \,.\, p(z))$

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The axiom of replacement
$(x, y, z,) (p(x, y) \,.\, p(x, z) : \supset y=z) \supset (x) (\exists\, y) (z) (z \in y \equiv (\exists\, u) (u \in x \,.\, p(u, z)))$.

The axiom of foundation $(\exists\, x)\, p(x) \supset (\exists\, x) (p(x) \,.\, (y) (y \in x \supset \sim p(y)))$.

If we regard as mathematical theorems of a theory Q formulated in the non-simple applied first order functional calulus only those theorems of Q which do not contain functional variables then it can be shown, by the method of *Rückverlegung der Einsetzungen* (compare [3], pp. 248-249) that the set of all the mathematical theorems of Q coincides with the set of all the theorems of the corresponding theory Q' formulated in the simple applied first order functional calculus (whose axioms are the axioms and the axiom schemata corresponding to the axioms of Q). Therefore Q and Q' could be regarded, from the mathematical point of view as the same theory. Nevertheless, we shall see that Q' is not obtained uniquely from Q if we disregard the actual axiomatic representation of Q.

We are interested in passing to set theories based on a finite set of axioms in the non-simple applied first-order functional calculus, since in this case we can define the notion of standard models for these theories in the sense of Henkin. A standard model of such a theory Q will be a model where the functional variables range over all the subsets of the universe set of the model. The statement that the universe u and the membership relation e (which are both taken to be sets) determine a standard model of Q can be easily formulated in set theory. This is done as follows: We take the conjunction of the axioms of Q and effect the following replacements[1]

$(x)($ by $(x)(x \in u \supset$ $\qquad$ $(\exists\, x)($ by $(\exists\, x)(x \in u.$

$x \in y$ by $\langle x\, y \rangle \in e$ $\quad$ $p_i(x_1, \cdots, x_{n_i})$ by $\langle x_1, \cdots, x_{n_i} \rangle \in f_i$

and

then we close the resulting formula with respect to the variables f_i by the prefix $(f_1, \cdots, f_j) (f_1 \subseteq u \,.\, \cdots \,.\, f_j \subseteq u : \supset$. Thus we obtain a formula which we shall denote with $Sm^Q(u, e)$.

Standard models for set theories for which $\langle x\, y \rangle \in e \equiv : y \in u \,.\, x \in y$ and $y \in u \,.\, x \in y : \supset x \in u$ are called standard complete models: $Scm^Q(u) \equiv : (y)(y \in u \supset y \subseteq u) \,.\, (e) (\langle x\, y \rangle \in e \equiv : y \in u \,.\, x \in y : .\, Sm^Q(u, e))$.

We denote by S the set theory which consists of the axioms of extensionality, pairing, sum-set, power-set, subsets and foundation. SF will denote the theory obtained from S by adding to it the axiom of replacement. Z (resp. ZF) will denote the theory obtained from S (resp. SF) by the addition of the axiom of infinity (axiom VII* of [2]). We shall assume that these theories are formulated in the simple first-order

[1] Alphabetic change of bound variables may also be needed.

functional calculus unless we are dealing with standard models of these theories, in which case we shall assume that we have passed to corresponding formalizations in the non-simple first-order functional calculus.

By the methods of Shepherdson [12] 1.5 and Mostowski [9] it is easy to prove (in SF) that each standard model of a set theory Q which includes the axioms of extensionality and foundation is isomorphic to some standard complete model of Q.

The function $R(\alpha)$ is defined by $R(\alpha) = \sum_{\beta<\alpha} \mathfrak{P}(R(\beta))$ ($\mathfrak{P}(x)$ is the power-set of x). The rank of an element x of $R(\alpha)$ is defined to be the first β such that $x \in R(\beta)$. We shall assume in the following that the properties of these functions are known.[2]

We can prove, in the same way as Shepherdson [12] 3.14 and 3.3 [3] that if Q contains the axioms and the axiom schemata of SF then each standard complete model of Q is of the form $R(\alpha)$, where α is some limit number. Thus we can conclude that each standard model of a theory Q which contains the axioms and axiom schemata of SF is isomorphic to some standard complete model of Q of the form $R(\alpha)$. If we regard as assertions of infinity those statements which assert the existence of standard models for strong set theories, we see now why all assertions of infinity reduce to statements about the existence of ordinal numbers with appropriate properties.

The (strongly) inaccessible numbers α are usually defined as regular initial numbers greater than ω which satisfy $(\lambda)(\lambda < \alpha \supset \overline{\overline{2^{\lambda}}} < \alpha)$. This definition leads to the expected consequence only if the axiom of choice is assumed, since, for example, if the cardinal of the continuum is not an aleph then according to this definition no ordinal is inaccesible. Shepherdson [12] established the close connection between the inaccessible numbers and what we call the standard complete models of ZF. These results of Shepherdson can serve to give a new definition of inaccessible numbers which will have a satisfactory meaning even if the axiom of choice is not assumed.

DEFINITION 1. α is called inaccessible if $R(\alpha)$ is a standard complete model of ZF.

$In(\alpha) \equiv Scm^{ZF}(R(\alpha))$

Shepherdson [12] proves, in effect, that this definition is equivalent to the usual definition if the axiom of choice is assumed. Without using the axiom of choice it can be proved that α is inaccessible if and only if

(1) $\alpha > \omega$

(2) α is regular

(3) $(z)(z \in R(\alpha) \supset \sim \overline{\overline{z}} \geq \overline{\overline{\alpha}})$[4]

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[2] See, for example, Shepherdson [**12**] 3.2– he denotes the function R by G.

[3] Shepherdson's super-complete models are our standard complete models of ZF.

We shall widely use in the following the fact that every inaccessible number is regular (this is proved by Shepherdson [12] 3.42).

Definition 1 shows clearly why such a number is called inaccessible, i. e., unobtainable from the smaller ordinal numbers by means of the set theory ZF. Following Specker [13] we can generalize this definition as follows:

DEFINITION 2. Let Q be a set theory formulated by a finite number of axioms in the non-simple applied first-order functional calculus. An ordinal number α is called inaccessible with respect to Q if $R(\alpha)$ is a standard complete model of Q.

$In^Q(\alpha) = Scm^Q(R(\alpha))$.

3. A strong axiom schema of infinity. The numbers inaccessible with respect to ZF are the inaccessible numbers. The numbers inaccessible with respect to the theory obtained from ZF by addition of the axiom $(\exists\, \sigma)\, In(\sigma)$ are all the inaccessible numbers except the first one. Thus we can go on and observe numbers inaccessible with respect to systems which require the existence of more and more inaccessible numbers. We can also observe the numbers inaccessible with respect to the extension of ZF which is obtained by adding $(\mu)\, (\exists\, \sigma)\, (\sigma > \mu\, .\, In(\sigma))$ to its axioms, etc. But if we want to have a really fast trip into the realm of infinity we shall use the means provided by the arithmetical approach to assertions of infinity.

Mahlo [4] defined a function $\pi_{\alpha,\beta}$ such that $\pi_{\alpha,0}$ counts the regular ordinal numbers, $\pi_{\alpha,1}$ counts the weakly inaccessible number and for increasing β $\pi_{\alpha,\beta}$, regarded as a function of α, counts ordinals which satisfy higher and higher requirements of weak inaccessibility. The whole hierarchy of Mahlo [4] is based on the class [5] of the regular numbers — the range of $\pi_{\alpha,0}$. If we replace $\pi_{\alpha,0}$ by a function $\pi'_{\alpha,0}$ whose range is a subclass of the class of the regular numbers we can define analogously functions $\pi'_{\alpha,\eta}$ and $\pi'_{\alpha,\eta,\xi}$ and prove theorems corresponding to Mahlo's theorems in [4, 5, 6,]. We shall take for the range of $\pi'_{\alpha,0}$ the class of the inaccessible numbers.

Our exposition will differ from Mahlo's also in a technical point: Whereas Mahlo uses any strictly increasing functions to count the members of given classes of ordinal numbers we shall use for this purpose normal functions (*Normalfunktionen*) [6] which are much easier to handle. A normal function at limit-number arguments may take values

[4] Since we do not assume that the cardinal numbers are formally defined $\sim \overline{\overline{z}} \geqq \overline{\overline{\alpha}}$ is an abbreviation of a statement about equivalence of sets.

[5] We shall use the word '*class*' instead of the word '*property*', e.g., instead of '*the property of being a regular number*' we shall say '*the class of the regular numbers*'.

outside the class whose members it counts, since the normal function counts the members of the closure (in the order topology) of the given class.

DEFINITION 3. The functions $P_\eta(\alpha)$[7] are defined by transfinite induction as follows: $P_0(0)$ is the first inaccessible number; $P_0(\beta+1)$ is the first inaccessible number greater than $P_0(\beta)$; for limit-number α $P_0(\alpha) = \lim_{\beta<\alpha} P_0(\beta)$. $P_\eta(\beta+1)$ (resp. $P_\eta(0)$) is the first inaccessible number σ greater than $P_\eta(\beta)$ (resp. the first inaccessible number) such that for each $\eta' < \eta$ $\sigma = P_{\eta'}(\gamma)$ for some limit number γ.

The functions $P_\eta(\alpha)$ are not assumed to be defined for evey η and α.

DEFINITION 4. $Q(\beta+1)$ (resp. $Q(0)$) is the first inaccessible number α greater than $Q(\beta)$ (resp. the first inaccessible number) such that $P_\alpha(0) = \alpha$. For a limit-number α $Q(\alpha) = \lim_{\beta<\alpha} Q(\beta)$.

We can also define functions $Q_\eta(\alpha)$ such that $Q_0(\alpha) = Q(\alpha)$, $Q_{\beta+1}$ is related to Q_β as Q_0 is related to P_0 and for limit-ordinal η Q_η counts the inaccessible numbers which are in the intersection of the ranges of all the functions $Q_{\eta'}$, $\eta' < \eta$. The numbers α for which $Q_\alpha(0) = \alpha$ we call Q^*-numbers.

We shall now consider the following axiom schema

M *Every normal function defined for all ordinals (d.f.a.o.) has at least one inaccessible number in its range*[8]

THEOREM 1. *M is equivalent to each of the following schemata*

M' *Every normal function d.f.a.o. has at least one fixed point which is inaccessible*

M'' *Every normal function d.f.a.o. has arbitrarily great fixed points which are inaccessible.*

Proof. Obviously M'' implies M' and M' implies M. We shall prove that M implies M''.

Let F be a normal function *d.f.a.o.* Let G be the derivative of F, i.e., the normal function which counts the fixed points of F. Since F is *d.f.a.o.* then by [1] § 8 G is also *d.f.a.o.* For any given γ let $H_\gamma(\xi) =$

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6 A function $F(\alpha)$ on the ordinal numbers into the ordinal numbers is called normal if:
(1) It is strictly increasing: $\alpha < \beta \supset F(\alpha) < F(\beta)$
(2) It is continous: For limit-number α $F(\alpha) = \lim_{\beta<\alpha} F(\beta)$.

7 These are the functions analogous to the functions $\pi_{\alpha,\eta}$ of Mohlo [4].

8 This schema is written formally as
$(\alpha,\beta,\gamma)(\varphi(\alpha,\beta) . \varphi(\alpha,\gamma) : \supset \beta = \gamma) . (\alpha)(\exists \beta)\varphi(\alpha,\beta) . (\alpha,\beta,\gamma,\delta)(\alpha < \gamma . \varphi(\alpha,\beta) . \varphi(\gamma,\delta) : \supset \beta < \delta) . (\alpha,\beta)(\sim(\exists \sigma)(\sigma + 1 = \alpha) . \alpha \neq 0 . \varphi(\alpha,\beta) : \subset (\gamma)(\gamma < \beta \supset (\exists \delta,\eta)(\delta < \alpha . \varphi(\delta,\eta) . \eta > \gamma))) : \supset (\exists \alpha,\beta)(\varphi(\alpha,\beta) . \mathrm{In}(\beta))$ where φ is a formula of set theory such that there is no confusion of variables in the corresponding instance of the schema.

$G(\gamma+\xi)$. H_γ is a normal function *d.f.a.o.* and hence by M there is an ordinal ξ such that $\beta = H_\gamma(\xi)$ is inaccessible. Since $\beta = G(\gamma+\xi)$, $F(\beta) = \beta$. By a well-known theorem the value of a normal function is not less than the argument and hence $\beta \geq \gamma+\xi \geq \gamma$.

In order to see how near M is to a purely arithmetical assertion it is interesting to note that M is equivalent to the conjunction of

(1) *There exist arbitrarily great inaccessible numbers*

(2) *Every normal function d.f.a.o. has at least one regular number in its range*

The proof of this makes use of the fact that every regular ordinal which is the limit of a set of inaccessible numbers is inaccessible (since an ordinal is inaccessible if and only if it is regular, greater than ω and $(z)(z \in R(\alpha) \supset \sim \bar{\bar{z}} \geq \bar{\alpha}))$. Let F be any normal function *d.f.a.o.* If there exist arbitrarily great inaccessible numbers then the function $P_0(\alpha))$ is *d.f.a.o.* and also the normal function $F(P_0(\alpha))$ is *d.f.a.o.* By (2), using the same reasoning as in Theorem 1, there is a regular ordinal β such that $F(P_0(\beta)) = \beta$, i.e., $P_0(\beta) = \beta$ and $F(\beta) = \beta$. Since β is a limit number and $P_0(\beta)=\beta$, β is the limit of a sequence of inaccessible numbers and since β is regular it is inaccessible.

ZM will denote the set theory obtained from ZF by the addition of M.

We shall now introduce a principle of reflection over ZF. This will be an axiom schema which will assert the existence of standard complete models of ZF which reflect in some sense the situation of the universe.

Let φ be a formula of set theory. We denote by $\mathrm{Rel}(u, \varphi)$ the formula obtained from φ by relativizing all the quantifiers in it to u, i.e., by replacing each occurrence $(z)\chi$ or $(\exists z)\chi$ by $(z)(z \in u \supset \chi)$ or $(\exists z)(z \in u \,.\, \chi)$, respectively.[9]

The principle of complete reflection over ZF

N $(\exists u)(Scm^{ZF}(u) \,.\, (x_1, \cdots, x_n)(x_1, \cdots x_n \in u \supset . \varphi \equiv \mathrm{Rel}(u, \varphi)))$
where φ is any formula which has no free variables except $x_1 \cdots, x_n$.

As seen from the formulation of N, it is closely connected with the notion of an arithmetical extension of Tarski and Vaught [17]. In the proofs of Theorems 2, 3, 5 and 6 we shall use the methods used by Montague and Vaught [8] for arithmetical extensions.

We shall see now that another principle of reflection, which seems at first sight to be stronger than N is equivalent to N.

THEOREM 2. *N is equivalent in S to the following schema*

[9] If φ contains u bounded then u is replaced in φ before the relativization by the first variable, in alphabetic order, which does not occur in φ.

N' $(\exists u)(z \in u \,.\, Scm^{ZF}(u).\ \ (x_1, \cdots, x_n)(x_1, \cdots, x_n \in u \supset :$
$\varphi_1 \equiv \mathrm{Rel}(u, \varphi_1). \cdots\cdots. \varphi_m \equiv \mathrm{Rel}(u, \varphi_m)))$
where m is any natural number and φ_i, $1 \leq i \leq m$, is a formula which has no free variables except $x_1, \cdots, x_n$.

Proof. Obviously N' implies N. Now we assume N and we shall prove first the schema N'' which is like N', only that it does not contain the part $z \in u$. Let $\emptyset$ be the formula $\bigvee_{i=1}^m t = i \,.\, \varphi_i$. Since the natural numbers $1, 2, \cdots, m$ are absolute with respect to standard complete models (see, e.g., [12] 2.320) we have $Scm(u) \supset :\, .\, \mathrm{Rel}(u, \emptyset) \equiv :$ $\bigvee_{i=1}^m t = i \,.\, \mathrm{Rel}(u, \varphi_i)$. We use now N with respect to $\emptyset$ and we obtain the existence of a set u such that $Scm^{ZF}(u)$ and

$$(t)(x_1, \cdots, x_n)(t, x_1, \cdots, x_n \in u \supset \therefore \bigvee_{i=1}^m t = i \,.\, \varphi_i$$
$$: \equiv : \bigvee_{i=1}^m t = i \,.\, \mathrm{Rel}(u, \varphi_i))$$

From $Scm^{ZF}(u)$ we can prove easily by induction that $\omega \subseteq u$, and therefore, substituting j for t in the above formula, $1 \leq j \leq m$, we get $x_1, \cdots, x_n \in u \supset .\, \varphi_j \equiv \mathrm{Rel}(u, \varphi_j)$, and thus we have proved N''. Now we shall prove N' from N''.

Given $\varphi_1, \cdots, \varphi_m$ we denote

$$\varphi_{m+1} = (\exists u)\Big(z \in u \,.\, Scm^{ZF}(u) \,.\, (x_1, \cdots, x_n)\Big(x_1, \cdots, x_n \in u \supset .$$
$$\bigwedge_{i=1}^{m} \varphi_i \equiv \mathrm{Rel}(u, \varphi_i)\Big)\Big)$$
$$\varphi_{m+2} = (z)\, \varphi_{m+1}$$

We use now N'' for $\varphi_1, \cdots, \varphi_{m+2}$. Thus we have the existence of u such that $Scm^{ZF}(u)$ and

$$x_1, \cdots, x_n \in u \supset .\, \varphi_i \equiv \mathrm{Rel}(u, \varphi_i) \qquad 1 \leq i \leq m \tag{3}$$
$$z \in u \supset .\, \varphi_{m+1} \equiv \mathrm{Rel}(u, \varphi_{m+1}) \tag{4}$$
$$\varphi_{m+2} \equiv \mathrm{Rel}(u, \varphi_{m+2})\,. \tag{5}$$

By $Scm^{ZF}(u)$ and (3) we have $(z)(z \in u \supset \varphi_{m+1})$, and hence, by (4), also $(z)(z \in u \supset \mathrm{Rel}(u, \varphi_{m+1}))$; but the latter formula is $\mathrm{Rel}(u, \varphi_{m+2})$ and hence, by (5), we have φ_{m+2}, which is the instance of N' corresponding to $\varphi_1, \cdots, \varphi_m$.

We note that Theorem 2 will remain valid if ZF is replaced in both N and N' by S or by any extension of S.

THEOREM 3. *In ZF the schema M is equivalent to the schema N and to the following schema*

N''' $(\exists\, \alpha)\, (In(\alpha)\, .\, (x_1, \cdots, x_n)\, (x_1, \cdots, x_n \in R(\alpha) \supset .\, \varphi \equiv \mathrm{Rel}\,(R(\alpha),\ \varphi)))$ *where* φ *is any formula which has no free variables except* $x_1, \cdots, x_n$.

Proof. As we have already mentioned in § 2 all the standard complete models of ZF are of the form $R(\alpha)$. Hence, by Definition 1, N and N''' are equivalent.

We shall now prove M from N'''. Let $\varphi(x, y\,; x_1, \cdots, x_n)$ be any formula. Let $\chi(x_1, \cdots, x_n)$ be the formula asserting that if $\varphi(\xi, \eta : x_1, \cdots, x_n)$ gives a function $\eta = F(\xi)$ which is normal and *d.f.a.o.* then $F(\xi)$ has at least one inaccessible number in its range. From N''' we shall pass, as in Theorem 2, to a schema which is like N''' only that $\varphi \equiv \mathrm{Rel}\,(R(\alpha), \varphi)$ is replaced by $\bigwedge_{i=1}^{4} \varphi_i \equiv \mathrm{Rel}\,(R(\alpha), \varphi_i)$. We shall take $\varphi_1 \equiv \varphi$, $\varphi_2 \equiv (\xi)\, (\exists\, \eta)\, \varphi\, (\xi, \eta)$, $\varphi_3 \equiv \chi$, $\varphi_4 \equiv (x_1, \cdots, x_n)\, \chi$. By the corresponding instance of that schema there exists an inaccessible number α such that

(6) $x_1, \cdots, x_n, x, y \in R\,(\alpha) \supset .\, \varphi \equiv \mathrm{Rel}\,(R(\alpha),\ \varphi)$
(7) $x_1, \cdots, x_n \in R\,(\alpha) \supset .\, (\xi)\, (\exists\, \eta)\, \varphi\, (\xi, \eta) \equiv \mathrm{Rel}\,(R(\alpha),\ (\xi)\, (\exists\, \eta)\, \varphi\, (\xi, \eta))$
(8) $x, \cdots, x_n \in R(\alpha) \supset .\, \chi \equiv \mathrm{Rel}\,(R(\alpha), \chi)$
(9) $(x_1, \cdots, x_n\,)\, \chi \equiv \mathrm{Rel}\,(R(\alpha),\ (x_1, \cdots, x_n)\chi)$.

We shall now assume that for certain $x_1, \cdots, x_n \in R(\alpha)$ $\varphi(\xi, \eta)$ gives a function $\eta = F(\xi)$ which is normal and *d.f.a.o.* The relativization of an ordinal-number-variable μ to the set $R(\alpha)$ is $\mu < \alpha$ (see Shepherdson [12] 2.316) and thus, since we assume the left-hand side of (7), we get

$$(\xi)\, (\xi < \alpha \supset (\exists\, \eta)\, (\eta < \alpha\, .\, \mathrm{Rel}\,(R(\alpha)\,,\ \varphi(\xi\,, \eta))))$$

and by (6) we have $(\xi)\, (\xi < \alpha \supset (\exists\, \eta)\, (\eta < \alpha\, .\, \varphi\, (\xi, \eta)))$. Since F is normal and α is a limit number we have $F(\alpha) = \alpha$, thus proving $x_1, \cdots, x_n \in R\,(\alpha) \supset \chi\, (x_1, \cdots, x_n)$. By (8) we have $x_1 \cdots, x_n \in R\,(\alpha) \supset \mathrm{Rel}\,(R(\alpha),\ \chi(x_1, \cdots, x_n))$ which is $\mathrm{Rel}\,(R(\alpha), (x_1, \cdots, x_n)\, \chi)$ and hence, by (9), we have $(x_1, \cdots, x_n)\, \chi$, thus proving M.

Now we shall prove N from M. In this we shall make use of ideas of Montague in [7]. Let φ be any formula of set theory. We write φ in prenex normal form. Let φ be of the form $(y)\, (\exists\, z)\, (u)\, (\exists\, t)\, \varphi^*$ where φ^* does not contain any quantifiers, and let φ have the two free variables x_1, x_2. For formulae φ of any other structure the treatment is analogous to the treatment of this case.

Given any x_1, x_2, y let $F_1(x_1, x_2, y)$ be the set of all the sets z which satisfy $(u)\, (\exists\, t)\varphi^*$ and which are of minimal rank among the sets satisfying this requirement. If there are z's satisfying $(u)\, (\exists\, t)\, \varphi^*$ then by the axiom of foundation they have certain ranks and hence $F_1(x_1, x_2, y) \neq 0$, otherwise $F_1(x_1, x_2, y) = 0$. $F_1(x_1, x_2, y)$ is a set since it is the subset of some set $R(\alpha)$ or it is the void set. Given any x_1, x_2, y, z, u we denote by $F_2(x_1, x_2, y, z, u)$ the set of all the sets t which satisfy φ^* and which are of the least rank among the sets t satisfying φ^*. $\sim \varphi = (\exists\, y)\,(z)\,(\exists\, u)\,(t) \sim \varphi^*$. We define for this formula corresponding functions $F_3(x_1, x_2)$ and $F_4(x_1, x_2, y, z)$.

$$H(x) = x + \sum_{x_1, x_2, y \in x} F_1(x_1, x_2, y) + \sum_{x_1, x_2 y, z, u \in x} F_2(x_1, x_2, y, z, u) + \sum_{x_1, x_2 \in x} F_3(x_1, x_2) + \sum_{x_1, x_2, y, z \in x} F_4(x_1, x_2, y, z)\,.$$

Let ξ be the rank of the set x, then $x \subseteq R(\xi - 1)$ (ξ cannot be a limit-number). Let us define

$$J(x) = R(\xi - 1), \quad K(x) = J(H(x)), \quad P(x) = \sum_{n \in \omega} K^n(x)\,.$$

It follows immediately from the definition of $P(x)$ that

$$x_1, x_2, y, z, u \in P(x) \supset : F_1(x_1, x_2, y) \subseteq P(x) \,.\, F_2(x_1, x_2, y, z, \mathrm{u}) \subseteq P(x) \,.\, F_3(x_1, x_2) \subseteq P(x) \,.\, F_4(x_1, x_2, y, z) \subseteq P(x)\,.$$

Denote

$$\varnothing(s) \equiv (x_1, x_2, y, z, u)\,(x_1, x_2, y, z, u \in s \supset : F_1(x_1, x_2, y) \subseteq s \,.\, F_2(x_1, x_2, y, z, u) \subseteq s \,.\, F_3(x_1, x_2) \subseteq s \,.\, F_4(x_1, x_2, y, z) \subseteq s).$$

Assume $\varnothing(s)$. We shall see that $x_1, x_2 \in s \supset . \varphi \equiv \mathrm{Rel}(s, \varphi)$. We have

$$\mathrm{Rel}\,(s, \varphi) \equiv (y)\,(y \in s \supset (\exists\, z)(z \in s \,.\, (u)\,(u \in s \supset (\exists\, t)\,(t \in s \,.\, \varphi^*))))$$

and by definition of $F_1 - F_4$

(10) $$(x_1, x_2, y, z, u, t)\,(x_1, x_2, y \in s \,.\, z \in F_1(x_1, x_2, y) \,.\, u \in s \,.\, t \in F_2(x_1, x_2, y, z, u) : \supset \varphi^*)$$

(11) $$(x_2, x_2, y, z, u, t)\,(x_1, x_2 \in s \,.\, y \in F_3\,(x_1, x_2) \,.\, z \in s \,.\, u \in F_4(x_1, x_2, y, z) \,.\, t \in s : \supset \sim \varphi^*)$$

If φ holds for $x_1, x_2 \in s$ then $F_1(x_1, x_2, y) \neq 0$ and $F_2(x_1, x_2, y, z, u) \neq 0$ for $y \in s$, $z \in F_1(x_1, x_2, y)$ and $u \in s$ and hence by (10) $\mathrm{Rel}\,(s, \varphi)$ holds for x_1, x_2. If $\sim \varphi$ holds for x_1, x_2 then we have by (11), in the same way, that $\mathrm{Rel}\,(s, \sim \varphi)$ holds, i.e., $\sim \mathrm{Rel}\,(s, \varphi)$ holds.

Since we have always $\sum_{\nu < \mu} R(\alpha_\nu) = R(\sup_{\nu < \mu} \alpha_\nu)$ and by the definition of the function K $K^n(x)$ is of the form $R(\beta)$ also $P(x) = \sum_{n \in \omega} K^n(x)$ is equal to $R(\alpha)$ for some α. Since $\varnothing(P(x))$ we have $\varnothing(R(\alpha))$. If we want α to be greater than μ it is enough to take $x = \{\mu\}$ and by $x \subseteq P(x)$ we have $\mu \in P(x) = R(\alpha)$, i.e., $\mu < \alpha$. Now let F be the normal function counting, in the order of their magnitude, the ordinals α which satisfy $\varnothing(R(\alpha))$. Since we have arbitrarily great ordinals α satisfying $\varnothing(R(\alpha))$ F is *d.f.a.o.* For ξ which is not a limit-number we have $\varnothing(R(F(\xi)))$. Let η be a limit-number, and let $x_1, x_2, y, z, u \in R(F(\eta))$. Let γ be the maximum of the ranks of x_1, x_2, y, z, u. Since η is a limit-number $F(\eta)$ is also a limit-number and therefore $\gamma < F(\eta)$. Since $F(\eta) = \lim_{\xi<\eta} F(\xi)$ there is an ordinal ξ, $\xi + 1 < \eta$, such that $\gamma \leq F(\xi + 1) < F(\eta)$, and hence $x_1, x_2, y, z, u \in R(F(\xi + 1))$. But, as we have already mentioned, $\varnothing(R(F(\xi + 1)))$ holds and therefore $F_1(x_1, x_2, y) \subseteq R(F(\xi + 1)) \subseteq R(F(\eta))$ and the same holds for $F_2 - F_4$. Thus we have proved $\varnothing(R(F(\eta)))$ also for limit-number η, hence $(\eta)\, \varnothing\, (R(F(\eta)))$.

By M the function $F(\eta)$ has in its range an inaccessible number α. Therefore we have $\varnothing(R(\alpha))$ and hence

$$(x_1, x_2)\,(x_1, x_2 \in R(\alpha) \supset . \,\varphi \equiv \mathrm{Rel}\,(R(\alpha), \varphi)).$$

N follows from Definition 1.

THEOREM 4. *In ZM it is provable that all the functions P_η are d.f.a.o. as well as the function Q.*

Proof. Let η be the least ordinal such that P_η is not *d.f.a.o.* and let α be the least ordinal for which $P_\eta(\alpha)$ is not defined. α cannot be a limit-number, since in that case $P_\eta(\alpha) = \lim_{\beta<\alpha} P_\eta(\beta)$. Let us "define" $P_\eta(\alpha)$ to be the class of all the ordinal numbers. By exactly the same arguments as those in the proof of Theorem 2 of Mahlo [4] (for the case $\alpha = \pi_{\mu,\nu}\,\mu, \nu < \alpha$) we can define a normal function "converging to $P_\eta(\alpha)$" which does not have inaccessible values at limit-number arguments, i.e., we have a normal function *d.f.a.o.* which does not satisfy M'. Now that we proved that for each η P_η is *d.f.a.o.* Let $Q(0)$ be undefined. As in the former case we "define" $Q(0)$ to be the class of all ordinals and use the arguments in the proof of Theorem 2 of Mahlo [4] (for the case of the least ξ such that $\xi = \pi_{1,\xi}$) to construct a normal function *d.f.a.o.* which does not satisfy M'. In the same way we prove, by transfinite induction, the existence of $Q(\alpha)$ for each α.

Arguments which are very similar to those of Theorem 4 can be used in order to prove in ZM that all the functions Q_η are *d.f.a.o.* as well as the normal function counting the Q^*-numbers, and so on.

4. **An hierarchy of set theories.** In analogy with Mahlo [4] we can give axioms of infinity stronger than M.

DEFINITION 5. α is call a hyper-inaccessible number of type 1 if it is inaccessible with respect to ZM, i.e., if it is inaccessible and each normal function whose domain is α and whose range is included in α

has at least one inaccessible number in its range. α is hyper-inaccessible of type $\mu+1$ if it is inaccessible and each normal function whose domain is α and whose range is included in α has at least one hyper-inaccessible number of type μ in its range. For a limit-number μ α is hyper-inaccessible of type μ if it is hyper-inaccessible of type λ for every $\lambda < \mu$.[10]

It follows immediately from Definition 5 that if α is hyper-inaccessible of type μ it is also hyper-inaccessible of type λ for every $\lambda < \mu$.

Let Λ be a definite ordinal number. To avoid going into details we assume that existence and uniqueness of Λ are provable in ZF and also that it is provable in ZF that the definition of Λ is absolute with respect to standard complete models of ZF. Observe the following axiom schema:

M_Λ (for $\Lambda \geq 2$) *Every normal function d.f.a.o. has for every $\mu < \Lambda$ at least one hyper-inaccessible number of type μ in its range.*

Obviously we have that if $ZF \vdash \Lambda < M$ then M_M implies M_Λ. Let ZM_Λ denote the theory obtained from ZF by addition of M_Λ. By Definition 5 α is a hyper-inaccessible number of type Λ if and only if $R(\alpha)$ is a standard complete model of M_Λ (here we use the absoluteness of Λ with respect to standard complete models of ZF).

In complete analogy to Theorem 3 we have:

THEOREM 5. *M_Λ is equivalent in ZF to the schemata*

N_Λ''' $(\mu)(\mu < \Lambda \supset (\exists \alpha)(\alpha$ *is hyper-inaccessible of type* $\mu \,.\, (x_1, \cdots, x_n)$ $(x_1, \cdots, x_n \in R(\alpha) \supset .\, \varphi \equiv \mathrm{Rel}(R(\alpha), \varphi))))$
where φ is any formula which has no free variables except $x_1, \cdots, x_n$.

and

N_Λ $(\mu)(\mu < \Lambda \supset (\exists u)(Scm^{ZM_\mu}(u)\,.$
$(x_1, \cdots, x_n)(x_1, \cdots, x_n \in u \supset .\, \varphi \equiv \mathrm{Rel}(u, \varphi))))$ *where φ is any formula which has no free variables except $x_1, \cdots, x_n$.*

By $Scm^{ZM_\mu}(u)$ we mean that u is a standard complete model of an axiom system like ZM_Λ only that in ZM_μ μ is taken as a parameter. Thus $Scm^{ZM_\mu}(u)$ is a formula with the two free variables μ and u.

By replacing Λ by $\Lambda+1$ in Theorem 5 we obtain easily that $M_{\Lambda+1}$ is equivalent to the schemata

$(\exists \alpha)(\alpha$ *is hyper-inaccessible of type* $\Lambda\,.$
$(x_1, \cdots, x_n)(x_1, \cdots, x_n \in R(\alpha) \supset .\, \varphi \equiv \mathrm{Rel}(\alpha), \varphi))$

[10] The hyper-inaccessible numbers of type 1 correspond to the ρ_0-numbers of Mahlo [4]. The hyper-inaccessible numbers of type λ correspond to the members of the range of $\pi_{\alpha,0,\lambda}$ of Mahlo [4].

and

$$(\exists u)(Scm^{ZM_A}(u) \,.\, (x_1, \cdots, x_n)(x_1, \cdots, x_n \in u \supset . \varphi \equiv \mathrm{Rel}(R(\alpha), \varphi))).$$

Now we shall see that the same relation which holds between ZF and ZM, and between ZM_A and ZM_{A+1} holds also between S and ZF.

THEOREM 6. *In S the axiom schema of replacement in conjunction with the axiom of infinity is equivalent to the schema*

N_0 $\quad (\exists u)(Scm^S(u) \,.\, (x_1, \cdots, x_n)(x_1, \cdots, x_n \in u \supset . \varphi \equiv \mathrm{Rel}(u, \varphi)))$
where φ *is a formula which does not contain free variables except* $x_1, \cdots, x_n$.

Proof. That N_0 is provable in ZF is Montague's theorem proved in [7] and it is proved by the same method as the corresponding part of Theorem 3.

Now we assume N_0 and prove the axioms of infinity and replacement. By N_0, taking any φ, we obtain $(\exists u)\, Scm^S(u)$. This u obviously satisfies the requirements of the axiom of infinity. Now, given $\varphi(v, w)$ with the only free variables $v, w, x_1, \cdots, x_n$ let χ denote the formula

$$(r, s, t)(\varphi(r, s) \,.\, \varphi(r, t) : \supset s = t) \supset (\exists y)(w)(w \in y \equiv (\exists v)(v \in x \,.\, \varphi(v, w)))$$

By N_0 we have, as in Theorem 2, that there exists a set u such that $Scm^S(u)$ and

(12) $\quad x_1, \cdots, x_n, v, w \in u \supset . \varphi \equiv \mathrm{Rel}(u, \varphi)$
(13) $\quad x_1, \cdots, x_n, v \in u \supset . (\exists w)\varphi \equiv \mathrm{Rel}(u, (\exists w)\varphi)$
(14) $\quad x_1, \cdots, x_n, x \in u \supset . \chi \equiv \mathrm{Rel}(u, \chi)$
(15) $\quad (x_1, \cdots, x_n)(x)\chi \equiv \mathrm{Rel}(u, (x_1, \cdots, x_n)(x)\chi)$

Since $\mathrm{Rel}(u,(\exists w)\varphi)$ is $(\exists w)(w \in u \,.\, \mathrm{Rel}(u, \varphi))$ we have by (12) and (13) $x_1, \cdots, x_n, v \in u \supset . (\exists w)(w \in u \,.\, \varphi) \equiv (\exists w)\varphi$; hence if $(r, s, t,)$ $(\varphi(r, s) \,.\, \varphi(r, t) : \supset s = t)$ then for $x \in u$, since $Scm^S(u)$ implies that then $x \subseteq u$, the function represented by $\varphi(v, w)$ maps the members of x on members of u, and therefore, by the axiom of subsets, that function maps x on some set y. Thus we have $x_1, \cdots, x_n, x \in u \supset \chi$ and by (14) $x_1, \cdots, x_n, x \in u \supset \mathrm{Rel}(u, \chi)$; but the closure of the latter formula is $\mathrm{Rel}(u, (x_1, \cdots, x_n)(x)\chi)$ and hence, by (15), we have χ.

By Theorem 6 we can view the axiom schemata M and M_A as natural continuations of the axioms of infinity and replacement. Therefore, although the consistency of ZF does not imply, even in ZM (if ZM is consistent), the consistency of ZM, it seems likely that if in the sequence $S, ZF, ZM, ZM_2, \cdots$ no inconsistency is introduced in the first step, from S to ZF, also no inconsistency is introduced in the further steps.

In the following definitions and statements we essentially follow Montague in [7].

Let the theory Q be an extension of the theory P. Let φ be any sentence of Q. $P+\{\varphi\}$ denotes the theory obtained from P by adding to it φ as a new axiom. $\mathrm{Con}\,(P+\{\varphi\})$ is the arithmetic sentence which asserts the consistency of $P+\{\varphi\}$. Q is called essentially reflexive over P if for every sentence φ of Q $\varphi \supset \mathrm{Con}\,(P+\{\varphi\})$ is a theorem of Q. Q is called an essentially infinite extension of P if no consistent extension of Q without new symbols is obtained from P by adding to it a finite number of axioms. If Q is essentially reflexive over P then Q is an essentially infinite extension of P. By the same argument as that of Montague in [7] each of the theories $S.\ ZF,\ ZM,\ \cdots$ is essentially reflexive over the preceding ones.

Let $E_{R(\alpha)}=\{\langle xy\rangle\,;\, x\in y\,.\, x, y\in R(\alpha)\}$, $A_\alpha=\langle R(\alpha), E_{R(\alpha)}\rangle$. Montague and Vaught proved in [8] that if $\beta<\alpha$ and $R(\alpha)$ is an arithmetical extension of $R(\beta)$ (i.e., for any formula φ with no free variables except $x_1, \cdots, x_n$

$$(x_1, \cdots, x_n)\,(x_1, \cdots, x_n \in R(\beta) \supset .\ \mathrm{Rel}\,(R(\alpha), \varphi) \equiv \mathrm{Rel}\,(R(\beta), \varphi)))$$

then both A_α and A_β are models of ZF (in the sense of models of the type S_4 of Tarski [16]).[11]

THEOREM 7. *If A_α and A_β are as mentioned above and β is inaccessible then both A_α and A_β are models of ZM. If β is hyper-inaccessible of type Λ then both A_α and A_β are models of $ZM_{\Lambda+1}$.*

The proof that A_α is a model as required is exactly like the second part of the proof of Theorem 3. A_β is also a model as required since if φ holds in A_α it holds in A_β.

Another aspect of the phenomena discovered by Montague and Vaught in [7] and [8] is the following theorem:

THEOREM 8. *Let Sb be a theory with the same language and axioms as S with the additional set-constant b and the additional axioms*

(16) $\quad Scm^S(b)$

(17) $\quad (x_1, \cdots, x_n)\,(x_1, \cdots, x_n \in b \supset .\ \varphi \equiv \mathrm{Rel}\,(b, \varphi))$ *where φ is any formula of S without free variables except $x_1, \cdots, x_n$.*

The theorems of Sb which do not contain the constant b are exactly the

[11] This and the following Theorem 7 can be read in two different ways. Either we take the theorems and proofs informally, in which case all the notions retain their verbal meaning; or that the theorems are taken to be formal theorems of S and then the notions of model and arithmetical extension are formal notions defined by means of the formal notion of satisfaction, which is given, for example, in Mostowski [**11**].

theorems of ZF; the theorems of $Sb + \{Scm^{ZF}(b)\}$ which do not contain b are exactly the theorems of ZM; and the theorems of $Sb + \{(\mu)(\mu < \Lambda \supset Scm^{ZM_\mu}(b))\}$ which do not contain b are exactly the theorems of ZM_Λ (the theorems of $Sb + \{Scm^{ZM_\Lambda}(b)\}$ which do not contain b are exactly the theorems of $ZM_{\Lambda+1}$).

Proof. Every theorem of ZF is provable in Sb since Sb contains the axioms of S and all the instances of N_0 are obviously provable in Sb. Now let the sentence χ be a theorem of Sb which does not contain b. Let $\varnothing(b)$ be the conjunction of all the instances of (16) and (17) used in the proof of χ. By the deduction theorem $\varnothing(b) \supset \chi$ is provable from the axioms of S, hence $(\exists u) \varnothing (u) \supset \chi$ is provable in S. But Montague's theorem (Theorem 6) $(\exists u) \varnothing (u)$ is a theorem of ZF, hence χ is provable in ZF.

The other statements of Theorem 8 follow in the same way from Theorems 3 and 5.

We see, by Theorem 8, that even though in the sequence $ZF, ZM, ZM_2, \cdots$ each theory is an essentially infinite extension of all the preceding ones we can get a corresponding sequence $Sb, Sb + \{Scm^{ZF}(b)\}, Sb + \{Scm^{ZM}(b)\}, \cdots$ in which the theories which are "almost the same" as the respective theories in the former sequence, and in which all the theories are obtained from the first one by the addition of respective single axioms.

5. **Peculiar behavior of models.** We shall now see examples illustrating the inadequacy for general use of the notion of standard model introduced in § 2. In our examples we shall use a formal satisfaction definition. The idea of using the formalized notion of satisfaction in these problems and the special way in which that notion is given here are due to Mostowski.[12] Our notations will be those of Mostowski [10].

Our first example will be an axiomatic representation ZF^* of ZF which has no standard model.

Let Φ_n be the nth formula in a given Gödelization of ZF. Given the functional variable $p(i, f)$ we shall construct a formula $\Psi(p)$ which asserts that $p(i, f)$ is a satisfaction definition.

$p(i, f)$ is a satisfaction definition if the following holds for every finite number i and every finite sequence of sets f:

(a) If Φ_i is the formula $x_k = x_j$ or $x_k \in x_j$ then $p(i, f)$ if and only if $D(f) = \{k, j\}$ and $f(k) = f(j)$ or $f(k) \in f(j)$, respectively.

(b) If $\Phi_i = \Phi_j \mid \Phi_h$ then
$p(i, f) \equi : D(f) = s_i \,.\sim p(j, f/s_j) \vee \sim p(h, f/s_h)$.

(c) If $\Phi_i = (\exists x_m) \Phi_j$ and x_m is free in Φ_j then $p(i, f) \equiv : D(f) = s_i \,. (\exists a)\, p(j, f + \{\langle ma \rangle\})$. If x_m is not free in Φ_j then $p(i, f) \equiv p(j, f)$.

[12] By oral communication.

This inductive definition can be replaced by an explicit one in the usual method and thus we get the required formula $\Psi(p)$ which asserts that p is a definition of satisfaction.

Now substitute for p in $\Psi(p)$ any formula φ of ZF. Assume $\Psi(\varphi)$, then by the usual methods, e,g., those of Mostowski [10] pp. 114-115, we obtain a truth definition for ZF in ZF and we arrive at the Tarski contradiction. Thus we have proved in $ZF \sim \Psi(\varphi)$ for any φ of ZF. Therefore we can add the axiom schema

$$\sim \Psi(\varphi) \quad \textit{for any } \varphi$$

to ZF without changing the theory and we call the new array of axioms ZF^*. The sets u and e form a standard model of ZF^* if $Sm^{ZF}(u, e)$ and there exists no subset v of u of ordered pairs $\langle if \rangle$ such that the formula obtained from the relativization of $\Psi(p)$ to the model by substituting $\langle if \rangle \in v$ for $p(i, f)$ holds. But form $Sm^{ZF}(u, e)$ it is easy to prove (in S) the existence of such a subset v of u, e.g., by the methods of Mostowski [11]. Hence ZF^* has no standard model. In other words, $\sim \Psi(p)$ is a true statement of set theory if p varies over the relations expressible in the set theory itself, but $\sim \Psi(p)$ is not a true statement if p varies over all the relations.

We shall now sketch briefly a second example. This will be a theory T which contains all the theorems of ZF, but has more standard complete models than ZF.

Mostowski defines in [10] when a class F of ordered pairs $\langle if \rangle$ is called an S-sequence for the formula Φ_j. This definition can be formulated without class variables, except F. Therefore, using the analogy between classes and functional variables, we can define, using only set variables beside p, when the functional variable $p(i, f)$ is an S-sequence for Φ_j. Let $Stf(u, i, f)$ be a formula which asserts that the finite sequence of sets f satisfies Φ_i in the complete model u (for the existence of such a formula cf. Mostowski [11]). We consider the following formula $\Omega(p)$ *p is an S-sequence for $\Phi_i \supset (\exists u)(Scm^S(u)\,.\,(f)(f$ is a finite sequence of sets whose range is in $u \supset .\, p(i, f) \equiv Stf(u, i, f)))$*. If we add to S the schema $\Omega(\varphi)$ where φ is any formula of S then we get a theory T which is an extension of ZF since all the instance of N_0 are provable in T (to prove the instance of N_0 corresponding to the formula φ with Gödel-number i we write down an S-sequence χ for Φ_i — this can be done by Mostowski [10] Σ_4 — and $\Omega(\chi)$ implies $x_1, \cdots, x_n \in u \supset . \varphi \equiv \mathrm{Rel}(u, \varphi)$). We shall now see that every standard complete model of ZF is a standard complete model of T but there are standard complete models of T with universes of smaller cardinality than that of and standard complete model of ZF. That every standard complete model of ZF is a standard complete model of T is the formal counterpart of Montague's theorem (that the axioms of infinity and replacement imply N_0). Now let τ be the first inaccessible number. By Montague and Vaught [8] there exists an ordinal number $\alpha < \tau$ such that $R(\alpha)$ is the union of the sets definable in the model A_τ and in conseqence A_τ is an arithmetical extension of A_α. In exactly the same way we can prove that there exists an ordinal β $\alpha < \beta < \tau$ such that $R(\beta)$ is the union of all the sets definable in the model A_τ by means of the new constant α, and in consequence A_τ is also an arithmetical extension of A_β. Hence, by Theorem 1.8 of [17], A_β is an arithmetical extension of A_α. It is easily seen that A_β is a standard model of T, where u required in the schema $\Omega(\varphi)$ is always taken to be $R(\alpha)$.

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
AND
HEBREW UNIVERSITY, JERUSALEM