PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 1, 1965

DEDEKIND DOMAINS AND RINGS OF QUOTIENTS

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We study the relation of the ideal class group of a Dedekind domain A to that of A_S, where S is a multiplicatively closed subset of A. We construct examples of (a) a Dedekind domain with no principal prime ideal and (b) a Dedekind domain which is not the integral closure of a principal ideal domain. We also obtain some qualitative information on the number of non-principal prime ideals in an arbitrary Dedekind domain.

If A is a Dedekind domain, S the set of all monic polynomials and T the set of all primitive polynomials of $A[X]$, then $A[X]_S$ and $A[X]_T$ are both Dedekind domains. We obtain the class groups of these new Dedekind domains in terms of that of A.

1. LEMMA 1-1. *If A is a Dedekind domain and S is a multiplicatively closed set of A such that A_S is not a field, then A_S is also a Dedekind domain.*

Proof. That A_S is integrally closed and Noetherian if A is, follows from the general theory of quotient ring formations. The primes of A_S are of the type PA_S, where P is a prime ideal of A such that $P \cap S = \phi$. Since height $PA_S =$ height P if $P \cap S = \phi$, $P \neq (0)$ and $P \cap S = \phi$ imply that height $PA_S = 1$.

PROPOSITION 1-2. If A is a Dedekind domain and S is a multiplicatively closed set of A, the assignment $C \to CA_S$ is a mapping of the set of fractionary ideals of A onto the set of fractionary ideals of A_S which is a homomorphism for multiplication.

Proof. C is a fractionary ideal of A if and only if there is a $d \in A$ such that $dC \subseteqq A$. If this is so, certainly $dCA_S \subseteqq A_S$, so CA_S is a fractionary ideal of A_S. Clearly $(B \cdot C)A_S = BA_S \cdot CA_S$, so the assignment is a homomorphism. Let D be any fractionary ideal of A_S. Since A_S is a Dedekind domain, D is in the free group generated by all prime ideals of A_S, i.e. $D = Q_i^{e_1} \cdots Q_k^{e_k}$. For each $i = 1, \cdots, k$ there is a prime P_i of A such that $Q_i = P_iA_S$. Set $E = P_1^{e_1} \cdots P_k^{e_k}$. Then using the fact that we have a multiplicative homomorphism of fractionary ideals, we get

Received December 13, 1963.

$$EA_S = (P_1A_S)^{e_1} \cdots (P_kA_S)^{e_k} = Q_1^{e_1} \cdots Q_k^{e_k}.$$

COROLLARY 1–3. *Let A be a Dedekind domain and S be a multiplicatively closed set of A. Let $\bar{C}$ (for C a fractionary ideal of A or A_S) denote the class of the ideal class group to which C belongs. Then the assignment $\bar{C} \rightarrow \overline{CA_S}$ is a homomorphism φ of the ideal class group of A onto that of A_S.*

Proof. It is only necessary to note that if $C = dA$, then $CA_S = dA_S$.

THEOREM 1–4. *The kernel of φ is generated by all $\bar{P}_\alpha$, where P_α ranges over all primes such that $P_\alpha \cap S \neq \phi$.*

If $P_\alpha \cap S \neq \phi$, then $P_\alpha A_S = A_S$. Suppose C is a fractionary ideal such that $\bar{C} = \bar{P}_\alpha$, i.e. $C = dP_\alpha$ for some d in the quotient field of A. Then $CA_S = dP_\alpha A_S = dA_S$, and thus $\overline{CA_S}$ is the principal class.

On the other hand, suppose that C is a fractionary ideal of A such that $CA_S = xA_S$. We may choose x in C. Then $C^{-1} \cdot xA$ is an integral ideal of A, and $(C^{-1} \cdot xA)A_S = A_S$. In other words, $C^{-1} \cdot xA = P_1^{f_1} \cdots P_l^{f_l}$, where $P_i \cap S \neq \phi$, $i = 1, \cdots, l$. Then $\bar{C} = \bar{P}_1^{-f_1}, \cdots, -\bar{P}_l^{-f_l}$, completing the proof.

EXAMPLE 1–5. There are Dedekind domains with no prime ideals in the principal class.

Let A be any Dedekind domain which is not a principal ideal domain. Let S be the multiplicative set generated by all Π_α, where Π_α ranges over all the prime elements of A. Then by Theorem 1–4, A_S will have the same class group as A but will have no principal prime ideals.

COROLLARY 1–6. *If A is a Dedekind domain which is not a principal ideal domain, then A has an infinite number of non-principal prime ideals.*

Proof. Choose S as in Example 1–5. Then A_S is not a principal ideal domain, hence has an infinite number of prime ideals, none of which are principal. These are of the form PA_S, where P is a (non-principal) prime of A.

COROLLARY 1–7. *Let A be a Dedekind domain with torsion class group and let $\{P_\alpha\}$ be a collection of primes such that the subgroup of the ideal class group of A generated by $\{\bar{P}_\alpha\}$ is not the entire* *class group. Then there are always an infinite number of non-principal primes not in the set* $\{P_\alpha\}$.

Proof. For each α, chose n_α such that $P_\alpha^{n_\alpha}$ is principal, say $= A \cdot a_\alpha$. Let S be the multiplicatively closed set generated by all a_α. By Theorem 1–4, A_S is not a principal ideal domain, hence A_S must have an infinite number of non-principal prime ideals by Corollary 1–6. These come from non-principal prime ideals of A which do not meet S. Each P_α does meet S, so there are an infinite number of non-principal primes outside the set $\{P_\alpha\}$.

COROLLARY 1–8. *Let A be a Dedekind domain with at least one prime ideal in every ideal class. Then for any multiplicatively closed set S, A_S will have a prime ideal in every class except possibly the principal class.*

Proof. By Corollary 1–3, every class of A_S is the image of a class of A. Let $\bar{D}$ be a non-principal class of A_S. $\bar{D} = \overline{CA_S}$, where C is a fractionary ideal of A. By assumption, there is a prime P of A such that $\bar{P} = \bar{C}$. If $PA_S = A_S$, then CA_S is principal and so $\bar{D}$ is the principal class of A_S. This is not the case, so PA_S is prime, and certainly $\overline{PA_S} = \overline{CA_S} = \bar{D}$.

EXAMPLE 1–9. There is a Dedekind domain which is not the integral closure of a principal ideal domain.

Let $A = Z[\sqrt{-5}]$. A is a Dedekind domain which is not a principal ideal domain. In A, $29 = (3 + 2\sqrt{-5}) \cdot (3 - 2\sqrt{-5})$. It follows from elementary algebraic number theory that $\Pi_1 = 3 + 2\sqrt{-5}$ and $\Pi_2 = 3 - 2\sqrt{-5}$ generate distinct prime ideals of A. Let $S = \{\Pi_1^k\}_{k \geqq 0}$. Then A_S is by Theorem 1–4 a Dedekind domain which is not a principal ideal domain. Let F denote the quotient field of A and Q the rational numbers. A_S cannot be the integral closure of a principal ideal domain whose quotient field is F since principal ideal domains are integrally closed. If A_S were the integral closure of a principal ideal domain C with quotient field Q, then C would contain Z, and Π_1 and Π_2 would be both units or nonunits in A_S (since Π_1 and Π_2 are conjugate over Q). But only Π_1 is a unit in A_S.

REMARK 1–10. Example 1–9 settles negatively a conjecture in Vol. I of *Commutative Algebra* [2, p. 284]. The following conjecture may yet be true: Every Dedekind domain can be realized as an A_S, where A is the integral closure of a principal ideal domain in a finite extension field and S is a multiplicatively closed set of A.

2. LEMMA 2–1. *Let A be a Dedekind domain. Let S be the multiplicatively closed set of $A[X]$ consisting of all monic polynomials of $A[X]$. Let T be the multiplicatively closed set of all primitive polynomials of $A[X]$ (i.e. all polynomials whose coefficients generate the unit ideal of A). Then $A[X]_S$ and $A[X]_T$ are both Dedekind domains.*

Proof. $A[X]$ is integrally closed and noetherian, and so both $A[X]_S$ and $A[X]_T$ are integrally closed and noetherian. Let P be a prime ideal of $A[X]$. If $P \cap A \neq (0)$, then $P \cap A = Q$ is a maximal ideal of A. If $P \neq QA[X]$, then passing to $A[X]/QA[X]$, it is easy to see that $P = QA[X] + f(X)\cdot A[X]$ where $f(X)$ is a suitably chosen monic polynomial of $A[X]$. In this case $P \cap S \neq \phi$, so $PA[X]_S = A[X]_S$. Thus if $P \cap A \neq (0)$ and $PA[X]_S$ is a proper prime of $A[X]_S$, then $P = QA[X]$ where $Q = P \cap A$. Then height $P =$ height $Q = 1$. If $P \cap A = (0)$, then $PK[X]$ is a prime ideal of $K[X]$ (where K denotes the quotient field of A). Certainly height $P =$ height $PK[X] = 1$, so in any case if a prime P of $A[X]$ is such that $P \cap S = \phi$, then height $P \leqq 1$. This proves that $A[X]_S$ is a Dedekind domain. Since $S \subseteqq T$, $A[X]_T$ is also a Dedekind domain by Lemma 1–1.

REMARK 2–2. $A[X]_T$ is customarily denoted by $A(X)$ [1, p. 18]. For the remainder of this article, $A[X]_S$ will be denoted by A^1.

PROPOSITION 2–3. A^1 *has the same ideal class group as A. In fact, the map $\bar{C} \rightarrow \overline{CA^1}$ is a one-to-one map of the ideal class group of A onto that of A^1.*

We can prove that $\bar{C} \rightarrow \overline{CA^1}$ is a one-to-one map of the ideal class of A into that of A by showing that if two integral ideals D and E of A are not in the same class, neither are DA^1 and EA^1. Suppose then that $\overline{DA^1} = \overline{EA^1}$. This implies that there are elements $f_i(X)$, $g_i(X)$, $i = 1,2$ in $A[X]$ with $g_i(X)$ monic for $i = 1,2$ such that

$$DA^1 \cdot \frac{f_1(X)}{g_1(X)} = EA^1 \cdot \frac{f_2(X)}{g_2(X)} .$$

Let a_i be the leading coefficient of $f_i(X)$ for $i = 1,2$, and let $d \in D$. Then we get a relation

$$d \cdot \frac{f_1(X)}{g_1(X)} = \frac{e(X)}{g(X)} \cdot \frac{f_2(X)}{g_2(X)} , \ g(X) \text{ monic,}$$

where $e(X)$ can be chosen as a polynomial in $A[X]$ all of whose coefficients are in E. This leads to $d\, g_2(X) \cdot f_1(X) \cdot g(X) = e(X) \cdot f_2(X) \cdot g_1(X)$. The leading coefficient on the right is in $a_2 \cdot E$. This shows that $a_1 \cdot D$

$D \subseteqq a_2 \cdot E$. Likewise $a_2 \cdot E \subseteqq a_1 \cdot D$, thus $a_1 \cdot D = a_2 \cdot E$ and $\bar{D} = \bar{E}$.

To prove the map is onto, the following lemma is needed.

LEMMA 2-4. *Let A be a Dedekind domain with quotient field K. To each polynomial $f(X) = a_n X^n + \cdots - + a_0$ of $K[X]$ assign the fractionary ideal $c(f) = (a_n, \cdots, a_0)$. Then $c(fg) = c(f)\, c(g)$.*

Proof. Let V_p (for each prime P of A) denote the P-adic valuation of A. It is immediate that $V_p(c(f)) = \min\ V_p(a_i)$. Because of the unique factorization of fractionary ideals in Dedekind domains, it suffices to show that $V_p(c(fg)) = V_p(c(f)) + V_p(c(g))$ for each prime P of A. This will be true if the equation is true in each $A_p[X]$. But A_p is a principal ideal domain, and the well-known proof for principal ideal domains shows the truth of the lemma.

To complete Prop. 2-3, let P be a prime ideal of A^1. The proof of Lemma 2-1 shows that if $P \cap A \neq (0)$, then $P = QA^1$ where Q is a prime of A. Thus $\bar{P} = \overline{QA^1}$ and ideal classes generated by these primes are images of classes of A. Suppose now that P is a prime of A^1 such that $P \cap A = (0)$. Let $P^1 = P \cap A[X]$. Then $P^1 \cap A = (0)$, and $P^1 \cdot K[X]$ is a prime ideal of $K[X]$. Let $P^1 \cdot K[X] = f(X)K[X]$; we may choose $f(X)$ in $A[X]$. Let $C = c(f)$. Suppose that $g(X) \cdot f(X) \in A[X]$. Then because $c(fg) = (c(f)) + (c(g)) \geqq 0$ for all P, $g(X) \in C^{-1} \cdot A[X]$. Conversely if $g(X) \in C^{-1} \cdot A[X]$, then $g(X)\, f(X) \in A[X]$. Thus $P^1 = f(X)K[X] \cap A[X] = C^{-1} \cdot A[X] \cdot f(X)A[X]$, and $P = P^1 A^1 = C^{-1} \cdot A^1 \cdot f(X)A^1$. This gives finally that $\bar{P} = \overline{C^{-1}A^1}$, and the class is an image of a class of A under our map. Since the ideal class group of A^1 is generated by all $\bar{P}$ where P is a prime of A^1, this finishes the proof.

COROLLARY 2-5. *A^1 has a prime ideal in each ideal class.*

Proof. Let w be any nonunit of A. Then $(wX + 1)\, K[X] \cap A^1$ ($= (wX + 1)A^1$) is a prime ideal in the principal class. Otherwise let C be any integral ideal in a nonprincipal class $\bar{D}^{-1}$. C can be generated by 2 elements, so suppose $C = (c_0, c_1)$; then $Q = (c_0 + c_1 X) \cdot K[X] \cap A^1$ is a prime ideal in $\overline{C^{-1}A^1} = \bar{D}$.

PROPOSITION 2-6. If A is a Dedekind domain, then $A(X)$ is a principal ideal domain.

Proof. Since $A(X) = A^1_T$, Corollary 1-3 and the proof of Corollary 2-5 show that each nonprincipal class of $A(X)$ contains a prime $QA(X)$, where Q is a prime ideal of A of the type $(c_0 + c_1 X)K[X] \cap A^1$. Clearly $Q \cap A[X] = (c_0 + c_1 X)K[X] \cap A[X] = C^{-1} \cdot A[X] \cdot (c_0 + c_1 X)A[X] \nsubseteqq$

$PA[X]$ for any prime P of A. Thus there is in $Q \cap A[X]$ a primitive polynomial of $A[X]$. Thus $QA(X) = A(X)$. Theorem 1–4 now implies that every class of A becomes principal in $A(X)$, i.e. $A(X)$ is a principal ideal domain.

REMARK 2–7. Proposition 2–6 is interesting in light of the fact that the primes of $A(X)$ are exactly those of the form $PA(X)$, where P is a prime of A [1, p. 18].

REMARK 2–8. If the conjecture given in Remark 1–10 is true for a Dedekind domain A, it is also true for A^1. For suppose $A = B_M$, where M is a multiplicatively closed set of B and B is the integral closure of a principal ideal domain B_0 in a suitable finite extension field. Let S, S^1, and T be the set of monic polynomials in $A[X]$, $B[X]$, and $B_0[X]$ respectively. Then $A^1 = A[X]_S = (B_M[X])_S = (B[X]_M)_S = (B[X])_{\langle M,S\rangle} = (B[X]_{S^1})_{\langle M,S\rangle}$. The last equality holds because $S^1 \subseteqq S \subseteqq \langle M, S\rangle$. It is easy to see that $B[X]_{S^1}$ is the integral closure of the principal ideal domain $B_0[X]_T$ in $K(X)$, where K is the quotient field of B.

REFERENCES

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