PACIFIC JOURNAL OF MATHEMATICS
Vol. 122, No. 2, 1986

HERMITE CHARACTER SUMS

RONALD J. EVANS

Analogues over finite fields are presented for the major formulas in the theory of classical Hermite functions.

1. Introduction. Character sum analogues over finite fields of the most important transformation and summation formulas for ${}_2F_1$ and ${}_3F_2$ hypergeometric series have recently been formulated by Greene [**11**], [**11A**]. The power of this theory is demonstrated, for example, by the evaluation it yields [**12**] of the double sum of Legendre symbols

$$\sum_{x,y \,(\mathrm{mod}\, p)} \left(\frac{xy(x+1)(y+1)(x+y)}{p} \right).$$

This evaluation proves a conjecture in [**9**, p. 370] and solves the problem of finding explicitly the number of rational points (mod p) on the surface $z^2 = (x^2+1)(y^2+1)(x^2+y^2)$, a problem some algebraic geometers had worked on without success.

Character sum analogues of the important formulas for orthogonal polynomials are potentially as useful as those for hypergeometric series, so a systematic study should be made. Indeed, many character sums studied in the literature are analogues of special functions, e.g., the generalized Kloosterman sum (see (2.5), Theorem 2.6, and, say, [**10**], [**21A**, p. 253]).

In this paper, the focus is on analogues of Hermite polynomials, namely Hermite character sums $H_N(x)$ defined in (2.1). Each of the theorems in §4 is an analogue over finite fields of a classical formula stated just above it. The classical formulas are stated without conditions of validity; such conditions are often unrelated to the unpredictable conditions of validity for the finite field formulas.

It is not always possible to give proofs of the finite field formulas which parallel classical proofs. This is because no satisfactory analogues of limits, first derivatives, logarithms, and three term recurrence relations are known. It would be of great importance to find a unified approach which simultaneously explains formulas for orthogonal polynomials and the analogues over finite fields. Perhaps this will be accomplished by connecting the polynomials with Lie groups having counterparts over finite fields.

Theorems 4.24 and 4.36 are particularly elegant and interesting. Surprisingly, the former holds with absolutely no restrictions on the characters A, B, and C.

In §3, multivariable Hermite sums are defined and a biorthogonality relation is proved. In §5, an analogue of the associated Hermite polynomial [**1**] is briefly discussed, and an example is given to show how finite field analysis may be used to construct explicit formulas for classical special functions. The mysterious fact that such a technique generally works reflects the beauty and unity of mathematics.

For some recent work related to finite field analogues of classical formulas for special functions, see references [**7**]–[**9**], [**11**]–[**16**], [**19**], and [**21**]. This subject of course dates back a long time. As early as 1837, Jacobi [**18**, p. 257] had been aware of a finite field analogue of the Gauss multiplication formula for the gamma function [**25**, p. 26]. Jacobi did not have available the tools needed to prove his formula, and over a century went by before a proof was provided by Davenport and Hasse [**5**]. No elementary proof is known, but see [**3**, §8], [**13**].

2. Definitions, notation, and preliminary results. Let q be a positive integral power of an odd prime p. The finite field of q elements is denoted by $GF(q)$. The capital letters A, B, C, M, N are reserved for multiplicative characters on $GF(q)$, but 1 and ϕ will denote the trivial and quadratic characters, respectively. Write Σ_x to denote the sum over all $x \in GF(q)$, and write Σ_N to denote the sum over all $q-1$ characters N on $GF(q)$. For $x \in GF(q)$, $\operatorname{Tr}(x)$ denotes the trace of x from $GF(q)$ to $GF(p)$, and ζ^x denotes $\exp(2\pi i \operatorname{Tr}(x)/p)$. If $x \in GF(q)$, let $\bar{x}$ be the multiplicative inverse of x when $x \neq 0$ and let $\bar{x} = 0$ when $x = 0$. The expression $\zeta^{x/2}$ means $\zeta^{x\bar{2}}$, not $\exp(2\pi i \operatorname{Tr}(x)/2p)$. Define $\overline{N}$ by $N\overline{N} = 1$.

Analogous to the gamma function

$$\Gamma(n) = \int_0^\infty x^n e^{-x} \frac{dx}{x}$$

is the Gauss sum

$$G(N) = \sum_x N(x)\zeta^x,$$

and analogous to the beta function

$$\beta(m,n) = \int_0^1 x^m (1-x)^n \frac{dx}{x(1-x)}$$

is the Jacobi sum

$$J(M,N) = \sum_x M(x)N(1-x).$$

For some basic properties of these sums, see [**17**, Ch. 8]; e.g., for $A \neq 1$, $MN \neq 1$, $G(A)G(\overline{A}) = qA(-1)$, $J(M,N) = G(M)G(N)/G(MN)$.

We wish to define character sum analogues of Hermite functions $H_n(x)$, Laguerre functions $L_n^a(x)$, Legendre functions $P_n(x)$, and Bessel functions $J_n(x)$, $K_n(x)$, motivated by the familiar integral representations

$$H_n(x) = \frac{1}{\Gamma(-n)} \int_0^\infty e^{-u^2-2ux} u^{-n} \frac{du}{u} \qquad [\mathbf{20}, (10.5.2)],$$

$$L_n^a(x) = \frac{1}{2\pi i} \int_C (u+1)^{n+a} u^{-n} e^{-ux} \frac{du}{u} \qquad [\mathbf{20}, \text{p. } 77],$$

$$P_n(x) = \frac{1}{2\pi i} \int_C (1 - 2xu + u^2)^{-1/2} u^{-n} \frac{du}{u} \qquad [\mathbf{20}, \text{p. } 45],$$

$$J_n(x) = \frac{1}{2\pi i} \int_C u^{-n} e^{x(u-u^{-1})/2} \frac{du}{u} \qquad [\mathbf{20}, (5.10.7)],$$

$$K_n(x) = \frac{1}{2} \int_0^\infty u^{-n} e^{-x(u+u^{-1})/2} \frac{du}{u} \qquad [\mathbf{20}, (5.10.25)].$$

Thus for $x \in GF(q)$, define the Hermite character sum

$$H_N(x) = \frac{1}{G(\overline{N})} \sum_u \overline{N}(u) \zeta^{u^2+2ux}, \tag{2.1}$$

the Laguerre character sum

$$L_N^A(x) = q^{-1} \sum_u \overline{N}(u) AN(1+u) \zeta^{xu}, \tag{2.2}$$

the Legendre character sum

$$P_N(x) = q^{-1} \sum_u N(u) \phi(1 - 2xu + u^2), \tag{2.3}$$

and the Bessel character sums

$$J_N(x) = q^{-1} \sum_u N(u) \zeta^{x(u-\bar{u})/2}, \tag{2.4}$$

$$K_N(x) = \sum_u N(u) \zeta^{x(u+\bar{u})/2}. \tag{2.5}$$

(Note that $K_N(x)$ is a generalized Kloosterman sum.) Confluent hypergeometric character sums $\Psi(A, B;\, x)$ and $\Phi(A, B;\, x)$ (cf. [**20**, (9.11.6), (9.11.1)]) can be defined as multiples of $L_{\overline{A}}^B(x)$, as follows:

$$\Psi(A, B;\, x) = \frac{q}{G(A)} L_{\overline{A}}^B(x), \tag{2.6}$$

$$\Phi(A, B;\, x) = \frac{qA(-1)}{J(A, B\overline{A})} L_{\overline{A}}^B(x). \tag{2.7}$$

Define an operator $D^N = D_x^N$ on the set of complex functions F on $GF(q)$ by

$$D^N F(x) = \frac{1}{G(\overline{N})} \sum_t \overline{N}(t) F(x-t). \tag{2.8}$$

Thus D_x^N is the analogue of the nth derivative with respect to x (cf. Cauchy's integral formula for $f^{(n)}(x)$). We next prove four theorems involving D^N. The first gives an analogue of composition of derivative operators.

THEOREM 2.1. *For a function* F: $GF(q) \to \mathbf{C}$,

$$D^1 F(x) = F(x) - \sum_t F(t), \tag{2.9}$$

$$D^N D^{\overline{N}} F(x) = F(x) - q^{-1} \sum_t F(t) \quad \textit{for } N \neq 1, \tag{2.10}$$

and

$$D^N D^M = D^{NM} \quad \textit{for } NM \neq 1. \tag{2.11}$$

Proof. By (2.8), one easily proves (2.9). Now,

$$\begin{aligned} L := G(\overline{N}) G(\overline{M}) D^N D^M F(x) &= \sum_{s,t} \overline{M}(s) \overline{N}(t) F(x-t-s) \\ &= \sum_{s,t} \overline{M}(s) \overline{N}(t-s) F(x-t) \\ &= \sum_{t \neq 0} \sum_s \overline{M}(s) \overline{N}(t-s) F(x-t) + N(-1) F(x) \sum_s \overline{MN}(s) \\ &= J(\overline{M}, \overline{N}) \sum_t \overline{MN}(t) F(x-t) + N(-1) F(x) \sum_s \overline{MN}(s). \end{aligned}$$

If $M = \overline{N} \neq 1$, then $J(\overline{M}, \overline{N}) = -N(-1)$ and $G(\overline{N}) G(\overline{M}) = N(-1) q$, so

$$\begin{aligned} L &= N(-1) \Big\{ -\sum_{t \neq 0} F(x-t) + (q-1) F(x) \Big\} \\ &= N(-1) \Big\{ q F(x) - \sum_t F(t) \Big\} \end{aligned}$$

and (2.10) follows. If $M \neq \overline{N}$, then $J(\overline{M}, \overline{N}) = G(\overline{M}) G(\overline{N}) / G(\overline{MN})$, so by (2.8), $L = G(\overline{M}) G(\overline{N}) D^{NM} F(x)$ and (2.11) follows.

The next theorem gives an analogue of Leibniz's rule.

THEOREM 2.2. *If E: $GF(q) \to \mathbf{C}$ and F: $GF(q) \to \mathbf{C}$, then*

$$D^N(E(x)F(x)) = \sum_M \frac{G(\overline{M})G(M\overline{N})}{(q-1)G(\overline{N})}(D^M E(x))(D^{N\overline{M}}F(x)).$$

Proof. By (2.8), the right side above equals

$$\begin{aligned}\sum_M \frac{1}{(q-1)G(\overline{N})} &\sum_{s,t\neq 0} \overline{M}(s)M\overline{N}(t)E(x-s)F(x-t)\\ &= \frac{1}{G(\overline{N})}\sum_{s,t\neq 0}\overline{N}(t)E(x-s)F(x-t)\cdot\frac{1}{(q-1)}\sum_M M(t/s)\\ &= \frac{1}{G(\overline{N})}\sum_t \overline{N}(t)E(x-t)F(x-t) = D^N(E(x)F(x)).\end{aligned}$$

The next theorem gives an analogue of n-fold integration by parts.

THEOREM 2.3. *Let E: $GF(q) \to \mathbf{C}$ and F: $GF(q) \to \mathbf{C}$. Then*

$$\sum_x E(x)D^N F(x) = N(-1)\sum_x F(x)D^N E(x).$$

Proof. By (2.8),

$$\begin{aligned}G(\overline{N})\sum_x E(x)D^N F(x) &= \sum_u E(u)\sum_t \overline{N}(t)F(u-t)\\ &= \sum_x F(x)\sum_u E(u)\overline{N}(u-x)\\ &= \sum_x F(x)\sum_t E(x-t)\overline{N}(-t) = G(\overline{N})N(-1)\sum_x F(x)D^N E(x).\end{aligned}$$

The next theorem is the analogue of the Taylor expansion.

THEOREM 2.4. *Let F: $GF(q) \to \mathbf{C}$ and fix $a \in GF(q)$. For $x \neq a$,*

$$F(x) = \sum_N \frac{G(\overline{N})}{q-1} D^N F(x)\Big|_{x=a} N(a-x).$$

Moreover, this expansion is unique in the sense that if

$$0 = \sum_N R(N)N(a-x)$$

for all x, then $R(N) = 0$ for all N.

Proof. For $x \neq a$,

$$\sum_N \frac{G(\overline{N})}{q-1} D^N F(x)\Big|_{x=a} N(a-x) = \sum_N \frac{N(a-x)}{q-1} \sum_t \overline{N}(t) F(a-t)$$

$$= \sum_{t \neq 0} F(a-t) \cdot \frac{1}{q-1} \sum_N N\left(\frac{a-x}{t}\right) = F(x).$$

To prove the statement on uniqueness, multiply both sides of the equality $0 = \Sigma_N R(N) N(a-x)$ by $\overline{M}(a-x)$ and then sum over x.

The next theorem gives an analogue of Fourier inversion. We omit the easy proof.

THEOREM 2.5. *Let F: $GF(q) \to \mathbf{C}$. Then $F(x) = \Sigma_a Q(a) \zeta^{ax}$, where $Q(a) = q^{-1} \Sigma_u F(u) \zeta^{-au}$. Moreover, this expansion is unique in the sense that if $0 = \Sigma_a R(a) \zeta^{ax}$ for all x, then $R(a) = 0$ for all a.*

The next theorem is the analogue of

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} \qquad [\mathbf{20}, (5.8.5)].$$

It evaluates Salié's sum over $GF(q)$; see Mordell [**23**], [**24**].

THEOREM 2.6. *For all x,*

$$K_\phi(x) = \phi(2x) G(\phi)(\zeta^x + \zeta^{-x}).$$

Proof. By the uniqueness assertion in Theorem 2.5, it suffices to show that for all a,

$$(2.12) \qquad Q(a) := q^{-1} \sum_x K_\phi(x) \zeta^{-ax}$$

$$= q^{-1} \phi(2) G(\phi) \sum_x \phi(x) (\zeta^{x(1-a)} + \zeta^{x(-1-a)}).$$

The left side of (2.12) equals

$$(2.13) \qquad q^{-1} \sum_t \phi(t) \sum_x \zeta^{x(t+\bar{t}-2a)/2} = \sum_{\substack{t \\ t+\bar{t}=2a}} \phi(t),$$

and the right side of (2.12) equals

$$(2.14) \qquad \begin{aligned} &q^{-1} \phi(2) G(\phi) \{ G(\phi) \phi(1-a) + G(\phi) \phi(-1-a) \} \\ &= \phi(2a+2) + \phi(2a-2). \end{aligned}$$

If $\phi(a^2-1)=-1$, then the expressions in (2.13) and (2.14) vanish; if $\phi(a^2-1)=0$, these expressions equal $\phi(a)$. Finally, assume that $\phi(a^2-1)=1$. It remains to show that $\phi(a+\sqrt{a^2-1})=\phi(2a+2)$. This follows because

$$\frac{2a+2\sqrt{a^2-1}}{a+1}=\left(1+\frac{\sqrt{a^2-1}}{a+1}\right)^2.$$

The next theorem generalizes Theorem 2.6; it is the analogue of

$$K_n(x)=\frac{\sqrt{\pi}}{\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n\int_1^\infty e^{-xw}(w^2-1)^{n-1/2}\,dw \qquad [\mathbf{20}, (5.10.24)].$$

THEOREM 2.7. *Unless $N=1$ and $x=0$,*

$$K_N(x)=\frac{N(x/2)G(\phi)}{G(N\phi)}\sum_w N\phi(w^2-1)\zeta^{wx}.$$

Proof. The result is clear for $x=0$, so assume $x\neq 0$. By (2.5),

$$K_N(x)G(N\phi)=\sum_{t,u\neq 0}N(t)N\phi(u)\zeta^{x(t+\bar{t})/2+u}.$$

Replace t by $tx/(2u)$ to get

$$K_N(x)G(N\phi)=\sum_{t,u\neq 0}\phi(u)N\left(\frac{tx}{2}\right)\zeta^{x(tx/2u+2u/tx)/2+u}.$$

Now replace u by ut to get

$$\overline{N}\left(\frac{x}{2}\right)K_N(x)G(N\phi)=\sum_{t,u}\phi(u)N\phi(t)\zeta^{u(t+1)+\bar{u}x^2/4}$$
$$=\sum_t N\phi(t)S(x,t),$$

where

$$S(x,t):=\sum_u\phi(u)\zeta^{u(t+1)+\bar{u}x^2/4}.$$

If $t+1=0$, $S(x,t)=G(\phi)$. If $\phi(t+1)=-1$, then replacement of u by $ux^2/(4t+4)$ yields

$$S(x,t)=\phi(t+1)\sum_u\phi(u)\zeta^{\bar{u}(t+1)+ux^2/4}=-S(x,t),$$

so $S(x,t)=0$. If $\phi(t+1)=1$, then replacement of u by $ux/(2\sqrt{t+1})$ yields

$$S(x,t)=\phi(2x\sqrt{t+1})\sum_u \phi(u)\zeta^{x\sqrt{t+1}(u+\bar{u})/2}$$

$$=\phi(2x\sqrt{t+1})K_\phi(x\sqrt{t+1})=G(\phi)\left(\zeta^{x\sqrt{t+1}}+\zeta^{-x\sqrt{t+1}}\right)$$

by Theorem 2.6. Thus, with $w^2=t+1$, $\overline{N}(x/2)K_N(x)G(N\phi)=G(\phi)\Sigma_w N\phi(w^2-1)\zeta^{xw}$, as desired.

Finally, we record the following well-known special case of the Hasse-Davenport multiplication formula mentioned in the Introduction:

$$G(A^2)G(\phi)=A(4)G(A)G(A\phi). \tag{2.15}$$

3. Multivariable Hermite sums. The following theorem is the analogue of Taylor's theorem in several variables. We omit the proof, as it is similar to that of Theorem 2.4. We shall write $\mathbf{a}$ for the vector $(a_1,\ldots,a_r)$ $\in GF(q)^r$ and $\mathbf{N}$ for the vector of characters $(N_1,\ldots,N_r)$.

THEOREM 3.1. *Let F: $GF(q)^r\to\mathbf{C}$ and fix $\mathbf{a}\in GF(q)^r$. If $u_i\neq a_i$ for each i, $1\le i\le r$, then*

$$F(\mathbf{u})=(q-1)^{-r}\sum_{\mathbf{N}} D_{u_1}^{N_1}\cdots D_{u_r}^{N_r}F(\mathbf{u})\Big|_{\mathbf{u}=\mathbf{a}}\prod_{i=1}^{r}G(\overline{N}_i)N_i(a_i-u_i).$$

Moreover, this expansion is unique in the sense that if for all $\mathbf{u}$,

$$0=\sum_{\mathbf{N}}R(\mathbf{N})\prod_{i=1}^{r}N_i(a_i-u_i),$$

then $R(\mathbf{N})=0$ for all $\mathbf{N}$.

Fix a symmetric $r\times r$ matrix D over $GF(q)$ with nonzero determinant $d\in GF(q)$. Given a row vector $\mathbf{x}$, let $\mathbf{x}^t$ denote its transpose. For the rest of this section, let $\mathbf{w}$ and $\mathbf{u}$ be vectors with $w_iu_i\neq 0$, $1\le i\le r$. In view of Theorem 3.1 with $\mathbf{a}=0$, we can define multivariable Hermite character sums $G_{\mathbf{N}}(\mathbf{x})$ and $H_{\mathbf{M}}(\mathbf{x})$ by

$$\zeta^{((\mathbf{x}-D^{-1}\mathbf{u})^tD(\mathbf{x}-D^{-1}\mathbf{u})-\mathbf{x}^tD\mathbf{x})/2} = (q-1)^{-r}\sum_{\mathbf{N}}G_{\mathbf{N}}(\mathbf{x})\prod_{i=1}^{r}G(\overline{N}_i)N_i(-u_i), \tag{3.1}$$

and

$$\zeta^{((\mathbf{x}-\mathbf{w})^tD(\mathbf{x}-\mathbf{w})-\mathbf{x}^tD\mathbf{x})/2} = (q-1)^{-r}\sum_{\mathbf{M}}H_{\mathbf{M}}(\mathbf{x})\prod_{i=1}^{r}G(\overline{M}_i)M_i(-w_i). \tag{3.2}$$

The following theorem is the analogue of the biorthogonality property of Hermite polynomials [**6**, p. 286, (1)].

THEOREM 3.2.

$$L(\mathbf{N},\mathbf{M}) := \sum_{\mathbf{x}} \zeta^{\mathbf{x}'D\mathbf{x}/2} G_{\mathbf{N}}(x) H_{\mathbf{M}}(\mathbf{x})$$

$$= \begin{cases} (q-1)^r \phi(2^r d) G^r(\phi) \prod_{i=1}^{r} \dfrac{M_i(-1)}{G(\overline{M}_i)}, & \textit{if } \mathbf{N} = \mathbf{M} \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. Multiplying the equalities in (3.1) and (3.2), we obtain

$$(q-1)^{-2r} \sum_{\mathbf{M},\mathbf{N}} L(\mathbf{M},\mathbf{N}) \prod_{i=1}^{r} G(\overline{M}_i) G(\overline{N}_i) M_i(-w_i) N_i(-u_i)$$

$$= \zeta^{(\mathbf{u}'D^{-1}\mathbf{u}+\mathbf{w}'D\mathbf{w})/2} \sum_{\mathbf{x}} \zeta^{(\mathbf{x}'D\mathbf{x}-2\mathbf{x}'(\mathbf{u}+D\mathbf{w}))/2}$$

$$= \zeta^{(\mathbf{u}'D^{-1}\mathbf{u}+\mathbf{w}'D\mathbf{w})/2} \sum_{\mathbf{x}} \zeta^{((\mathbf{x}-\mathbf{w}-D^{-1}\mathbf{u})'D(\mathbf{x}-\mathbf{w}-D^{-1}\mathbf{u})-(\mathbf{u}+D\mathbf{w})'D^{-1}(\mathbf{u}+D\mathbf{w}))/2}$$

$$= \zeta^{(\mathbf{u}'D^{-1}\mathbf{u}+\mathbf{w}'D\mathbf{w})/2} \sum_{\mathbf{x}} \zeta^{(\mathbf{x}'D\mathbf{x}-(\mathbf{u}+D\mathbf{w})'D^{-1}(\mathbf{u}+D\mathbf{w}))/2}$$

$$= \zeta^{-\mathbf{w}'\mathbf{u}} \sum_{\mathbf{x}} \zeta^{\mathbf{x}'D\mathbf{x}/2}.$$

Since q is odd, there exists an invertible matrix Q over $GF(q)$ such that $Q'DQ$ is diagonal [**4**, p. 253, Theorem 15]. Thus, replacing $\mathbf{x}$ by $Q\mathbf{x}$, we find that

$$\zeta^{-\mathbf{w}'\mathbf{u}} \sum_{\mathbf{x}} \zeta^{\mathbf{x}'D\mathbf{x}/2} = \zeta^{-\mathbf{w}'\mathbf{u}} \phi(2^r d) G^r(\phi)$$

$$= \phi(2^r d) G^r(\phi)(q-1)^{-r} \sum_{\mathbf{M}} \prod_{i=1}^{r} G(\overline{M}_i) M_i(-w_i u_i),$$

by Theorem 3.1. Comparing coefficients of $\prod_{i=1}^{r} M_i(-w_i u_i)$, we easily obtain the result.

4. Hermite sums. In this section, we catalogue the theorems (in somewhat arbitrary order) corresponding to what we believe to be the most important classical formulas for Hermite functions. In many cases, it is more difficult to construct an elegant analogue (and find general conditions of validity) than it is to give proofs. If for example one had made the reasonable guess that the analogue of the binomial coefficient in (4.29) is

$$qG(N)/((q-1)G(B)G(N\overline{B}))$$

(instead of $G(\overline{B})G(B\overline{N})/((q-1)G(\overline{N}))$, used in Theorem 4.29), unnecessary complications would have resulted.

Corresponding to the Rodriguez formula

$$(4.1)\qquad H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}\qquad [\mathbf{6}, \text{p. } 193, (7)],$$

we have

THEOREM 4.1. $H_N(x)=N(-1)\zeta^{-x^2}D^N\zeta^{x^2}$.

Proof. By (2.8) and (2.1),

$$D^N\zeta^{x^2}=\frac{1}{G(\overline{N})}\sum_u \overline{N}(u)\zeta^{(x-u)^2}=\frac{N(-1)}{G(\overline{N})}\sum_u \overline{N}(u)\zeta^{(x+u)^2}$$

$$=N(-1)\zeta^{x^2}H_N(x).$$

Corresponding to

$$(4.2)\qquad \frac{d^m}{dx^m}H_n(x)=(-2)^m\frac{\Gamma(m-n)}{\Gamma(-n)}H_{n-m}(x)\qquad [\mathbf{6}, \text{p. } 119, (15)],$$

we have

THEOREM 4.2.

$$D^M H_N(x)=\frac{M(-2)G(M\overline{N})}{G(\overline{N})}H_{N\overline{M}}(x).$$

Proof.

$$G(\overline{N})D^M H_N(x)=\frac{G(\overline{N})}{G(\overline{M})}\sum_t \overline{M}(t)H_N(x-t)$$

$$=\frac{1}{G(\overline{M})}\sum_{t,u\neq 0}\overline{M}(t)\overline{N}(u)\zeta^{u^2+2u(x-t)}=\sum_u M(-2u)\overline{N}(u)\zeta^{u^2+2ux}$$

$$=M(-2)G(M\overline{N})H_{N\overline{M}}(x).$$

Corresponding to

$$(4.3)\qquad \frac{d^m}{dx^m}e^{-x^2}H_n(x) = (-1)^m e^{-x^2}H_{n+m}(x) \qquad [\mathbf{6}, \text{p. } 119, (16)],$$

we have

THEOREM 4.3. *If* $MN \neq 1$, *then*

$$D^M \zeta^{x^2} H_N(x) = M(-1)\zeta^{x^2} H_{MN}(x).$$

Proof. Since $MN \neq 1$, $D^M D^N = D^{MN}$ by Theorem 2.1. Thus, by Theorem 4.1,

$$\begin{aligned} D^M \zeta^{x^2} H_N(x) &= N(-1) D^M D^N \zeta^{x^2} \\ &= N(-1) D^{MN} \zeta^{x^2} = N(-1)\zeta^{x^2} MN(-1) H_{MN}(x). \end{aligned}$$

Corresponding to

$$(4.4)\qquad H_0(x) = 1 \qquad [\mathbf{6}, \text{p. } 193, (8)],$$

we have

THEOREM 4.4. $H_N(x) = 1 - \zeta^{-x^2}G(\phi)$ *when* $N = 1$.

Proof. By Theorem 2.1, $D^1\zeta^{x^2} = \zeta^{x^2} - G(\phi)$, so the result follows from Theorem 4.1. Alternatively, put $N = 1$ in (2.1).

Corresponding to

$$(4.5)\qquad H_n(-x) = (-1)^n H_n(x) \qquad [\mathbf{6}, \text{p. } 193, (14)],$$

we have

THEOREM 4.5. $H_N(-x) = N(-1)H_N(x)$.

Proof. Replace u by $-u$ in (2.1).

Corresponding to

$$(4.6)\qquad H_{2m}(0) = (-1)^m (2m)!/m! \qquad [\mathbf{6}, \text{p. } 193, (15)],$$

we have

THEOREM 4.6.

$$H_N(0) = \begin{cases} 0, & \textit{if } N \textit{ is not a square} \\ \dfrac{G(\overline{M}) + G(\overline{M}\phi)}{G(\overline{M}^2)}, & \textit{if } N = M^2. \end{cases}$$

Proof. $G(\overline{N})H_N(0) = \sum_u \overline{N}(u)\zeta^{u^2}$, which vanishes when N is not a square. If $N = M^2$, then

$$G(\overline{N})H_N(0) = \sum_u \overline{M}(u^2)\zeta^{u^2} = \sum_u \overline{M}(u)\zeta^u\{1 + \phi(u)\}$$
$$= G(\overline{M}) + G(\overline{M}\phi).$$

Corresponding to the generating function formula

$$(4.7) \qquad e^{2xz-z^2} = \sum_{n=0}^{\infty} \frac{H_n(x)z^n}{n!} \qquad [\mathbf{6}, \text{p. } 194, (19)],$$

we have

THEOREM 4.7. *For* $z \neq 0$, $\zeta^{z^2-2xz} = \sum_N N(-z)G(\overline{N})H_N(x)/(q-1)$.

Proof. By (2.1), the right side above equals

$$\sum_N \frac{N(-z)}{q-1} \sum_u \overline{N}(u)\zeta^{u^2+2ux} = \sum_{u=-z} \zeta^{u^2+2ux} = \zeta^{z^2-2zx}.$$

Corresponding to the polynomial expansion

$$(4.8) \qquad H_n(x) = n! \sum_{m=0}^{[n/2]} \frac{(-1)^m(2x)^{n-2m}}{m!(n-2m)!} \qquad [\mathbf{6}, \text{p. } 193, (9)],$$

we have

THEOREM 4.8. *For* $x \neq 0$,

$$H_N(x) = \frac{1}{G(\overline{N})(q-1)} \sum_M N\overline{M}^2(2x)G(\overline{M})G(M^2\overline{N}).$$

Proof. By Theorem 2.4 with $a = 0$,

$$(4.8\text{a}) \qquad H_N(x) = \frac{1}{q-1} \sum_A G(\overline{A})D^A H_N(x)\Big|_{x=0} A(-x).$$

By Theorem 4.2,

$$D^A H_N(x)|_{x=0} = \frac{A(-2)G(A\overline{N})}{G(\overline{N})} H_{N\overline{A}}(0).$$

Thus, by Theorem 4.6,

$$D^A H_N(x)|_{x=0} = \begin{cases} 0, & \text{if } N\overline{A} \text{ is not a square} \\ \dfrac{A(-2)}{G(\overline{N})}\big(G(\overline{M}) + G(\overline{M}\phi)\big), & \text{if } N\overline{A} = M^2. \end{cases}$$

Replacing A by $N\overline{M}^2$ in (4.8a), we obtain the desired result.

Corresponding to the integral representation

$$(4.9)\qquad H_n(x) = \frac{(-2i)^n e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{2itx-t^2} t^n \, dt \qquad [\mathbf{20}, (4.11.4)],$$

we have

THEOREM 4.9. *If $N \neq 1$, then*

$$H_N(x) = \frac{\zeta^{-x^2} G(\phi)}{q} \sum_u N(2u) \zeta^{2ux-u^2}.$$

Proof.

$$\begin{aligned} L := {} & G(\overline{N}) \zeta^{-x^2} G(\phi) q^{-1} \sum_u N(2u) \zeta^{2ux-u^2} \\ = {} & G(\phi) q^{-1} \sum_{u,t \neq 0} N\left(\frac{2u}{t}\right) \zeta^{t-(u-x)^2}. \end{aligned}$$

Replace t by $2tu$ to obtain

$$L = G(\phi) q^{-1} \sum_{u \neq 0} \sum_t \overline{N}(t) \zeta^{2tu-(u-x)^2}.$$

The condition $u \neq 0$ may be dropped since $N \neq 1$. Replace u by $u + x$ to obtain

$$\begin{aligned} L &= G(\phi) q^{-1} \sum_t \overline{N}(t) \zeta^{2tx} \sum_u \zeta^{2tu-u^2} \\ &= G(\phi) q^{-1} \sum_t \overline{N}(t) \zeta^{2tx+t^2} \phi(-1) G(\phi) \\ &= \sum_t \overline{N}(t) \zeta^{2tx+t^2} = G(\overline{N}) H_N(x), \end{aligned}$$

as desired.

COROLLARY 4.10. *If $N \neq 1$, then*

$$\overline{H}_N(x) = \frac{\overline{N}(2) q \zeta^{x^2}}{G(\phi) G(N)} H_N(x),$$

where the bar on the left denotes complex conjugation.

Proof. By Theorem 4.9, the right side above equals

$$\frac{1}{G(N)} \sum_u N(u) \zeta^{2ux-u^2} = \frac{N(-1)}{G(N)} \sum_u N(u) \zeta^{-u^2-2ux} = \overline{H}_N(x).$$

Corresponding to

$$\int_{-\infty}^{\infty} e^{-x^2+ax} H_n(x) = \sqrt{\pi}\, a^n e^{a^2/4} \qquad [\mathbf{20}, \text{p. } 74], \tag{4.11}$$

we have

THEOREM 4.11. *Unless both $a = 0$ and $N = 1$,*

$$\sum_x \zeta^{x^2-2ax} H_N(x) = G(\phi) N(2a) \zeta^{-a^2}.$$

Proof.

$$\begin{aligned} G(\overline{N}) \sum_x \zeta^{x^2-2ax} H_N(x) &= \sum_{x,t} \overline{N}(t) \zeta^{t^2+2x(t-a)+x^2} \\ &= \zeta^{-a^2} \sum_t \overline{N}(t) \zeta^{2ta} \sum_x \zeta^{(x+t-a)^2} = \zeta^{-a^2} G(\phi) \sum_t \overline{N}(t) \zeta^{2ta} \end{aligned}$$

and the result follows.

Corresponding to the integral equation

$$H_n(x) = \frac{e^{x^2/2}}{i^n \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy-y^2/2} H_n(y)\, dy \qquad [\mathbf{20}, (4.12.3)], \tag{4.12}$$

we have

THEOREM 4.12. *If $N \neq 1$, then*

$$H_N(x) = \frac{\phi(2) G(\phi) N(-1)}{q} \zeta^{-x^2/2} \sum_u \zeta^{-xu-u^2/2} \overline{H}_{\overline{N}}(u).$$

Proof. By Theorem 4.9, the right side above equals

$$\begin{aligned} &\frac{\phi(2) G(\phi)}{q} \overline{G(\phi)} q^{-1} \sum_u \zeta^{-xu+(u^2-x^2)/2} \sum_t N(-2t) \zeta^{t^2-2tu} \\ &= \phi(2) q^{-1} \sum_t N(-2t) \zeta^{t^2-x^2/2} \sum_u \zeta^{(u^2-2xu-4tu)/2} \\ &= q^{-1} G(\phi) \sum_t N(-2t) \zeta^{t^2-x^2/2-(x+2t)^2/2} \\ &= \zeta^{-x^2} G(\phi) q^{-1} \sum_t N(-2t) \zeta^{-t^2-2tx} = H_N(x), \end{aligned}$$

again by Theorem 4.9.

Corresponding to

(4.13) $\qquad H_{2n}(x) = (-4)^n n! L_n^{-1/2}(x^2) \qquad$ [**6**, p. 193, (2)],

we have

THEOREM 4.13. *If $x \neq 0$ and $N \neq \phi$, then*

$$H_{N^2}(x) = \frac{qN(4)}{G(\overline{N})} L_N^{\phi}(x^2).$$

Proof. By Theorem 4.8,

$$H_{N^2}(x) = \frac{1}{(q-1)G(\overline{N}^2)} \sum_M N^2\overline{M}^2(2x)G(\overline{M})G(M^2\overline{N}^2).$$

Replace M by $N\overline{M}$ to obtain

$$\text{(4.13a)} \qquad H_{N^2}(x) = \frac{1}{(q-1)G(\overline{N}^2)} \sum_M M^2(2x)G(M\overline{N})G(\overline{M}^2).$$

By Theorem 2.4, for $z \neq 0$,

$$\begin{aligned} qL_N^A(z) &= \sum_M \frac{G(\overline{M})}{q-1} D^M qL_N^A(z)\Big|_{z=0} M(-z) \\ &= \sum_M \frac{M(-z)}{q-1} \sum_t \overline{M}(t) qL_N^A(-t). \end{aligned}$$

Thus, by the definition (2.2) of L_N^A,

$$\begin{aligned} qL_N^A(z) &= \sum_M \frac{M(-z)}{q-1} \sum_t \overline{M}(t) \sum_u NA(1+u)\overline{N}(u)\zeta^{-tu} \\ &= \sum_M \frac{M(z)}{q-1} G(\overline{M}) \sum_u M\overline{N}(u)NA(1+u) \\ &= \frac{N(-1)}{q-1} \sum_M M(-z)G(\overline{M})J(M\overline{N}, NA). \end{aligned}$$

Therefore, with $A = \phi$ and $z = x^2$,

$$\begin{aligned} \text{(4.13b)} \qquad & \frac{N(4)}{G(\overline{N})} qL_N^{\phi}(x^2) \\ &= \frac{N(-4)}{G(\overline{N})(q-1)} \sum_M M(-x^2)G(\overline{M})J(M\overline{N}, N\phi). \end{aligned}$$

Comparing (4.13a) and (4.13b), we see that it remains to show that

$$\text{(4.13c)} \qquad \frac{M(4)G(M\overline{N})G(\overline{M}^2)}{G(\overline{N}^2)} = \frac{N(-4)M(-1)G(\overline{M})J(M\overline{N}, N\phi)}{G(\overline{N})}.$$

If $M = \phi$, then since $N \neq \phi$, we have $J(M\overline{N}, N\phi) = -N\phi(-1)$, so (4.13c) follows from (2.15) with $A = \overline{N}$. If $M \neq \phi$, then $J(M\overline{N}, N\phi) = G(M\overline{N})G(N\phi)/G(M\phi)$, and again (4.13c) follows from (2.15).

Corresponding to

$$H_{-1/2}(z) = e^{z^2/2}\sqrt{\frac{z}{2\pi}}\,K_{1/4}\left(\frac{z^2}{2}\right) \tag{4.14}$$

(which is stated incorrectly in both [**6**, p. 119, (20)] and [**20**, p. 298, #6]), we have

THEOREM 4.14. *Let $x \neq 0$ and let N be quartic, so that $N^4 = 1$. Then*

$$H_\phi(x) = \frac{\zeta^{-x^2/2}\phi(2x)}{G(\phi)}K_N\left(\frac{x^2}{2}\right).$$

Proof. By Theorem 4.13 and the definition of L_N^ϕ,

$$\begin{aligned} H_\phi(x) = H_{N^2}(x) &= \frac{qN(4)}{G(\overline{N})}L_N^\phi(x^2) \\ &= \frac{N(4)}{G(\overline{N})}\sum_t \overline{N}(t)N\phi(1+t)\zeta^{tx^2} = \frac{\phi(2)}{G(\overline{N})}\sum_t \overline{N}(t+t^2)\zeta^{tx^2} \\ &= \frac{\phi(2)}{G(\overline{N})}\sum_t \overline{N}(t^2-\bar{4})\zeta^{(t-\bar{2})x^2} = \frac{\zeta^{-x^2/2}}{G(\overline{N})}\sum_t \overline{N}(t^2-1)\zeta^{tx^2/2}. \end{aligned}$$

Thus, by Theorem 2.7,

$$H_\phi(x) = \frac{\zeta^{-x^2/2}}{G(\overline{N})}\frac{G(\overline{N})\overline{N}(x^2/4)}{G(\phi)}K_N\left(\frac{x^2}{2}\right),$$

as desired.

Corresponding to

$$\int_{-\infty}^{\infty} e^{-w^2}w^nH_n(xw)\,dw = \sqrt{\pi}\,n!P_n(x) \qquad [\mathbf{6}, \text{p. } 195, (29)], \tag{4.15}$$

we have

THEOREM 4.15. *Let $N \neq 1$. Then*

$$\sum_w \zeta^{w^2}N(w)H_N(xw) = \frac{qN(-1)G(\phi)}{G(\overline{N})}P_N(x) + \frac{qN(-1)}{G(\overline{N})}\sum_{\substack{w \\ w^2+1=2xw}} N(w).$$

(Note that the last sum vanishes if $\phi(x^2-1) = -1$.)

Proof. By (2.3),

$$G(\overline{N})\sum_w \zeta^{w^2}N(w)H_N(xw) = \sum_u \overline{N}(u)\zeta^{u^2(1-x^2)}\sum_w N(w)\zeta^{(w+ux)^2}$$

$$= \sum_{u\neq 0}\zeta^{u^2(1-x^2)}\sum_w N(w)\zeta^{(wu+ux)^2} = \sum_w N(w)\sum_u \zeta^{u^2(1+w^2+2wx)}$$

$$= \sum_w N(w)\phi(1+w^2+2wx)G(\phi) + q\sum_{\substack{w\\ w^2+1=-2wx}} N(w)$$

$$= qN(-1)G(\phi)P_N(x) + qN(-1)\sum_{\substack{w\\ w^2+1=2wx}} N(w).$$

Corresponding to the addition theorem

$$H_n(ax+by) = \sum_{m=0}^{n}\binom{n}{m}a^m b^{n-m}H_m(x)H_{n-m}(y) \tag{4.16}$$

[**6**, p. 196, (40)],

we have

THEOREM 4.16. *If $a, b, x, y \in GF(q)$ with $a^2+b^2=1$ and $ab\neq 0$, then*

$$H_N(ax+by) = \sum_M \frac{G(\overline{M})G(M\overline{N})}{(q-1)G(\overline{N})}M(a)N\overline{M}(b)H_M(x)H_{N\overline{M}}(y).$$

Proof. Let $w \neq 0$. By Theorem 4.7,

$$\zeta^{a^2w^2+2wax} = \sum_M \frac{M(aw)G(\overline{M})}{q-1}H_M(x)$$

and

$$\zeta^{b^2w^2+2wby} = \sum_A \frac{A(bw)G(\overline{A})}{q-1}H_A(y).$$

Multiply to obtain

$$\zeta^{w^2+2w(ax+by)} \tag{4.16a}$$

$$= (q-1)^{-2}\sum_{A,M} M(a)A(b)MA(w)G(\overline{M})G(\overline{A})H_M(x)H_A(y).$$

Also, by Theorem 4.7,

$$\zeta^{w^2+2w(ax+by)} = \sum_N \frac{G(\overline{N})}{q-1}N(w)H_N(ax+by). \tag{4.16b}$$

The result now follows from Theorem 2.4 upon comparing the coefficients of $N(w)$ in (4.16a) and (4.16b).

Corresponding to (cf. (4.15))

$$(4.17) \qquad \int_{-\infty}^{\infty} e^{-aw^2} H_{2m}(w)\,dw = \frac{\sqrt{\pi}\,(2m)!}{m!} a^{-1/2}(a^{-1}-1)^m$$

[**20**, p. 75],

we have

THEOREM 4.17. *Let $a \neq 0, 1$. Then*

$$\sum_w \zeta^{aw^2} H_N(w)$$
$$= \begin{cases} 0, & \text{if } N \text{ is not a square} \\ \dfrac{G(\phi)}{G(\overline{M}^2)} \phi(a)\big(M(1-a^{-1})G(\overline{M}) + M\phi(1-a^{-1})G(\overline{M}\phi)\big), & \text{if } N = M^2. \end{cases}$$

Proof. Let $x = a^{-1}$. Then

$$G(\overline{N}) \sum_w \zeta^{aw^2} H_N(w) = \sum_w \zeta^{w^2 a} \sum_u \overline{N}(u) \zeta^{u^2+2uw}$$
$$= \sum_u \overline{N}(u) \zeta^{u^2(1-x)} \sum_w \zeta^{a(w+ux)^2} = \phi(a) G(\phi) \sum_u \overline{N}(u) \zeta^{u^2(1-x)},$$

and the result easily follows.

A generalization of the formula

$$\int_0^{\infty} e^{-x^2} H_n(x)^2 \cos(xy\sqrt{2})\,dx = e^{-y^2/2}\sqrt{\pi}\,2^{n-1} n! L_n^0(y^2),$$

which is incorrectly stated in [**6**, p.195, (33)], is

$$(4.18) \qquad \int_{-\infty}^{\infty} e^{-x^2-2ixy} H_n(x) H_m(x)\,dx$$
$$= L_m^{n-m}(2y^2) e^{-y^2}(-iy)^{n-m} 2^n \sqrt{\pi}\, m!,$$

which is stated incorrectly in [**22**, (4.166)]. Corresponding to (4.18), we have (cf. Theorem 4.11)

THEOREM 4.18. *Let $N \neq 1$, $a \neq 0$. Then*

$$L := \sum_x \zeta^{x^2+2xa} H_N(x) H_M(x) = \frac{\zeta^{-a^2} q G(\phi) N(-2a) \overline{M}(a)}{G(\overline{M})} L_M^{N\overline{M}}(-2a^2).$$

Proof. By Theorems 2.3 and 4.1,

$$L = \sum_x H_M(x)\zeta^{2ax}\left(N(-1)D^N\zeta^{x^2}\right) = \sum_x \zeta^{x^2}D^N\left(\zeta^{2ax}H_M(x)\right).$$

Thus,

$$\begin{aligned} L &= \frac{1}{G(\overline{N})}\sum_x \zeta^{x^2}\sum_t \overline{N}(t)\zeta^{2a(x-t)}H_M(x-t) \\ &= \frac{1}{G(\overline{N})G(\overline{M})}\sum_{s,t}\overline{N}(t)\overline{M}(s)\zeta^{s^2-2st-2at}\sum_x \zeta^{x^2+2x(a+s)} \\ &= \frac{G(\phi)}{G(\overline{N})G(\overline{M})}\sum_{s,t}\overline{N}(t)\overline{M}(s)\zeta^{s^2-2st-2at-(a+s)^2} \\ &= \frac{G(\phi)\zeta^{-a^2}}{G(\overline{N})G(\overline{M})}\sum_{s,t}\overline{N}(t)\overline{M}(s)\zeta^{-2(st+at+as)}. \end{aligned}$$

Since $N \neq 1$, the term with $s = -a$ may be excluded. Replace t by $-t/(a+s)$ to get

$$\text{(4.18a)} \qquad \begin{aligned} L &= \frac{G(\phi)\zeta^{-a^2}}{G(\overline{N})G(\overline{M})}\sum_{s,t}N(-a-s)\overline{N}(t)\overline{M}(s)\zeta^{2t-2as} \\ &= \frac{G(\phi)\zeta^{-a^2}N(-2)}{G(\overline{M})}\sum_s N(a+s)\overline{M}(s)\zeta^{-2as}. \end{aligned}$$

Since $a \neq 0$,

$$L = \frac{G(\phi)\zeta^{-a^2}N(-2a)\overline{M}(a)}{G(\overline{M})}\sum_s N(s+1)\overline{M}(s)\zeta^{-2a^2s}$$

and the result follows by the definition of $L_N^A(x)$.

Corresponding to the orthogonality relation

$$\text{(4.19)} \qquad \int_{-\infty}^{\infty} e^{-x^2}H_m(x)H_n(x)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ 2^n n!\sqrt{\pi}, & \text{if } m = n \ [\mathbf{20}, (4.13.1), (4.13.4)], \end{cases}$$

we have

THEOREM 4.19.

$$\sum_x \zeta^{x^2}H_M(x)H_N(x) = \begin{cases} 0, & \textit{if } M \neq N \\ \dfrac{N(-2)G(\phi)(q-1)}{G(\overline{N})}, & \textit{if } M = N. \end{cases}$$

Proof. By (4.18a) with $a = 0$,

$$\sum_x \zeta^{x^2} H_M(x) H_N(x) = \frac{G(\phi)N(-2)}{G(\overline{M})} \sum_s \overline{M}N(s)$$

and the result follows.

Corresponding to the Gauss transform formula

$$(4.20)\quad (2\pi u)^{-1/2}\int_{-\infty}^{\infty} H_n(v)\exp\left(-(x-v)^2/2u\right)dv$$
$$= \begin{cases} (2x)^n, & \text{if } u = 1/2 \\ (1-2u)^{n/2} H_n\left(x/\sqrt{1-2u}\right), & \\ & \text{if } 0 \le u < 1/2 \text{ [6, p. 195, (30), (31)]}, \end{cases}$$

we have

THEOREM 4.20. *Let $N \neq 1$, $u \neq 0$, and assume $1 - 2u = a^2 \in GF(q)$. Then*

$$L := \frac{\phi(2u)}{G(\phi)} \sum_v H_N(v) \zeta^{(x-v)^2/2u} = \begin{cases} N(2x), & \text{if } a = 0 \\ N(a) H_N(x/a), & \text{if } a \neq 0. \end{cases}$$

Proof.

$$L = \frac{\phi(2u)}{G(\overline{N})G(\phi)} \sum_v \zeta^{(x-v)^2/2u} \sum_w \overline{N}(w) \zeta^{w^2+2wv}$$
$$= \frac{\phi(2u)}{G(\overline{N})G(\phi)} \sum_w \overline{N}(w) \zeta^{w^2+2wx} \sum_s \zeta^{2ws+s^2/2u},$$

where we have replaced v by $s + x$. Now,

$$\sum_s \zeta^{2ws+s^2/2u} = \zeta^{-2w^2u}\phi(2u)G(\phi).$$

Therefore,

$$L = \frac{1}{G(\overline{N})} \sum_w \overline{N}(w) \zeta^{w^2a^2+2wx}.$$

If $a = 0$, clearly $L = N(2x)$. Suppose $a \neq 0$. Then

$$L = \frac{N(a)}{G(\overline{N})} \sum_w \overline{N}(w) \zeta^{w^2+2xw/a} = N(a) H_N(x/a).$$

Corresponding to

$$(4.21)\quad H_n(x) H_{-n-1}(x) = e^{x^2} \int_0^{\infty} J_{n+1/2}\left(\frac{t^2}{2}\right) \cos\left(xt - \frac{\pi n}{2}\right) e^{-xt}\, dt$$

[**6**, p. 120, (7)],

we have

THEOREM 4.21. *Let $N \neq 1$ and $x \neq 0$. Then*

$$H_N(x)H_{\overline{N}}(x) = \frac{\zeta^{-x^2}\phi(2)N(-1)}{q}\sum_u K_{N\phi}(u^2)\zeta^{2xu}.$$

Proof.

$$L := H_N(x)H_{\overline{N}}(x)qN(-1) = H_N(x)H_{\overline{N}}(x)G(N)G(\overline{N})$$

$$= \sum_{t,u\neq 0} \overline{N}(u)N(t)\zeta^{t^2+u^2+2x(t+u)}.$$

Since $N \neq 1$, we obtain, upon replacing t by tu,

$$L = \sum_t N(t)\sum_u \zeta^{u^2(t^2+1)+2xu(t+1)}$$

$$= \sum_t N(t)\zeta^{-x^2(t+1)^2/(t^2+1)}\phi(t^2+1)G(\phi)$$

$$= \zeta^{-x^2}G(\phi)\sum_t N(t)\phi(t^2+1)\zeta^{-2tx^2/(t^2+1)}$$

$$= \zeta^{-x^2}G(\phi)\sum_t N\phi(t)\phi(t+\bar{t})\zeta^{-2x^2/(t+\bar{t})}.$$

It remains to show that

$$\text{(4.21a)} \quad \sum_u K_{N\phi}(u^2)\zeta^{2xu} = \phi(2)G(\phi)\sum_t N\phi(t)\phi(t+\bar{t})\zeta^{-2x^2/(t+\bar{t})}.$$

By (2.5),

$$K_{N\phi}(u^2) = \sum_t N\phi(t)\zeta^{u^2(t+\bar{t})/2},$$

so

$$\sum_u K_{N\phi}(u^2)\zeta^{2xu} = \sum_t N\phi(t)\sum_u \zeta^{2xu+u^2(t+\bar{t})/2}.$$

Since the rightmost sum on u equals

$$\zeta^{-2x^2/(t+\bar{t})}\phi(t+\bar{t})\phi(2)G(\phi),$$

(4.21a) follows.

Corresponding to

$$(4.22)\quad H_n\left(\frac{z(1+i)}{2}\right)H_n\left(\frac{z(1-i)}{2}\right) = \frac{2^{n+3/2}}{\sqrt{\pi}\,\Gamma(-n)}\int_0^\infty K_{n+1/2}(t^2)\cos\left(zt-\frac{\pi n}{2}\right)e^{-zt}\,dt$$

[**6**, p. 120, (9)],

we have

THEOREM 4.22. *If $N \neq 1$ and $x \neq 0$, then*

$$H_N(x)\overline{H}_{\overline{N}}(x) = \frac{\phi N(2)G(N)}{qG(\phi)}\sum_t K_{N\phi}(t^2)\zeta^{2xt}.$$

Proof. This follows from Theorem 4.21 and Corollary 4.10.

Corresponding to (cf. (4.15))

$$(4.23)\quad \int_0^\infty e^{-t^2}t^m H_n(t)\,\frac{dt}{t} = \frac{\sqrt{\pi}\,2^{n-m}\Gamma(m)}{\Gamma((m-n+1)/2)}$$

[**6**, p. 122, (20)],

we have

THEOREM 4.23. *If $N \neq M$,*

$$\sum_t \zeta^{t^2}M(t)H_N(t) = \begin{cases} 0, & \text{if } M\overline{N} \text{ is not a square},\\ \dfrac{G(M)}{G(M\overline{N})}(G(A)+G(A\phi)), & \text{if } M\overline{N} = A^2.\end{cases}$$

Proof.

$$\begin{aligned}\sum_t \zeta^{t^2}M(t)H_N(t) &= \frac{1}{G(\overline{N})}\sum_{t,w}\overline{N}(w)M(t)\zeta^{w^2+2tw+t^2}\\ &= \frac{1}{G(\overline{N})}\sum_{t,w\neq 0}M(t)\overline{N}(w-t)\zeta^{w^2}\\ &= \frac{1}{G(\overline{N})}\sum_{w\neq 0}M\overline{N}(-w)\zeta^{w^2}\sum_t M(t)\overline{N}(1-t)\\ &= \frac{J(M,\overline{N})}{G(\overline{N})}\sum_w M\overline{N}(w)\zeta^{w^2} = \frac{G(M)}{G(M\overline{N})}\sum_w M\overline{N}(w)\zeta^{w^2},\end{aligned}$$

and the result follows.

By [**27**, p. 564, (14)], [**26**, Problem 87], we have

$$(4.24)\qquad \int_{-\infty}^{\infty} e^{-x^2}H_a(x)H_b(x)H_c(x)\,dx = \frac{\sqrt{\pi}\,2^{-n}a!b!c!}{(-n-a)!(-n-b)!(-n-c)!},$$

when $n = -(a+b+c)/2$ is an integer and a, b, c are positive integers. Corresponding to (4.24) is (cf. Theorem 4.19)

THEOREM 4.24.

$$L := \sum_x \zeta^{x^2} H_A(x)H_B(x)H_C(x)$$
$$= \begin{cases} 0, & \text{if } ABC \text{ is not a square} \\ \dfrac{G(\phi)}{G(\bar{A})G(\bar{B})G(\bar{C})}(\bar{N}(-2)G(AN)G(BN)G(CN) \\ \qquad + \bar{N}\phi(-2)G(AN\phi)G(BN\phi)G(CN\phi)), & \text{if } \overline{ABC} = N^2. \end{cases}$$

Proof. Successively applying Theorems 4.1, 2.3, 2.2, and 4.2, we have

$$L = \sum_x H_A(x)H_B(x)C(-1)D^C\zeta^{x^2} = \sum_x \zeta^{x^2}D^C H_A(x)H_B(x)$$
$$= \sum_x \zeta^{x^2}\sum_M \frac{G(\bar{M})G(M\bar{C})}{(q-1)G(\bar{C})}D^M H_A(x)D^{C\bar{M}}H_B(x)$$
$$= \sum_M \frac{G(\bar{M})G(M\bar{C})}{(q-1)G(\bar{C})}\sum_x \zeta^{x^2}\frac{C(-2)G(M\bar{A})G(C\bar{M}\bar{B})}{G(\bar{A})G(\bar{B})}H_{A\bar{M}}(x)H_{BM\bar{C}}(x).$$

By Theorem 4.19, $L = 0$ if ABC is not a square, while if $ABC = \bar{N}^2$, only $M = ACN$ and $M = ACN\phi$ contribute to the sum; these contributions are easily seen to be respectively the two required terms in Theorem 4.24.

Corresponding to Mehler's formula

$$(4.25)\qquad (1-u^2)^{-1/2}\exp\left\{\frac{2xyu-(x^2+y^2)u^2}{1-u^2}\right\} = \sum_{n=0}^{\infty}\frac{H_n(x)H_n(y)(u/2)^n}{n!} \quad [\mathbf{6}, \text{p. } 194, (22)],$$

we have

THEOREM 4.25. *Let*

$$r(x,y,z) = \begin{cases} G(\phi)\zeta^{-x^2}, & \text{if } x = yz \text{ and } z = \pm 1 \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$L := \sum_N \frac{G(\overline{N})}{q-1} H_N(x) H_N(y) \overline{N}(-2z)$$

$$= \phi(z^2-1)\zeta^{(2xyz-x^2-y^2)/(1-z^2)} - \phi(-1)\zeta^{-x^2-y^2} + r(x,y,z).$$

(*The first term on the right is interpreted as* 0 *when* $z^2 = 1$.)

Proof. The result is clear for $z = 0$, so assume $z \neq 0$. Then

$$L = \frac{1}{(q-1)} \sum_N \frac{\overline{N}(-2z)}{G(\overline{N})} \sum_{s,t\neq 0} \overline{N}(st)\zeta^{s^2+t^2+2sx+2ty}$$

$$= \frac{1}{q(q-1)} \sum_{N\neq 1} \sum_{s,t,u\neq 0} \overline{N}(2zst/u)\zeta^{s^2+t^2+u+2sx+ty}$$

$$- \frac{1}{q-1} \sum_{s,t\neq 0} \zeta^{s^2+t^2+2sx+2ty}$$

$$= \frac{1}{q(q-1)} \sum_N \sum_{s,t,u\neq 0} \overline{N}(2zst/u)\zeta^{s^2+t^2+u+2sx+2ty}$$

$$- \frac{1}{q} \sum_{s,t\neq 0} \zeta^{s^2+t^2+2sx+2ty}$$

$$= \frac{1}{q} \sum_{s,t\neq 0} \left(\zeta^{s^2+t^2+2zst+2sx+2ty} - \zeta^{s^2+t^2+2sx+2ty}\right).$$

The restrictions $s \neq 0$, $t \neq 0$ can now be dropped, so

$$L = \frac{-G^2(\phi)}{q}\zeta^{-x^2-y^2} + \frac{1}{q}\sum_s \zeta^{s^2+2sx-(y+sz)^2}\sum_t \zeta^{t^2+t(2y+2zs)+(y+sz)^2}$$

$$= -\phi(-1)\zeta^{-x^2-y^2} + \frac{G(\phi)}{q}\zeta^{-y^2}\sum_s \zeta^{s^2(1-z^2)+2s(x-yz)}.$$

If $z^2 = 1$ and $x = yz$, then

$$L = -\phi(-1)\zeta^{-x^2-y^2} + G(\phi)\zeta^{-x^2},$$

as desired. If $z^2 = 1$ and $x \neq yz$, then

$$L = -\phi(-1)\zeta^{-x^2-y^2},$$

as desired. Finally, suppose $z^2 \neq 1$. Then

$$L = -\phi(-1)\zeta^{-x^2-y^2} + \frac{G(\phi)}{q}\zeta^{-y^2-(x-yz)^2/(1-z^2)}\phi(1-z^2)G(\phi)$$

$$= -\phi(-1)\zeta^{-x^2-y^2} + \phi(z^2-1)\zeta^{(2xyz-x^2-y^2)/(1-z^2)},$$

as desired.

COROLLARY 4.26. *If $N \neq 1$, and*

$$F(x, y) = \begin{cases} N(-2x/y), & \textit{if } x = \pm y \neq 0 \\ N(2) + N(-2), & \textit{if } x = y = 0 \\ 0, & \textit{otherwise}, \end{cases}$$

then

$$H_N(x)H_N(y) = \frac{F(x, y)G(\phi)\zeta^{-x^2}}{G(\overline{N})} + \frac{N(2)}{G(\overline{N})}\sum_t \overline{N}(t)\phi(1-t^2)\zeta^{(2xyt+t^2x^2+t^2y^2)/(1-t^2)}.$$

Proof. By Theorems 4.25 and 2.4,

$$\begin{aligned} &H_N(x)H_N(y)\overline{N}(2) \\ &= D_z^N\Big\{\phi(1-z^2)\zeta^{(2xyz-x^2z^2-y^2z^2)/(z^2-1)} \\ &\qquad -\phi(-1)\zeta^{-x^2-y^2} + r(x, y, z)\Big\}\Big|_{z=0} \\ &= \frac{1}{G(\overline{N})}\sum_t \overline{N}(t)\phi(1-t^2)\zeta^{(2xyt+x^2t^2+y^2t^2)/(1-t^2)} \\ &\quad + \frac{1}{G(\overline{N})}\sum_t \overline{N}(t)r(x, y, -t), \end{aligned}$$

and the result follows since

$$N(2)\sum_t \overline{N}(t)r(x, y, -t) = F(x, y)G(\phi)\zeta^{-x^2}.$$

Under certain conditions [**20**, p. 71, Theorem 2], a function $f(x)$ has an expansion of the form

$$f(x) = \sum_{n=0}^{\infty} e_n H_n(x), \tag{4.27}$$

where

$$e_n = \frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} f(x) H_n(x)\, dx.$$

Corresponding to (4.27), we have

THEOREM 4.27. *Let F: $GF(q) \to \mathbf{C}$. Then*

$$F(u) = \sum_N e_N H_N(u) + q^{-1}\zeta^{-u^2} G(\phi) \sum_t F(t),$$

where

$$e_N = \frac{G(\overline{N})\overline{N}(-2)}{G(\phi)(q-1)} \sum_x \zeta^{x^2} F(x) H_N(x).$$

Proof. By Theorem 4.25 with $z = 1$,

$$\begin{aligned} \sum_N e_N H_N(u) &= \frac{1}{G(\phi)} \sum_x \zeta^{x^2} F(x) \left\{ -\phi(-1)\zeta^{x^2-u^2} + r(x,u,1) \right\} \\ &= \frac{-\phi(-1)}{G(\phi)} \zeta^{-u^2} \sum_x F(x) + F(u). \end{aligned}$$

Corresponding to

$$H_b(x)H_c(x) = \sum_{m=0}^{\infty} 2^m m! \binom{b}{m}\binom{c}{m} H_{b+c-2m}(x) \tag{4.28}$$

[**6**, p. 195, (37)],

we have

THEOREM 4.28. *Let*

$$g(B,C) = \begin{cases} 0, & \text{if } BC \text{ is not a square} \\ W(2)G(\overline{W}) + \phi W(2) G(\overline{W}\phi), & \text{if } BC = W^2. \end{cases}$$

Then

$$\begin{aligned} H_B(x)H_C(x) = {} & \frac{C(-1)\zeta^{-x^2} G(\phi) g(B,C)}{G(\overline{B})G(\overline{C})} + \frac{1}{G(\overline{B})G(\overline{C})(q-1)} \\ & \times \sum_M M(-2) G(\overline{M}) G(M\overline{B}) G(M\overline{C}) H_{BC\overline{M}^2}(x). \end{aligned}$$

Proof. By Theorem 4.27 with $F(x) = H_B(x)H_C(x)$,

$$H_B(x)H_C(x) = \sum_A e_A H_A(x) + q^{-1}\zeta^{-x^2}G(\phi)\sum_t H_B(t)H_C(t), \tag{4.28a}$$

where

$$e_A = \frac{G(\overline{A})\overline{A}(-2)}{G(\phi)(q-1)}\sum_x \zeta^{x^2}H_A(x)H_B(x)H_C(x). \tag{4.28b}$$

When ABC is a square, then as in Theorem 4.24, let $N^2 = \overline{A}\,\overline{B}\,\overline{C}$, and put $M = NBC$, so that $A = BC\overline{M}^2$. Then by (4.28b) and Theorem 4.24,

$$\sum_A e_A H_A(x)$$
$$= \frac{1}{G(\overline{B})G(\overline{C})(q-1)}\sum_M M(-2)G(\overline{M})G(M\overline{B})G(M\overline{C})H_{BC\overline{M}^2}(x).$$

It is easily checked that

$$\sum_t H_B(t)H_C(t) = \frac{qC(-1)}{G(\overline{B})G(\overline{C})}g(B,C),$$

so the result follows from (4.28a).

The formula

$$H_n(x+a) = \sum_{k=0}^{n}\binom{n}{k}(2a)^k H_{n-k}(x) \tag{4.29}$$

is stated in [**25**, p. 252, Ex. 1] and, in slightly different form, in [**25**, p. 253, Ex. 8]. Miller [**22**, p. 106] states (4.29) incorrectly. Corresponding to (4.29), we have

THEOREM 4.29. *Let $a \neq 0$. Then*

$$H_N(x+a) = \frac{1}{q-1}\sum_B \frac{G(\overline{B})G(B\overline{N})}{G(\overline{N})}B(2a)H_{N\overline{B}}(x).$$

Proof. The right side above equals

$$\frac{1}{(q-1)G(\overline{N})}\sum_B G(\overline{B})B(2a)\sum_u B\overline{N}(u)\zeta^{u^2+2ux}$$
$$= \frac{1}{G(\overline{N})}\sum_{u\neq 0}\overline{N}(u)\zeta^{u^2+2ux}\frac{1}{q-1}\sum_B G(\overline{B})B(2ua)$$
$$= \frac{1}{G(\overline{N})}\sum_u \overline{N}(u)\zeta^{u^2+2ux+2ua} = H_N(x+a).$$

LEMMA 4.30. *Let $z \neq 0$. Then (cf. Theorem 4.19)*

$$L := \sum_t \zeta^{t^2} H_M(t) H_N(t+z) = G(M\overline{N}) G(\phi) N(2z) \overline{M}(-z)/G(\overline{N}).$$

Proof.

$$\begin{aligned} G(\overline{M})G(\overline{N})L &= \sum_{s,t,u} \overline{M}(s)\overline{N}(u)\zeta^{t^2+s^2+2st+u^2+2u(t+z)} \\ &= G(\phi) \sum_{s,u\neq 0} \overline{M}(s)\overline{N}(u)\zeta^{2uz-2us} \\ &= G(\phi) \sum_{s,u\neq 0} \overline{M}(s)M\overline{N}(u)\zeta^{2uz-2s} \\ &= G(\phi)M(-2)G(\overline{M})N\overline{M}(2z)G(M\overline{N}), \end{aligned}$$

as desired.

Corresponding to (cf. (4.3) and (4.7))

$$(4.31) \quad e^{2xz-z^2} H_m(x-z) = \sum_{n=0}^{\infty} \frac{H_{n+m}(x) z^n}{n!} \qquad [**25**, \text{p. } 197, (1)],$$

we have

THEOREM 4.31. *Let $z \neq 0$. Then*

$$\begin{aligned} F(x) &:= \zeta^{z^2-2xz} H_M(x-z) \\ &= \frac{\overline{M}(z)\zeta^{-x^2}G(\phi)}{G(\overline{M})} + \sum_N \frac{G(\overline{N})N(-z)}{q-1} H_{NM}(x). \end{aligned}$$

Proof. Define e_N as in Theorem 4.27. By Lemma 4.30, $e_N = N\overline{M}(-z)G(M\overline{N})/(q-1)$. It is easily proved (cf. Theorem 4.11) that

$$\sum_t \zeta^{-z^2-2tz} H_M(t) = q\overline{M}(z)/G(\overline{M}).$$

Thus the result follows from Theorem 4.27.

COROLLARY 4.32. *Let $z \neq 0$. Then*

$$\zeta^{z^2} H_M(z) = \frac{\overline{M}(-z)G(\phi)}{G(\overline{M})} + \sum_A \frac{G(\overline{A})G(\overline{A}^2 M)A^2\overline{M}(z)}{(q-1)G(\overline{A}^2)}.$$

Proof. In Theorem 4.31, set $x = 0$, replace z by $-z$, and apply Theorem 4.6.

LEMMA 4.33. *Let $a \neq 0, \pm 1$. Then (cf. Lemma 4.30, Theorem 4.19)*

$$L := \sum_x \zeta^{x^2} H_M(x) H_N(ax)$$

$$= \begin{cases} 0, & \text{if } \overline{M}N \text{ is not a square} \\ \dfrac{M(-2a)G(\phi)}{G(\overline{N})}\big(A(1-a^2)G(\overline{A}) + \phi A(1-a^2)G(\overline{A}\phi)\big), & \\ & \text{if } \overline{M}N = A^2. \end{cases}$$

Proof.

$$L = \frac{1}{G(\overline{N})G(\overline{M})} \sum_{s,t} \overline{M}(s)\overline{N}(t)\zeta^{s^2+t^2} \sum_x \zeta^{x^2+2x(s+at)}$$

$$= \frac{G(\phi)}{G(\overline{N})G(\overline{M})} \sum_{s,t \neq 0} \overline{M}(s)\overline{N}(t)\zeta^{t^2(1-a^2)-2ast}$$

$$= \frac{M(-2a)G(\phi)}{G(\overline{M})} \sum_t M\overline{N}(t)\zeta^{t^2(1-a^2)},$$

and the result easily follows.

Corresponding to

$$(4.34) \qquad a^n H_n\left(\frac{x}{a}\right) = n! \sum_{k=0}^{[n/2]} \frac{H_{n-2k}(x)(1-a^2)^k}{(n-2k)!k!} \qquad [\mathbf{1}, (4.16)]$$

(of which the special case $a = \sqrt{2}$ is given in [**25**, p. 253, Ex. 7]), we have (cf. Theorem 4.20)

THEOREM 4.34. *Let $a \neq 0, \pm 1$. Then*

$$N(a)H_N\left(\frac{x}{a}\right) = \frac{1}{(q-1)G(\overline{N})} \sum_A H_{N\overline{A}^2}(x) G(\overline{A}) G(\overline{N}A^2) A(a^2-1).$$

Proof. By Theorem 4.27 with $F(x) = H_N(x/a)$,

$$H_N\left(\frac{x}{a}\right) = \sum_M H_M(x) \frac{G(\overline{M})\overline{M}(-2)}{G(\phi)(q-1)} \sum_t \zeta^{t^2} H_M(t) H_N\left(\frac{t}{a}\right).$$

Applying Lemma 4.33 and replacing M by $N\overline{A}^2$, we obtain the result.

Corresponding to the formula

$$(4.35) \qquad e^{-t^2-2xt} H_n\left(x + t + \frac{a}{t}\right) = \sum_{m=0}^{\infty} L_m^{n-m}(2a) H_m(x)\left(\frac{t}{2a}\right)^{m-n},$$

a version of which is incorrectly stated in [**22**, (4.76)], we have (cf. Theorems 4.29 and 4.31)

THEOREM 4.35. *For $at \neq 0$ and $N \neq 1$,*

$$F(t) := \zeta^{t^2+2xt} H_N(x+t+a\bar{t}) = \frac{\overline{N}(-t)G(\phi)\zeta^{-x^2-2a}}{G(\overline{N})} + \frac{q}{q-1}\sum_M L_M^{N\overline{M}}(2a)H_M(x)M\overline{N}\left(\frac{t}{2a}\right).$$

Proof. By (2.8),

$$\begin{aligned}
G(\overline{A})D_t^A F(t)\big|_{t=0} &= \sum_u \overline{A}(u)\zeta^{u^2-2ux}H_N(x-u-a\bar{u}) \\
&= \frac{1}{G(\overline{N})}\sum_{u,v}\overline{A}(u)\overline{N}(v)\zeta^{u^2-2ux+v^2+2vx-2vu-2av\bar{u}} \\
&= \frac{1}{G(\overline{N})}\sum_{u,v}\overline{A}(u)\overline{N}(v+u)\zeta^{v^2+2vx-2av\bar{u}-2a} \\
&= \frac{\zeta^{-2a}}{G(\overline{N})}\left(\sum_u \overline{A}\,\overline{N}(u) - \sum_{u,v\neq 0}\overline{A}(uv)\overline{N}(v+uv)\zeta^{v^2+2vx-2a\bar{u}}\right) \\
&= \frac{\zeta^{-2a}}{G(\overline{N})}\left(\sum_u \overline{AN}(u) + \sum_{u,v}\overline{AN}(v)AN(u)\overline{N}(1+u)\zeta^{v^2+2vx-2au}\right) \\
&= \frac{\zeta^{-2a}}{G(\overline{N})}\left(\sum_u \overline{AN}(u) + H_{AN}(x)G(\overline{AN})\sum_u AN(u)\overline{N}(1+u)\zeta^{-2au}\right).
\end{aligned}$$

Now, since $N \neq 1$ and $a \neq 0$,

$$\begin{aligned}
G(\overline{AN})\sum_u AN(u)\overline{N}(1+u)\zeta^{-2au} &= \sum_{u,v\neq 0}\overline{AN}(v)\overline{N}(1+u)\zeta^{u(v-2a)} \\
&= -\sum_v \overline{AN}(v) + G(\overline{N})\sum_v \overline{AN}(v)N(v-2a)\zeta^{2a-v} \\
&= -\sum_v \overline{AN}(v) + \zeta^{2a}G(\overline{N})\overline{A}(-2a)\sum_v \overline{AN}(v)N(v+1)\zeta^{2av} \\
&= -\sum_v \overline{AN}(v) + \zeta^{2a}G(\overline{N})\overline{A}(-2a)qL_{AN}^{\overline{A}}(2a).
\end{aligned}$$

Thus,

$$\begin{aligned}
G(\overline{A})D_t^A F(t)\big|_{t=0} = \frac{\zeta^{-2a}}{G(\overline{N})}&\sum_u \overline{AN}(u)(1-H_{AN}(x)) \\
&+ H_{AN}(x)\overline{A}(-2a)qL_{AN}^{\overline{A}}(2a).
\end{aligned}$$

Therefore, by Theorem 2.4,

$$F(t) = \frac{\zeta^{-2a}}{(q-1)G(\overline{N})} \sum_A (1 - H_{AN}(x)) A(-t) \sum_u \overline{AN}(u) + \frac{q}{q-1} \sum_A H_{AN}(x) A\left(\frac{t}{2a}\right) L_{AN}^{\overline{A}}(2a).$$

The result now follows with use of Theorem 4.4.

Let $G_n(x)$ denote the Hermite function of the second kind defined in [**2**, (2.44)]. In view of [**20**, (9.13.8) and (1.2.3)], it is reasonable to consider $\phi(-1)H_N(x)$ as the finite field analogue of $G_n(x)$.

Corresponding to (cf. (4.24))

$$\int_{-\infty}^{\infty} G_{m+l+2k+1}(x) H_m(x) H_l(x) e^{-2x^2}\, dx = \frac{(-1)^k 2^{k+l+m}(k+m)!(k+l)!}{k!} \quad [\mathbf{2}, (2.47)], \tag{4.36}$$

we have

THEOREM 4.36. *Let A, B, C, and ABC be nontrivial. Then*

$$L := \sum_x \zeta^{2x^2} H_A(x) H_B(x) H_C(x)$$
$$= \begin{cases} 0, & \textit{if } ABC \textit{ is not a square}, \\ q^{-1}\phi(-1)\overline{N}(2)G(\overline{AN})G(\overline{BN})G(\overline{CN}) \\ \quad + q^{-1}\phi(-1)\overline{N}\phi(2)G(\overline{AN}\phi)G(\overline{BN}\phi)G(\overline{CN}\phi), & \textit{if } \overline{A}\,\overline{B}\,\overline{C} = N^2. \end{cases}$$

Proof. We have

$$L = \frac{1}{G(\overline{A})G(\overline{B})G(\overline{C})} \sum_{r,s,t,x} \overline{A}(r)\overline{B}(s)\overline{C}(t)\zeta^{r^2+s^2+t^2+2x^2+2x(r+s+t)}$$
$$= \frac{G(\phi)\phi(2)}{G(\overline{A})G(\overline{B})G(\overline{C})} \sum_{r,s,t\neq 0} \overline{A}(r)\overline{B}(s)\overline{C}(t)\zeta^{(r^2+s^2+t^2-2rs-2rt-2st)/2}$$
$$= \frac{G(\phi)\phi(2)}{G(\overline{A})G(\overline{B})G(\overline{C})} \sum_{r,s,t\neq 0} \overline{ABC}(t)\overline{B}(s)\overline{C}(r)\zeta^{t^2(1+r^2+s^2-2rs-2r-2s)/2}.$$

Clearly $L = 0$ if ABC is not a square, so assume that $\overline{ABC} = N^2$. Then, since $N^2 \neq 1$,

$$L = \frac{G(\phi)\phi(2)}{G(\overline{A})G(\overline{B})G(\overline{C})}(N(2)G(N)Z(N) + N\phi(2)G(N\phi)Z(N\phi)), \tag{4.36a}$$

where

(4.36b) $$Z(N) = \sum_{r,s} \bar{B}(s)\bar{C}(r)\bar{N}(1 + r^2 + s^2 - 2rs - 2r - 2s).$$

By [7, (20) and (27)],

(4.36c) $$Z(N) = \frac{N\phi(-1)G(\bar{N}^2)G(\bar{B})G(\bar{C})J(\bar{B}\bar{N}\phi, \bar{C}\bar{N}\phi, N\phi)}{qG(A)},$$

where

$$J(M_1, M_2, M_3) := \sum_{x+y+z=1} M_1(x)M_2(y)M_3(z).$$

It is well-known [17, p. 100] that

(4.36d) $$J(M_1, M_2, M_3) = G(M_1)G(M_2)G(M_3)\overline{G(M_1M_2M_3)}/q$$

when $M_3 \neq 1$. By (4.36c) and (4.36d),

(4.36e)

$$Z(N) = \frac{N\phi(-1)G(\bar{N}^2)G(\bar{B})G(\bar{C})G(\bar{B}\bar{N}\phi)G(\bar{C}\bar{N}\phi)G(N\phi)\overline{G(\bar{B}\bar{C}\bar{N}\phi)}}{q^2G(A)},$$

and $Z(N\phi)$ is found by replacing N by $N\phi$ in (4.36e). Thus, from (4.36a) and (2.15), the result follows.

5. Associated Hermite sums. Let $H_n(x;\, c)$ denote the associated Hermite polynomial defined in [1, p. 16]. In view of the generating function formulas (4.7) and

$$\sum_{m=0}^{\infty} \frac{t^{m+c}H_m(x;\, c)}{(c)_{m+1}} = e^{2xt-t^2}\int_0^t u^{c-1}e^{u^2-2xu}\,du \qquad [\mathbf{1}, (4.14)],$$

it is reasonable to define the finite field analogue $H_M(x;\, C)$ of $H_m(x;\, c)$, for a nontrivial character C, by

$$H_M(x;\, C) = H_{MC}(x)\frac{1}{G(C)}\sum_u C(u)\zeta^{2xu-u^2}.$$

Thus,

$$H_M(x;\, C) = H_{MC}(x)\overline{H_C(x)} = H_{MC}(x)H_C(x)\bar{C}(2)\zeta^{x^2}q/(G(\phi)G(C)),$$

where the last equality follows from Corollary 4.10. In view of Theorem 4.19, the character sums $H_M(x;\, C)$ clearly satisfy an orthogonality relation with weight function $\zeta^{x^2}/\overline{H_C(x)}^2$ (cf. [1, (4.7)]).

To obtain the analogue of the polynomial representation of $H_m(x;\, c)$ in powers of x (see [1, (2.8) and §5]), one must, according to Theorem 2.4, compute $D^N H_M(x;\ C)$ at $x = 0$. This may be accomplished by applying Theorems 4.28, 4.3, and 4.6. We omit the details.

We close with an example to show how finite field analysis might be used to *conjecture* explicit formulas for special functions, which then may be proved by complex analytic methods.

Suppose it is desired to find a formula for $H_m(x;\ c)$ as a line4ar combination of Hermite polynomials. By Theorem 4.27, we have essentially

$$H_M(x;\ C) \overset{*}{=} \frac{\overline{C}(2)\phi(-1)}{(q-1)G(C)} \times \sum_A H_A(x) G(\overline{A})\overline{A}(-2) \sum_t \zeta^{2t^2} H_A(t) H_C(t) H_{MC}(t),$$

where the asterisk signifies that a few isolated terms have been ignored. Using Theorem 4.36 to evaluate the inner sum on t, we have

$$H_M(x;\ C) \overset{*}{=} \frac{\overline{C}(2)}{q(q-1)G(C)} \sum_N H_{\overline{BCN^2}}(x) G(BCN^2) \times BCN^2(-2)\overline{N}(2) G(BCN) G(\overline{BN}) G(\overline{CN}),$$

where $B = MC$. Replace N by $N\overline{B}$ to obtain

$$H_M(x;\ C) \overset{*}{=} \frac{M(-1)}{q(q-1)G(C)} \times \sum_N H_{M\overline{N}^2}(x) G(N^2\overline{M}) N(2) G(NC) G(\overline{N}) G(M\overline{N}).$$

From this one might conjecture that

$$H_m(x;\ c) = \sum_{n=0}^{\infty} \frac{(-2)^n (c)_n (m-n)!}{n!(m-2n)!} H_{m-2n}(x),$$

which is in agreement with [**1**, (4.18)].

Acknowledgment. The author is grateful to Richard Askey for several helpful suggestions.

References

[1] R. Askey and J. Wimp, *Associated Laguerre and Hermite polynomials*, Proc. Royal Soc. Edinburgh, **96A** (1984), 15–37.

[2] R. Askey, T. Koornwinder, and M. Rahman, *An integral of products of ultraspherical functions and a q-extension*, to appear.

[3] B. Berndt and R. Evans, *Sums of Gauss, Eisenstein, Jacobi, Jacobsthal, and Brewer*, Illinois J. Math., **23** (1979), 374–437.

[4] G. Birkhoff and S. MacLane, *A Survey of Modern Algebra*, 3rd edition, Macmillan, N. Y., 1965.

[5] H. Davenport and H. Hasse, *Die Nullstellen der Kongruenzzetafunktionen in gewissen zyklischen Fällen*, J. reine angew. Math., **172** (1934), 151–182.

[6] A. Erdélyi, *Higher Transcendental Functions*, vol. 2, McGraw Hill, N. Y., 1953.

[7] R. Evans, *Identities for products of Gauss sums over finite fields*, Enseignement Math., **27** (1981), 197–209.

[8] ______, *Character sum analogues of constant term identities for root systems*, Israel J. Math., **46** (1983), 189–196.

[9] R. Evans, J. Pulham, and J. Sheehan, *On the number of complete subgraphs contained in certain graphs*, J. Combinatorial Theory, Series B, **30** (1981), 364–371.

[10] D. Goldfeld and P. Sarnak, *Sums of Kloosterman sums*, Inventiones Math., **71** (1983), 243–250.

[11] J. Greene, *Character sum analogues for hypergeometric and generalized hypergeometric functions over finite fields*, Ph.D. thesis, University of Minnesota, 1984.

[11A] J. Greene, *Hypergeometric functions over finite fields*, preprint.

[12] J. Greene and D. Stanton, *A character sum evaluation and Gaussian hypergeometric series*, J. Number Theory, (to appear).

[13] ______, *The triplication formula for Gauss sums*, preprint.

[14] A. Helversen-Pasotto, *L'identité de Barnes pour les corps finis*, C. R. Acad. Sci. Paris, Série A, **286** (1978), 297–300.

[15] ______, *L'identité de Barnes pour les corps finis*, Séminaire Delange-Pisot-Poitou (Théorie des Nombres), 19e année, 22(1977/78).

[16] ______, *Darstellungen von $Gl(3, F_q)$ und Gaußsche summen*, Math. Ann., **260** (1982), 1–21.

[17] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer-Verlag, N. Y., 1982.

[18] C. Jacobi, *Über die Kreisteilung und ihre Anwendung auf die Zahlentheorie*, J. reine angew. Math., **30** (1846), 166–182; Werke, vol. 6, Chelsea, N. Y., 1969, pp. 254–264.

[19] N. Koblitz, *The number of points on certain families of hypersurfaces over finite fields*, Compositio Math., **48** (1983), 3–23.

[20] N. Lebedev, *Special Functions and Their Applications*, Dover, N. Y., 1972.

[21] W. Li and J. Soto-Andrade, *Barnes' identities and representations of $GL(2)$*, part I: *finite field case*, J. reine angew Math., **344** (1983), 171–179.

[21A] R. Lidl and H. Niederreiter, *Finite Fields*, Addison-Wesley, Reading, Mass., 1983.

[22] W. Miller, *Lie Theory and Special Functions*, Academic Press, N. Y., 1968.

[23] L. Mordell, *On some exponential sums related to Kloosterman sums*, Acta Arith., **21** (1972), 65–69.

[24] ______, *On Salié's sum*, Glasgow Math. J., **14** (1973), 25–26.

[25] E. Rainville, *Special Functions*, Macmillan, N. Y., 1960.

[26] G. Szegö, *Orthogonal polynomials*, AMS Colloq. Publ. 23, 4th ed., Providence, R. I., 1975.

[27] N. Vilenkin, *Special functions and the theory of group representations*, Amer. Math. Soc., Providence, 1968.

Received March 15, 1985.

UNIVERSITY OF CALIFORNIA, SAN DIEGO
LA JOLLA, CA 92093