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FACTORIZATIONS OF NATURAL EMBEDDINGS OF l_p^n INTO L_r, II

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This is a continuation of the paper by Figiel, Johnson and Schechtman with a similar title. Several results from there are strengthened, in particular: 1. If T is a "natural" embedding of l_2^n into L_1 then, for any well-bounded factorization of T through an L_1 space in the form $T = uv$ with v of norm one, u well-preserves a copy of l_1^k with k exponential in n. 2. Any norm one operator from a $C(K)$ space which well-preserves a copy of l_2^n also well-preserves a copy of l_∞^k with k exponential in n. As an application of these and other results we show the existence, for any n, of an n-dimensional space which well-embeds into a space with an unconditional basis only if the latter contains a copy of l_∞^k with k exponential in n.

Introduction. In this continuation of [FJS], we show that in some situations considered in [FJS], conclusions of certain theorems can be strengthened. More explicitly, suppose that T is an operator from some Banach space into L_1 which factors through some L_1-space Z as uw and normalized so that $\|w\| = 1$. In Corollary 12.A we show that if T is the inclusion mapping from a "natural" n-dimensional Hilbert-ian subspace of L_1 into L_1, then u well-preserves a copy of l_1^k with k exponential in n (where "well" and the base of the exponent depend on $\|u\|$ and on a quantitative measure of "naturalness"). This improves the result of [FJS] that the same hypotheses yield that l_1^k well-embeds into uZ. (Actually Corollary 12.A requires only that T be a "good" isomorphism from a "natural" n-dimensional Hilbertian subspace of L_1). Corollary 12.B gives a similar improvement of Corollary 1.5 in [FJS]; in Corollary 12.B the operator T is assumed to be a mapping from a space whose dual has controlled cotype into L_1 which acts like a quotient mapping relative to a "natural" Hilbertian subspace of L_1.

Corollary 20 strengthens the conclusion of proposition 4.3 in [FJS] in a similar manner; it states that an operator from a $C(K)$ space which well-preserves a copy of l_2^n also well-preserves a copy of l_∞^k with k exponential in n (rather than just have rank which is exponential in n). This can be viewed as a finite dimensional analogue of a particular case of a result of Pełczyński **[Pe1]** stating that every non weakly compact operator from a $C(K)$ space preserves a copy of c_0. Although the statements of Corollary 20 and Corollary 12.A are very similar, the results themselves do not seem to follow from each other via standard duality arguments.

In Theorem 21 we apply the earlier results in order to prove that for each m there is an m-dimensional normed space G such that any superspace of G with a good unconditional basis must contain a copy of l_∞^k with k exponential in m.

We thank J. Bourgain for pointing us in the right direction on the material presented here. After proving the results in **[FJS]**, we suggested to him that there might be a translation invariant operator T of bounded norm on L_1 of the circle which is the identity on the span of the first n Rademacher functions and which does not preserve l_1^k with k exponential in n. By disproving this conjecture, Bourgain started us thinking that Corollary 12.A was true.

We use standard Banach space theory notation, as can be found in **[LT]** and **[T-J]**. In particular, $d(X, Y)$ is used for the Banach-Mazur distance between the normed spaces X and Y, while $\pi_p(T)$ is the p-absolutely summing norm of the linear operator T. As usual, t^* denotes $\frac{t}{t-1}$, the conjugate index to t.

Most nonstandard notation is used only "locally" and is introduced when needed. However, the following two definitions are used throughout the paper and are important for understanding the formulations of the main L_1-results, Corollaries 12A and 12B: Given $1 \leq p < q \leq \infty$ and an operator $u\colon Z \to L_p(\mu)$, we define

$$C_{p,q}(u) = \inf\{\|h\|_s \|h^{-1}u\colon Z \to L_q\|\}$$

where the inf is over all changes of measure h; i.e., over all $0 < h \in L_s(\mu)$ where $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$. Also recall that for a pair of linear operators $T\colon X \to Y$ and $U\colon X_1 \to Y_1$, the factorization constant of U through T is defined to be

$$\gamma_T(U) = \inf\{\|A\|\,\|B\|\colon A\colon X_1 \to X,\ B\colon Y \to Y_1,\ U = BTA\}.$$

We let $\gamma_T(U) = \infty$ if no such factorization exists. We also put

$$\gamma_T(Z) = \gamma_T(\mathrm{id}_Z\colon Z \to Z).$$

This should be compared with the classical concept of $\gamma_p(U)$, which in our notation can be defined (say, for an operator U from a separable space) by

$$\gamma_p(U) = \gamma_{\mathrm{id}_{L_p}}(U).$$

A quantitative version of Rosenthal's lemma. In this section we prove, in Proposition 1 below, a quantitative version of Maurey's formulation [M] of Rosenthal's lemma [R] stating that an operator into $L_1(\mu)$ either factors through an $L_p(\nu)$ space for some $p > 1$ via a change of density or is of type no better than 1. Our approach is in fact close in spirit to Rosenthal's original argument which (unlike some later arguments) was basically quantitative in nature.

Proposition 1 combines with the essentially known Proposition 4 to yield the central result, Theorem 8, a specialized version of which (Theorem 11) gives our main L_1-results, Corollaries 12A and 12B.

PROPOSITION 1. *Let $1 < p < q \le \infty$ and $\sigma = 1 - q^*/p^*$. Let $T: Z \to L_1(\mu)$, where μ is a probability measure. If $C_{1,p}(T) = K$ and $\|T: Z \to L_p(\mu)\| = \mathscr{C}K$, then for some*

$$m > (K/2^{2+1/p}\mathscr{C}\|T\|)^{p^*}$$

there exist $z_1, \dots, z_m$ in the unit ball of Z and mutually disjoint measurable sets $F_1, \dots, F_m$ such that for $i = 1, \dots, m$ one has

$$\|1_{F_i}Tz_i\|_1^\sigma \|1_{F_i}Tz_i\|_q^{1-\sigma} \ge 2^{-(1/p+1)}K. \tag{1}$$

For the proof we need two lemmas.

LEMMA 2. *Let $g \in L_1$ with $\|g\| \le 1$. Suppose E is a μ-measurable set, $1 < p < q \le \infty$ and*

$$\|1_{\sim E}g\|_p > \kappa > 0.$$

Then there exists a measurable set F, $F \cap E = \varnothing$, such that

$$\mu(F) < \left(\frac{2^{1/p}}{\kappa}\right)^{p^*},$$

$$\|1_F g\|_1^\sigma \|1_F g\|_q^{1-\sigma} > 2^{-1/p}\kappa.$$

Proof. Without loss of generality we can assume that $g \ge 0$. Set $F = [g > \gamma] \sim E$, where $\gamma > 0$ is defined below. Observe that

$$\int_{\sim F} g^p \le \gamma^{p-1} \int_{\sim F} g \le \gamma^{p-1}$$

and hence, by Hölder's inequality,

$$\kappa^p - \gamma^{p-1} < \int_F g^p \le \left(\int_F g\right)^{1-t} \left(\int_F g^q\right)^t,$$

where $t = (p-1)/(q-1)$. Since $\mu(F) < 1/\gamma$, we can fulfill both conditions of the lemma by choosing γ so that $\gamma^{p-1} = \frac{1}{2}\kappa^p$. □

LEMMA 3. *Let $T\colon Z \to L_1(\mu)$, μ being a probability measure. Suppose $1 < p < \infty$ and*

$$C_{1,p}(T) = K, \quad \|T\colon Z \to L_p(\mu)\| = \mathscr{C}K.$$

If $0 < \kappa < K$, $\eta \ge 0$ and $\mathscr{C}\eta^{1/p^*} \le 1 - \kappa/K$, then $\mu(E) \le \eta$ implies that there exists $z \in \mathrm{Ball}(Z)$ such that $\|1_{\sim E}Tz\|_p > \kappa$.

Proof. Without loss of generality we may assume that $\mu(E) > 0$. Observe that for any measurable set A one has

$$C_{1,p}(1_A T\colon Z \to L_1(\mu)) \le \mu(A)^{1/p^*}\|1_A T\colon Z \to L_p(\mu)\|.$$

The lemma follows by using this observation for $A = \sim E$ and $A = E$, because

$$\begin{aligned}\|1_{\sim E}T\colon Z \to L_p(\mu)\| &> C_{1,p}(1_{\sim E}T) \ge C_{1,p}(T) - C_{1,p}(1_E T)\\ &\ge K - \mu(E)^{1/p^*}\|1_E T\colon Z \to L_p(\mu)\| \ge K(1 - \mathscr{C}\eta^{1/p^*}) \ge \kappa.\end{aligned}$$ □

Proof of Proposition 1. Clearly, we may assume that $\|T\| = 1$. Put $\kappa = \frac{1}{2}K$, $\eta = (2\mathscr{C})^{-p^*}$, $\delta = (2^{1/p}/\kappa)^{p^*}$ and let us start with $E = \varnothing$. Since

$$\|1_{\sim E}T\colon Z \to L_p(\mu)\| > \kappa,$$

using Lemma 2 we can define z_1 and F_1 so that $\mu(F_1) < \delta$ and $\|1_{F_1}Tz_1\| - 1$ satisfies (1). Suppose now that, for some $i \ge 1$, we have already defined $z_1, \dots, z_i$ and $F_1, \dots, F_i$. Let $E = \bigcup_{j \le i} F_j$. As long as $\mu(E) \le \eta$, Lemma 3 guarantees that we can use Lemma 2 again in order to choose z_{i+1} and F_{i+1} so that $F_{i+1} \cap E = \varnothing$, $\mu(F_{i+1}) < \delta$ and $\|1_{F_{i+1}}Tz_{i+1}\|_1$ satisfies the estimate (1). Therefore this procedure can be applied more than η/δ times. Since we have been assuming $\|T\| = 1$, this yields the promised lower estimate for m and completes the proof of the proposition. □

The next proposition shows that we can actually get a somewhat stronger conclusion to Proposition 1; namely, for some k proportional to m, the identity on l_1^k can be factored through T.

PROPOSITION 4. *Let $T: Z \to L_1(\mu)$ be a bounded linear operator. Suppose that $z_1, \dots, z_m$ are in the unit ball of Z and $F_1, \dots, F_m$ are mutually disjoint μ-measurable sets such that, for $i = 1, \dots, m$,*

$$\|1_{F_i} T z_i\| \geq \delta \|T\| > 0.$$

Then for some $k \geq \frac{1}{8}\delta m$ there exist linear operators $A: l_1^k \to Z$ and $B: L_1(\mu) \to l_1^k$ such that $BTA = \mathrm{id}_{l_1^k}$ and $\|A\| \, \|B\| \, \|T\| \leq 2\delta^{-1}$; i.e., $\gamma_T(l_1^k) \leq 2\delta^{-1}\|T\|^{-1}$ for some $k \geq \frac{1}{8}\delta m$.

For the proof we need two basically known lemmas (see **[JS]**).

LEMMA 5. *Let $x_1, \dots, x_m$ be elements of $L_1(\mu)$ and let $A_1, \dots, A_m$ be mutually disjoint μ-measurable sets. If $1 < k \leq \frac{1}{2}m$, then there exists a subset $D \subset \{1, \dots, m\}$ with $|D| = k$ such that for each $i \in D$*

$$\sum_{j \in D \sim \{i\}} \int_{A_j} |x_i| \, d\mu \leq (2k-1) \binom{m}{2}^{-1} \sum_{i=1}^{m} \|x_i\|.$$

Proof. Setting $a_{ij} = \int_{A_j} |x_i| \, d\mu$, for $i, j = 1, \dots, m$, and $\alpha = \sum_{1 \leq i \neq j \leq m} a_{ij}$, we have

$$\alpha \leq \sum_{i=1}^{m} \|x_i\|.$$

Put $s = 2k$, $\mathscr{E} = \{E \subset \{1, \dots, m\} : |E| = s\}$. Write, for $E \in \mathscr{E}$,

$$\alpha(E) = \sum_{i \in E} \sum_{j \in E \sim \{i\}} a_{ij}.$$

It is easy to see that $\sum_{S \in \mathscr{E}} \alpha(E) = \binom{m-2}{s-2}\alpha = |\mathscr{E}| \binom{s}{2} \binom{m}{2}^{-1} \alpha$; hence we can pick $E_0 \in \mathscr{E}$ so that $\alpha(E_0) \leq \binom{s}{2}\binom{m}{2}^{-1}\alpha$. Let $F = \{i \in E_0 : \sum_{j \in E_0 \sim \{i\}} a_{ij} \geq \frac{1}{k}\alpha(E_0)\}$. Then F has at most k elements and, clearly, each k-element subset $D \subseteq E_0 \sim F$ has the required property. □

LEMMA 6. *Let $U: l_1^k \to L_1(\mu)$ be a linear operator, $\|U\| \leq 1$. Suppose there exist μ-measurable sets $F_1, \dots, F_k$ such that for $i = 1, \dots, k$*

$$\|1_{F_i} U e_i\| \geq \delta, \qquad \sum_{1 \leq i \neq j \leq k} \|1_{F_j} U e_i\| \leq \gamma,$$

where $0 < \gamma < \delta$. *Then there exists a linear operator* $Q\colon L_1(\mu) \to l_1^k$ *such that* $QU = \mathrm{id}_{l_1^k}$ *and* $\|Q\| \le (\delta - \gamma)^{-1}$.

Proof. Define $W\colon L_1(\mu) \to l_1^k$ by the formula $Wf = (\int f g_i \, d\mu)_{i=1}^k$, where

$$g_i = 1_{F_i} \operatorname{sgn}(Ue_i) \quad \text{for } i = 1, \dots, k.$$

It is easy to check that $\|W\| \le 1$ and for $x \in l_1^k$ one has $\|WUx\| \ge (\delta - \gamma)\|x\|$. Therefore the operator $Q = (WU)^{-1}W$ has the required properties. □

Proof of Proposition 4. We may assume that $\|T\| = 1$. Apply Lemma 5 with $\eta = \frac{1}{2}\delta$ and $x_i = Tz_i$ for $i = 1, \dots, m$. This yields a set $D \subset \{1, \dots, m\}$, with $|D| = k \ge \frac{1}{8}\delta m$, which satisfies the assertion of Lemma 5. Writing $\{z_i\colon i \in D\} = \{f_1, \dots, f_m\}$ we can define the operator A by the formula $Ae_i = f_i$, for $i = 1, \dots, k$. The existence of the operator B follows then immediately from Lemma 6. □

We shall combine Propositions 1 and 4 in Theorem 8 below. Before doing that we would like to state a dual version of Proposition 4. Note that, if $\dim X_1$, $\dim Y_1 < \infty$, then $\gamma_T(U) = \gamma_{T^*}(U^*)$. This follows from the principle of local reflexivity ([**LT**], p. 33).

COROLLARY 7. *Let* $V\colon l_\infty^m \to X$ *be an operator of norm* 1 *such that* $\|Ve_i\| \ge \delta > 0$ *for* $i = 1, \dots, m$. *Then* $\gamma_V(l_\infty^k) \le 2\delta^{-1}$ *for some* $k \ge \frac{1}{8}\delta m$.

Proof. Let $Z = X^*$. Pick norm one elements $z_1, \dots, z_m$ in Z such that $z_i(Ve_i) \ge \delta$ for $i = 1, \dots, m$. Using Proposition 4 we obtain $\gamma_{V^*}(l_1^k) \le 2\delta^{-1}$ for some $k \ge \frac{1}{8}\delta m$. Since $\gamma_V(l_\infty^k) = \gamma_{V^*}(l_1^k)$ this completes the proof. □

The L_1 result. The main results of this section are Theorem 11 and Corollary 12 below. Corollary 12.A states roughly that in any good factorization of a natural embedding of l_2^n into $L_1(0, 1)$ (for example, the embedding sending the unit vector basis of l_2^n to the first n Rademacher functions) through an L_1 space, the operator between the two L_1 spaces preserves an l_1^k space with k exponential in n. We begin however with a theorem of a more general nature which is a corollary to Propositions 1 and 4. The assumptions in both this theorem and Theorem 11 are stated in terms of factorization constants of an operator into an L_1 space through L_p spaces via changes of densities. The relation between these constants and factorizations of natural embeddings was one of the main tools in **[FJS]**. We shall return to this relation in the proof of Corollary 12. For the moment we just note that for Theorem 8 to be useful, we need the parameters Δ and σ in Theorem 8 to be such that Δ is bounded and σ is bounded away from 0 in order to make $\gamma_T(l_1^k)$ bounded. Subject to that restriction, we want k to be large. In practice, this is done by setting $q = \infty$ and choosing p so that $C_{1,p}(T) \sim 16\|T\|$; under certain conditions this choice makes p^* large enough to guarantee that k is large.

THEOREM 8. *Let $T: Z \to L_1(\mu)$ be a linear operator such that $T \neq 0$ and $C_{1,q}(T) < \infty$, where $1 < p < q \leq \infty$. Set $\sigma = 1 - \frac{q^*}{p^*}$, $\Delta = \|T\|^\sigma C_{1,q}(T)^{1-\sigma}/C_{1,p}(T)$. Then*

$$\gamma_T(l_1^k) \leq 2(4\Delta)^{1/\sigma}\|T\|^{-1} \quad \textit{for some } k \geq \tfrac{1}{8}(4\Delta)^{-1/\sigma}\left(\frac{C_{1,p}(T)}{8\|T\|}\right)^{p^*}.$$

Proof. By Maurey's result **[Ma]** quoted in **[FJS]**, for each $r \in (1, \infty]$ there is a nonnegative function $\phi_r \in L_1(\mu)$ such that $\int \phi_r \, d\mu = 1$ and

$$\|\phi_r^{-1}T: Z \to L_r(\phi_r \, d\mu)\| = C_{1,r}(T)$$

(this uses the convention $\frac{0}{0} = 0$). Set $\phi = \frac{1}{2}(\phi_p + \phi_q)$. Then for $r = q$ and $r = p$

$$\|\phi^{-1}T: Z \to L_r(\phi \, d\mu)\| \leq 2^{1/r^*} C_{1,r}(T).$$

Consider the operator $T_1 = \phi^{-1}T: Z \to L_1(\phi \, d\mu)$. Since $C_{1,r}(T_1) = C_{1,r}(T)$ for $r \in (1, \infty]$, applying Proposition 1 to the operator T_1, we have $\mathscr{C} \leq 2^{1/p^*}$. We can estimate for each i

$$\|1_{F_i} T_1 z_i\|_{L_q(\phi \, d\mu)} \leq \|T_1: Z \to L_q(\phi \, d\mu)\| \leq 2^{1/q^*} C_{1,q}(T).$$

Hence we obtain elements $z_1, \dots, z_m$ in $\mathrm{Ball}(Z)$ and sets $F_1, \dots, F_m$ so that

$$\|1_{F_i} T_1 z_i\|^\sigma_{L_1(\phi \, d\mu)} \geq 2^{-\frac{1}{p}=1}(2^{1/q^*} C_{1,q}(T))^{\sigma-1} C_{1,p}(T) = 2^{-2}\Delta^{-1}\|T\|^\sigma$$

and $m > (C_{1,p}(T)/8\|T\|)^{p^*}$. Therefore, we have

$$\|1_{F_i} T z_i\|_1 = \|1_{F_i} T_1 z_i\|_{L_1(\phi \, d\mu)} \geq (4\Delta)^{-\frac{1}{\sigma}}\|T\|.$$

Now we simply apply Proposition 4 to T, $z_1, \dots, z_m$ and $F_1, \dots, F_m$. □

We next state two corollaries to Theorem 8 concerning the dual situation.

COROLLARY 9. *Let* $U: C(K) \to X$ *be a linear operator such that* $0 < \pi_r(U) < \infty$, *where* $1 \le r < t < \infty$. *Set* $\sigma = 1 - \frac{r}{t}$, $\Delta = \|U\|^\sigma \pi_r(U)^{1-\sigma}/\pi_t(U)$. *Then*

$$\gamma_U(l_\infty^k) \le 2(4\Delta)^{1/\sigma}\|U\|^{-1} \quad \textit{for some } k \ge \frac{1}{8}(4\Delta)^{-1/\sigma}\left(\frac{\pi_t(U)}{8\|U\|}\right)^t .$$

Proof. This follows easily from Theorem 8, because $C(K)^*$ is an L_1 space, $\gamma_{U^*}(l_1^k) = \gamma_U(l_\infty^k)$ and $C_{1,p}(U^*) = \pi_{p^*}(U)$ for $1 < p \le \infty$ (see **[R]**). □

COROLLARY 10. *If* $U: l_\infty^N \to X$, $t > 1$ *and* $\pi_t(U) = c\|U\| > 0$, *then* $\gamma_U(l_\infty^k) \le 2(\frac{4}{c})^{t^*} N^{t^*-1}\|U\|^{-1}$ *for some* $k \ge 2^{t^*-3}(\frac{c}{8})^{t^*+t} N^{1-t^*}$.

Proof. This follows by using Corollary 9 with $r = 1$, because $\pi_1(U) \le N\|U\|$. □

The next theorem and Corollary 12 below are the main results of this section.

THEOREM 11. *Let* $T: Z \to L_1(\mu)$ *be a bounded linear operator and let* $1 < p < \infty$. *Let* $Z_0 \subseteq Z$. *Suppose that* $n = \dim TZ_0 < \infty$ *and that*

$$C_{1,p}(u) \ge c\|T\| > 0,$$

for each finite rank operator $u: Z \to L_1(\mu)$ *such that* $u|_{Z_0} = T|_{Z_0}$. *If* $c \ge 2^5$ *and* $\delta = (p-1)n$, *then* $\gamma_T(l_1^k) < 5^\delta$ *for some* $k > 5^{-\delta}(2^{1/\delta})^n$.

Before proving Theorem 11, we use it to derive Corollary 12. The first part of this corollary, Corollary 12.A, generalizes Corollary 1.5 in **[FJS]**.

COROLLARY 12.A. *Let* X *be an* n*-dimensional subspace of* L_1 *for which* $C_{1,p^*}(X) \le C\sqrt{p^*}$ *for all* $2 \le p^* < \infty$. *If* $T: L_1 \to L_1$ *is a linear operator of norm one and* $\|Tx\| \ge \tau\|x\|$ *for* $x \in X$ (*with* $\tau > 0$), *then* $\gamma_T(l_1^k) \le 5^{2D}$ *for some* $k \ge 5^{-2D}2^{n/(2D)}$, *where* $D = 2^{16}C^2\tau^{-4}$.

Proof. If $n < 2D$ the conclusion is obvious, so we may assume that $n \ge 2D$. Define p by $p^* = \frac{n}{D}$.

Set $S = (T|_X)^{-1}: TX \to X$, so that $\|S\| \le \frac{1}{\tau}$.

In order to apply Theorem 11, we need to obtain a lower estimate of $\inf C_{1,p}(u)$, where the inf ranges over all finite rank extensions

$u\colon L_1 \to L_1$ of $T|_X$. Given an extension $u\colon L_1 \to TX$ of $T|_X$ into TX (or even into L_1), we have

$$\sqrt{n} \le \pi_2(\mathrm{id}\,|_X) = \pi_2(ST|_X) \le \frac{1}{\tau}\pi_2(T|_X) \le \frac{1}{\tau}\pi_2(u) \le \frac{2}{\tau}\gamma_2(u)$$

(the last inequality follows from a weak form of Grothendieck's inequality). Theorem 1.3 in **[FJS]** then gives that

$$\inf C_{1,p}(u) \ge \frac{\tau}{8}\sqrt{n}[C_{1,p^*}(TX)]^{-1}.$$

But $C_{1,p^*}(TX)$ is estimated from above by $\|S\|C_{1,p^*}(X)$. Indeed, letting $I\colon X \to L_1$ and $J\colon TX \to L_1$ denote the inclusion maps, we have by **[R]** that $C_{1,p^*}(X) = \pi_p(I^*)$ and $C_{1,p^*}(TX) = \pi_p(J^*)$. Thus

$$\begin{aligned} C_{1,p^*}(TX) &= \pi_p(J^*) = \pi_p(S^*I^*T^*) \\ &\le \|S\|\pi_p(I^*)\|T\| \le \|S\|C_{1,p^*}(X) \le \frac{C}{\tau}\sqrt{p^*}. \end{aligned}$$

Therefore our choice $p^* = \frac{n}{D}$ yields $\inf C_{1,p}(u) \ge 2^5$, and the conclusion of Corollary 12.A follows from Theorem 11. □

Corollary 12.B strengthens a specialization of Corollary 12.A in the same way that Theorem 5.1 in **[FJS]** strengthens Corollary 1.5 in **[FJS]**.

COROLLARY 12.B. *Suppose that $X \subset L_1$, $\dim X = n$, and $C_{1,p^*}(X) \le C\sqrt{p^*}$ for all $2 \le p^* < \infty$. Let Y be a Banach space whose dual has finite cotype q constant $C_q(Y^*)$ and let $Q\colon Y \to L_1$ be an operator for which*

$$Q(\mathrm{Ball}(Y)) \supseteq \mathrm{Ball}(X).$$

Let $Q = UW$ be any factorization of Q through an L_1 space with $\|W\| \le 1$. Then for some absolute constant η, $\gamma_U(l_1^k) \le 5^D$ for some $k \ge 5^{-D}2^{n/D}$, where

$$D = \eta C^2 q C_q^2(Y^*)\|U\|^2.$$

Sketch of proof. Follow the proof of Theorem 5.1 in **[FJS]** (with r replaced by p) up to the place on p. 98 where it is proved that $C_{1,p}(U) \ge 2$. Of course, now we need and can assure that $C_{1,p}(\widetilde{U}) \ge 2^5\|U\|$ for any extension $\widetilde{U}\colon Z \to L_1$ of the restriction of U to $U^{-1}(X)$. Then apply Theorem 11. □

To prove Theorem 11 we need to introduce some notation and some preliminary results. Given two Banach spaces Z and W the space of all bounded linear operators between them is denoted by $B(Z, W)$, while $F(Z, W)$ is the set of those $u \in B(Z, W)$ such that rank $u < \infty$. By α we denote a norm on $F(Z, W)$ such that $\alpha(u) \leq \|u\|$ if rank $u = 1$.

Given a Banach space W and numbers $n, \beta \geq 1$, let us denote by $q_W(n, \beta)$ the least number k such that, whenever $v: W \to E$ is a continuous linear operator with rank $v \leq n$, there exists $P \in F(W, W)$ such that $vP = v$, $\|P\| \leq \beta$ and rank $P \leq k$. (Of course, we let $q_W(n, \beta) = \infty$, if no such k exists.) The reader who is familar with the uniform approximation property should note that this parameter is connected with the uniformity function for X^*; specifically, in the extension of terminology introduced in **[FJS]** for L_p, $q_W(n, \beta)$ is essentially the same as (even exactly the same as, for reflexive W) $k_{W^*}(n, \beta)$.

PROPOSITION 13. *Let $T \in B(Z, W)$ and let $Z_0 \subseteq Z$, $\dim T(Z_0) = n < \infty$. If $1 \leq \beta < \infty$ and $q_W(n, \beta) < \infty$, then there exists $P \in F(W, W)$ such that $\|P\| \leq \beta$, $\operatorname{rank} P \leq q_W(n, \beta)$ and*

$$\alpha(PT) \geq \inf\{\alpha(u) : u \in F(Z, W), u|_{Z_0} = T|_{Z_0}\}.$$

Proof. Write

$$Y = \{u \in F(Z, W) : u|_{Z_0} = T|_{Z_0}\} \quad \text{and} \quad A = \inf\{\alpha(u) : u \in Y\}.$$

By the Hahn-Banach theorem there is a norm one functional Φ on $(F(Z, W), \alpha)$ such that $\Phi(u) = A$ for each $u \in Y$. Observe that if $S: W \to Z^{**}$ is the linear operator defined by

$$(Sw)(z^*) = \Phi(z^* \otimes w),$$

then for all $u \in F(Z, W)$ one has

$$\Phi(u) = \operatorname{Tr}(Su) = \operatorname{Tr}(u^{**}S).$$

Clearly, our assumption on α yields $\|S\| \leq 1$. Moreover, for all $u \in Y$ one has

$$(T - u)^{**}S = 0. \tag{2}$$

Indeed, since $\operatorname{Ker}((T-u)^{**}) \supseteq (\operatorname{Ker}(T-u))^{\perp\perp} \supseteq Z_0^{\perp\perp}$, it suffices to verify that $SW \subseteq Z_0^{\perp\perp}$. The latter inclusion is obvious, because if $w \in W$, $z_0^* \in Z_0^{\perp}$ then we have $(Sw)(z_0^*) = \Phi(z_0^* \otimes w) = 0$, since z_0^* annihilates Z_0. Since $\operatorname{rank} u_0 = \dim TZ_0 = n$, for some $u_0 \in Y$, and since (2) implies that $T^{**}S = u^{**}S$ for each $u \in Y$, we obtain that $\operatorname{rank} T^{**}S \le n$. Hence, by the definition of $q_W(n, \beta)$, there is a $P \in F(W, W)$ such that $T^{**}SP = T^{**}S$, $\|P\| \le \beta$ and $\operatorname{rank} P \le q_W(n, \beta)$. Observe that, if u is any element of Y, then

$$\begin{aligned}\alpha(PT) &\ge \Phi(PT) = \operatorname{Tr}(SPT) = \operatorname{Tr}(T^{**}SP) = \operatorname{Tr}(T^{**}S) \\ &= \operatorname{Tr}(u^{**}S) = \Phi(u) = A.\end{aligned} \qquad \square$$

The following known lemma is equivalent (via a standard duality argument) to Lemma 17 below.

LEMMA 14. *If* $W = L_1(\mu)$ *and* $0 < \varepsilon < 1$, *then* $q_W(n, (1-\varepsilon)^{-1}) < \frac{1}{2}(\frac{2}{\varepsilon} + 1)^n$.

Proof. Let $u: W \to E$ have rank n. Write $u = UQ_0$, where $Q_0: W \to W/(\operatorname{Ker} u)$ is the quotient map and let $F = W/(\operatorname{Ker} u)$. Set $\beta = (1-\varepsilon)^{-1}$. Suppose first that for some k there exists an operator $Q: l_1^k \to F$ such that $\|Q\| \le \beta$ and $Q(\operatorname{Ball}(l_1^k)) \supseteq \operatorname{Ball}(F)$. By the lifting property of l_1^k there is $Q_1: l_1^k \to W$ such that $\|Q_1\| \le \|Q\| \le \beta$ and $Q = Q_0Q_1$. By the lifting property of $W = L_1(\mu)$ there is $Q_2: W \to l_1^k$ such that $\|Q_2\| \le \|Q_0\| \le 1$ and $Q_0 = QQ_2$. Let $P = Q_1Q_2$. Then $\|P\| \le \beta$, $\operatorname{rank} P \le k$ and

$$u = UQ_0 = UQQ_2 = UQ_0Q_1Q_2 = uP.$$

Now the well-known volume argument shows that the unit sphere of F contains an ε-net (where $(1-\varepsilon)^{-1} = \beta$) of cardinality $k < \frac{1}{2}(\frac{2}{\varepsilon}+1)^n$. Using this fact one easily constructs the operator $Q: l_1^k \to F$ with the two properties which we have used above. $\square$

Theorem 11 can now be obtained by letting $\varepsilon = \frac{1}{2}$ and replacing c by 2^5 in the following proposition.

PROPOSITION 15. *Let* $T: Z \to L_1(\mu)$ *satisfy the assumptions of Theorem* 11, *and let* $0 < \varepsilon < 1$. *Then*

$$\gamma_T(l_1^k) < 4\left(\frac{2}{(1-\varepsilon)c}\right)^p \left(\frac{2}{\varepsilon} + 1\right)^{n(p-1)} \|T\|^{-1}$$

for some $k > 4^{p-2}(\frac{(1-\varepsilon)c}{8})^{pp^*}(\frac{2}{\varepsilon}+1)^{-n(p-1)}$.

Proof. Let $\beta = (1-\varepsilon)^{-1}$, $W = L_1(\mu)$, $\alpha = C_{1,p}$. Thanks to Lemma 14, we can apply Proposition 13 which yields an operator P on $L_1(\mu)$ such that $\|P\| \le \beta$, $\operatorname{rank} P \le N = \frac{1}{2}(\frac{2}{\varepsilon}+1)^n$ and $C_{1,p}(PT) \ge c\|T\|$. Clearly,

$$C_{1,\infty}(PT) \le (\operatorname{rank} PT)\|PT\| \le N\|PT\|.$$

Let $q = \infty$. We can now estimate $\gamma_{PT}(l_1^k)$, using Theorem 8. The resulting inequality, combined with the obvious relation $\gamma_T(l_1^k) \leq \|P\|\gamma_{PT}(l_1^k)$, yields the desired estimates for $\gamma_T(l_1^k)$ and k. □

The $C(K)$ result. The main result of this section is Theorem 16 and in particular its Corollary 20 which gives a local version of a result of Pełczyński **[Pe1]** by showing that an operator from a $C(K)$ space which preserves a copy of l_2^n also preserves a copy of l_∞^k with k an exponent of n.

THEOREM 16. *Let $U: C(K) \to X$ be a bounded linear operator and let $1 < t < \infty$. Suppose that $E \subseteq C(K)$, $\dim E = n < \infty$ and let*

$$\pi_t(U|_E) = c\|U\| > 0.$$

If $c \geq 2^5$ and $\alpha = \frac{n}{t-1}$, then $\gamma_U(l_\infty^k) < 5^\alpha\|U\|^{-1}$ for some $k > 5^{-\alpha}(2^{1/\alpha})^n$.

For the proof we need a lemma and a proposition. Lemma 17, the dual statement of Lemma 14, is a weak version of Theorem 4.1 in **[FJS]**.

LEMMA 17. *Let F be a subspace of $C(K)$, $\dim F = n$. Let $0 < \varepsilon < 1$ and $N = \frac{1}{2}(\frac{2}{\varepsilon} + 1)^n$. Then there is $Q: C(K) \to C(K)$ such that $Qf = f$ for $f \in F$, $\|Q\| < (1-\varepsilon)^{-1}$ and $\operatorname{rank} Q \leq N$.*

Proof. There exists $k \leq N$ and an operator $J: F \to l_\infty^k$ such that $\|J\| < (1-\varepsilon)^{-1}$ and $\|Jf\| \geq \|f\|$ for $f \in F$. By the extension property of $C(K)$, there is $J_1: l_\infty^k \to C(K)$ such that $\|J_1\| < (\|J\|(1-\varepsilon))^{-1}$ and $J_1(Jf) = f$ for $f \in F$. We let $Q = J_1 J_2$, where $J_2: C(K) \to l_\infty^k$ is a linear extension of J with $\|J_2\| = \|J\|$. □

Theorem 16 follows easily from the next proposition by letting $\varepsilon = \frac{1}{2}$ and replacing c by 2^5.

PROPOSITION 18. *Let $U: C(K) \to X$ satisfy the assumptions of Theorem 16, and let $0 < \varepsilon < 1$. Then*

$$\gamma_U(l_\infty^k) < 4\left(\frac{2}{(1-\varepsilon)c}\right)^{t^*}\left(\frac{2}{\varepsilon}+1\right)^{n/(t-1)}\|U\|^{-1}$$

for some $k > 4^{t^*-2}(\frac{(1-\varepsilon)c}{8})^{tt^*}(\frac{2}{\varepsilon}+1)^{-n/(t-1)}$.

Proof. By Lemma 17, there exists $P: C(K) \to C(K)$ such that $\|P\| < (1-\varepsilon)^{-1}$, $\operatorname{rank} P < N = \frac{1}{2}(\frac{2}{\varepsilon}+1)^n$ and $Pe = e$ for $e \in E$. It follows that

$$C_{1,t^*}(P^*U^*) \geq \pi_t(UP) \geq \pi_t(UP|_E) = c\|U\|.$$

Since also $C_{1,\infty}(P^*U^*) \leq (\operatorname{rank} UP)\|(UP)^*\| \leq N\|UP\|$, we can finish the proof by applying an argument similar to that in the proof of Proposition 15 and dualizing. □

COROLLARY 19. *Let X be a Banach space. Suppose that $t, n > 1$ and there is an operator $U\colon C(K) \to X$ and a subspace $E \subseteq C(K)$, $\dim E = n < \infty$ such that $\pi_t(U|_E) = c\|U\| > 0$, where $c \geq 2^5$. Write $\alpha = \frac{n}{t-1}$, $c_1 = (\log 2)(\log \frac{4}{3})/\log 5$. Then, for all $j \leq \min\{5^{-\alpha}(2^{1/\alpha})^n, \frac{3}{4}\exp(c_1 n/\alpha^2)\}$, X contains a subspace X_j such that $d(X_j, l_\infty^j) < 2$.*

Proof. Since $c \geq 32$ using Theorem 16 we obtain that, if $\alpha = \frac{n}{t-1}$, then $\gamma_U(l_\infty^k) < 5^\alpha\|U\|^{-1}$ for some $k > 5^{-\alpha}(2^{1/\alpha})^n$. Consequently, we obtain the inequality

$$\gamma_{\mathrm{id}_X}(l_\infty^k) \leq \|U\|\gamma_U(l_\infty^k) < 5^\alpha,$$

from which we shall deduce a lower estimate for the number

$$j_0 = \min\{m\colon \gamma_{\mathrm{id}_X}(l_\infty^m) \geq 2\}.$$

Put for brevity $g_i(X) = \gamma_{\mathrm{id}_X}(l_\infty^i)$. We shall employ the estimate

$$g_{ij}(X) \geq g_i(X)\frac{2}{1 + g_j(X)^{-1}}, \tag{3}$$

for $i, j = 1, 2, \dots$, which is the quantitative statement of results of James **[J]** and Giesy **[G]** (see, e.g., **[F]**).

Suppose that $j_0 \leq k$ and let m be an integer such that $j_0^{m+1} > k \geq j_0^m$. Write $A = g_{j_0}(X) \geq 2$, $B = 2/(1 + A^{-1})$. Using repeatedly (3), one obtains

$$g_k(X) \geq g_{j_0}^m(X) \geq AB^{m-1} \geq \frac{A}{B^2}B^{\log k/\log j_0}.$$

Since $A \geq 2$, we have $A/B^2 \geq \frac{9}{8}$, $B \geq \frac{4}{3}$. Taking logarithms of both sides we obtain the estimate

$$\alpha \log 5 \geq \log(g_k(X)) > \left(\log\frac{4}{3}\right)\frac{\log k}{\log j_0}.$$

Using now the estimate $k > 5^{-\alpha}(2^{1/\alpha})n$, we get easily

$$\log j_0 > (\alpha \log 5)^{-1}\left(\log\frac{4}{3}\right)\left(\frac{n}{\alpha}\log 2 - \alpha\log 5\right)$$
$$= \left(\log\frac{4}{3}\right)\left(\frac{n}{\alpha^2}\frac{\log 2}{\log 5} - 1\right) =: \log M.$$

The latter estimate implies that if $1 < j \le \min\{k, M\}$ then $g_j(X) < 2$. This implies that X contains a subspace X_j with $d(X_j, l_\infty^j) < 2$ and completes the proof. □

COROLLARY 20. *Let $U\colon C(K) \to X$ be a linear operator of norm 1. Suppose that for some subspace $E \subseteq C(K)$ such that $d(E, l_2^n) = a$, $n \ge 2$, one has $\|Ux\| \ge b\|x\|$ for $x \in E$, where $b > 0$. Then U is bounded from below by $A_1^{-(a/b)^2}$ on a subspace $G \subseteq C(K)$ such that $d(G, l_\infty^j) < 2$ and $j \ge A_2^{(b/a)^4 n}$, where A_1, A_2 are absolute constants > 1.*

Proof. Since $\pi_t(l_2^n) \ge \sqrt{\frac{n}{t}}$ for $t \ge 1$ **[Pe2]**, we can estimate

$$\pi_t(U|_E) \ge \frac{b}{a}\pi_t(\mathrm{id}_{l_2^n}) \ge \frac{b}{a}\sqrt{\frac{n}{t}}\|U\|.$$

Assume first that $n \ge (2^5 a/b)^4$. Letting $t = (2^{-5}b/a)^2 n$ we obtain the estimate $\pi_t(U|_E) \ge 2^5\|U\|$. Let $\alpha = \frac{n}{t-1}$. Using Theorem 16 we obtain, for some $k > 5^{-\alpha}2^{n/\alpha}$, a pair of operators $A\colon l_\infty^k \to C(K)$ and $B\colon X \to l_\infty^k$ such that $\|A\|\,\|B\| < 5^\alpha$ and $BUA = \mathrm{id}_{l_\infty^k}$. Let $F = A(l_\infty^k)$. Clearly, one has $\|Ux\| \ge 5^{-\alpha}\|x\|$ for $x \in F$, and $d(F, l_\infty^k) < 5^\alpha$. Now, if $d(F, l_\infty^k) \ge 2$, then the argument used in the proof of Corollary 19 can be applied to F. Since $g_k(F) < 5^\alpha$ this will produce a subspace $G \subseteq F$ such that $j = \dim G > \frac{3}{4}\exp(c_1 n/\alpha^2) - 1$ and $d(G, l_\infty^j) < 2$. This yields the following conditions on numbers A_1, A_2

$$5^{n/(t-1)} \le A_1^{(a/b)^2}, \quad j \ge A_2^{(b/a)^4 n}.$$

If $n < (2^5 a/b)^4$, then we let G be any 2-dimensional subspace of E, so that $d(G, l_\infty^2) < 2$. This gives the following conditions on numbers A_1, A_2

$$b \ge A_1^{-(a/b)^2}, \quad 2 \ge A_2^{(2^5)^4}.$$

It is not difficult to check that one can find $A_1, A_2 > 1$ which satisfy all the above conditions. □

A space with very non-unconditional structure. The main result here is Theorem 21 which, roughly speaking, shows the existence of an m-dimensional space G which is contained in an n-dimensional space Z with an unconditional basis only if n is an exponent of m. Moreover any such Z must contain l_∞^k with k an exponent of m. This solves part of problem 11.4(b) in **[Pe3]**.

Recall that the gl norm of a linear operator $T\colon X \to Y$ is defined by the formula

$$\mathrm{gl}(T) = \sup\{\gamma_1(UT)\colon U : Y \to l_1,\ \pi_1(U) \le 1\},$$

and that one writes $\mathrm{gl}(X) = \mathrm{gl}(\mathrm{id}_X)$. Recall **[GL]** also that the unconditional constant of X is greater than or equal to $\mathrm{gl}(X)$.

THEOREM 21. *There is $\delta > 0$ such that for each $m \ge 2$ there is a Banach space G_m, $\dim G_m = m$, with the following property. If Z is a Banach space which contains an isometric copy of G_m, then there is a subspace Z_1 of Z such that $d(Z_1, l_\infty^k) < 2$, where $k \ge \exp(\delta m\, \mathrm{gl}(Z)^{-4})$.*

In fact, a somewhat stronger version of this theorem follows by applying Lemma 23 to the space obtained in Lemma 22. A stronger version of Lemma 22 appears as Theorem 7.1 in **[P]**.

LEMMA 22. *There is a constant $B < \infty$ such that for $n = 1, 2, \ldots$ there is a Banach space F_n, $\dim F_n = 2n$, and a linear operator $v\colon l_2^n \to F_n$ such that $\|ve\| \ge \|e\|$ for $e \in l_2^n$ and $\pi_1(v^*) \le B$.*

Proof. Consider a linear isometry $U\colon L_2^{3n} \to L_2^{3n}$. Write

$$E_1 = U([e_1, \ldots, e_{2n}]), \quad E_2 = U([e_{n+1}, \ldots, e_{3n}]).$$

It is well known (see e.g. **[K]**, **[S]**, or **[P]** Cor. 7.4) that for "most choices" of U one has

$$\|f\|_{L_2^{3n}} \le b\|f\|_{L_1^{3n}},$$

for $f \in E_1 \cup E_2$, where b can be taken independent of n. Let us fix a pair E_1, E_2 with the latter property. Put $F_n = L_1^{3n}/E_2^\perp$ and let

$$u\colon E_1/E_2^\perp \to L_1^{3n}/E_2^\perp = F_n$$

be the natural map. If $E_1/E_2^\perp$ is given the norm induced from $L_2^{3n}/E_2^\perp$, then $E_1/E_2^\perp$ is isometric to l_2^n and our choice of E_1 yields the estimate $\|e\| \le b\|ue\|$ for $e \in E_1/E_2^\perp$. Now u^* can be regarded as the composition of the embedding map $i_{\infty,2}^{E_2}\colon (E_2)_\infty \to (E_2)_2$ with the orthogonal projection P from L_2^{3n} onto $E_1 \cap E_2$. Hence our choice of E_2 yields

$$\pi_1(u^*) \le \|P\|\pi_1(i_{\infty,2}^{E_2}) \le \pi_1(i_{\infty,1}^{E_2})\|i_{1,2}^{E_2}\| \le b.$$

This shows that the operator $v = bu$ has the required properties, if $B = b^2$. $\square$

LEMMA 23. *Let F be a Banach space and let $v\colon l_2^n \to F$, $1 < t < \infty$. Suppose that*

$$\pi_t(v) \geq c\|v\| > 0, \quad \pi_1(v^*) \leq B\|v\|.$$

Let $j\colon F \to Z$ be a linear operator such that $\|jf\| \geq \|f\|$ for $f \in v(l_2^n)$. If $c \geq 2^5 B\,\mathrm{gl}(j)$, then $Z \supset Z_1$ such that $d(Z_1, l_\infty^k) < 2$ and $k = \dim Z_1 \geq \min\{5^{-\alpha}(2^{1/\alpha})^n, \frac{3}{4}\exp(c_1 n/\alpha^2)\}$, where α and c_1 are as in Corollary 19.

Proof. Observe that, since $\mathrm{gl}(j^*) = \mathrm{gl}(j)$, one has

$$\gamma_\infty(jv) = \gamma_1((jv)^*) \leq \pi_1(v^*)\,\mathrm{gl}(j^*) \leq B\,\mathrm{gl}(j)\|v\|.$$

Consider a $C(K)$-factorization of $jv\colon l_2^n \to Z^{**}$, say $jv = Ui$, where

$$\|i\colon l_2^n \to C(K)\| = 1, \quad \|U\colon C(K) \to Z^{**}\| \leq B\,\mathrm{gl}(j)\|v\|.$$

Put $E = i(l_2^n)$; then

$$\pi_t(U|_E) \geq \pi_t(Ui) \geq \pi_t(v) \geq c\|v\| \geq 2^5\|U\|.$$

Since $\gamma_{\mathrm{id}_{Z^{**}}}(l_\infty^k) = \gamma_{\mathrm{id}_Z}(l_\infty^k)$, the conclusion follows from Corollary 19. □

Proof of Theorem 21. We may assume that $m > 2(2^5 B\,\mathrm{gl}(Z))^4$, where B is the constant from Lemma 22. (If not, we just let $\delta = \frac{1}{4}(2^5 B)^{-4}$ and G_m can be an arbitrary space of dimension m.) Let $G_{2n} = F_n$ and $G_{2n+1} = F_n \oplus l_1^1$ for $n \geq (2^5 B)^4$. Fix an m and let Z be a Banach space and $j\colon G_m \to Z$ an isometric embedding, so that $\mathrm{gl}(j) \leq \mathrm{gl}(Z)$. Let $v\colon l_2^n \to G_m$ be the operator obtained from that in Lemma 22 (where $m = 2n$ or $m = 2n+1$). Observe that, for $t \geq 1$,

$$\pi_t(v) \geq \pi_t(\mathrm{id}_{l_2^n}) \geq \sqrt{\frac{n}{t}}.$$

Letting $t = (2^5 B\,\mathrm{gl}(j))^{-2} n$ and applying Lemma 23, one can easily find the absolute constant δ needed in Theorem 21. □

REMARK. It follows from ([**P**], Th. 7.1) that the spaces G_n in Theorem 21 can be chosen to have uniform cotype 2 constants.

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