

## A computer-assisted proof of existence of a periodic solution

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(Communicated by Masaki KASHIWARA, M.J.A., Nov. 12, 2014)

**Abstract:** We consider a three-dimensional dynamical system proposed in *Physica D*, **164**, (2002), 168–186. It is a conservative system and is unusual in that most of the solutions are unbounded. The paper presented a conjecture that an unstable periodic orbit determines directions of unbounded orbits of helical form. In the present paper we prove existence and local uniqueness of the conjectured periodic orbit by a method of numerical verification.

**Key words:** Three-dimensional dynamical system; periodic orbit; numerical verification.

**1. Introduction.** We consider the following system of ordinary differential equations

$$(1) \quad \dot{x} = y(z-1), \quad \dot{y} = z(1-x), \quad \dot{z} = x(y-1),$$

for real functions  $x(t)$ ,  $y(t)$ , and  $z(t)$  of a time-like variable  $t$ . This system was proposed by [2] as a model for a conservative dynamical system describing water-waves, etc.: see [2] and the references therein. It has the following interesting properties: (i) the phase-volume is conserved, (ii) most of the solutions are unbounded (but do not blow up in finite time), (iii) they bend in  $90^\circ$  or not, depending on initial data. [2] found an unstable periodic orbit which acts as a separator of direction, which means that solutions near the periodic orbit stay nearby for a while and are thrown away either straightly without a bend or to a direction of  $90^\circ$ -bend. See Fig. 1. In this sense the periodic orbit may be viewed as an organising centre, and plays a very important role in (1). However, at the time of publication of [2] we had no means to prove existence of a periodic orbit of that sort.

Our aim in the present paper is to give a computer-assisted proof of the existence and local uniqueness of a periodic orbit for (1). The proof uses a Poincaré map for (1), and a fixed point is proved to exist by verifying some technical conditions through a computer programme.

The present paper is organised as follows. In Section 2, we define the Poincaré map and state

our main theorem. Section 3 briefly introduces the interval arithmetic on which our method of numerical verification is based. We prove existence of a fixed point of the Poincaré map in Section 4, and we introduce in Section 5 a method for computing the linearised eigenvalues of the fixed point. In Section 6, we show a method for enlarging the region in which the fixed point of the Poincaré map can be proved to be unique. In Section 7, we present the computational details. Finally, we give concluding remarks in Section 8.

**2. Main results.** The numerical computation of [2] shows that the periodic orbit, which is reproduced in Fig. 2, passes through

$$(x, y, z) \approx (8.043, 0.5, -7.043).$$

To prove the existence of the periodic orbit, we first characterise a periodic orbit as a fixed point of a Poincaré map. To define the Poincaré map we introduce the following half-plane:

$$\{(x, y, z) \in \mathbf{R}^3 \mid x + z = 1, \dot{x} + \dot{z} < 0\},$$

where  $\dot{x}$  and  $\dot{z}$  are those in (1). For the sake of easy implementation of our computer programme, we define new coordinates  $x_1, x_2, x_3$  by

$$(2) \quad x_1 = x, \quad x_2 = y, \quad x_3 = x + z - 1.$$

In these new variables the half-plane above becomes the following

$$(3) \quad S = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_3 = 0, x_1 > 0\}.$$

The equations for  $(x_1, x_2, x_3)$  become

$$(4) \quad \begin{cases} \dot{x}_1 = x_2(x_3 - x_1), \\ \dot{x}_2 = (1 - x_1)(1 - x_1 + x_3), \\ \dot{x}_3 = x_2x_3 - x_1. \end{cases}$$

Since (2) defines a diffeomorphism between (1) and

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2000 Mathematics Subject Classification. Primary 37C27, 65G20.

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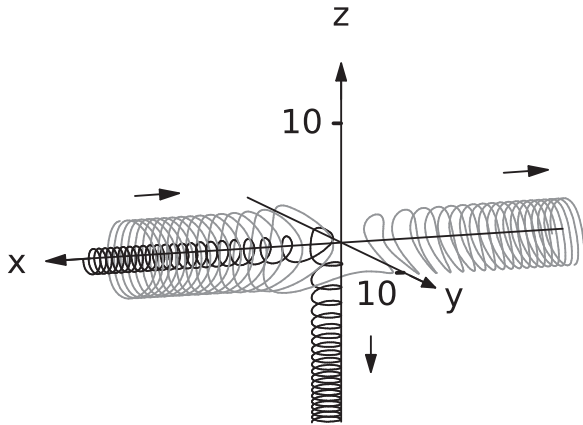


Fig. 1. Two unbounded orbits. The grey orbit passing through  $(x, y, z) = (15, 0, 3)$  does not bend but the radius reduces. The black orbit passing through  $(x, y, z) = (17.5, 0, 0)$  changes its course by  $90^\circ$ .

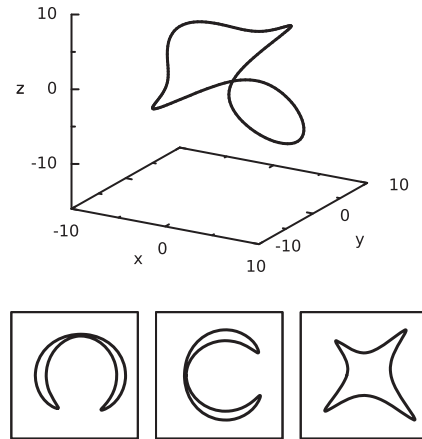


Fig. 2. The periodic orbit through  $(x, y, z) \approx (8.043, 0.5, -7.043)$ , and its projections onto  $(x, y)$ ,  $(y, z)$ , and  $(x, z)$ -planes, respectively from left to right. The squares in the three figures denote the boundary of  $[-10, 10]^2$ .

(4), we consider (4) in the rest of the paper. If an orbit of (4) passes through  $S$ , the sign of  $x_3$  changes from positive to negative. We consider the Poincaré map, which is defined on  $S$  and denoted by  $\tilde{P}$ , in the usual way. Let  $\phi : S \rightarrow \mathbf{R}^2$  be the chart on  $S$  defined by

$$\phi : (x_1, x_2, 0) \mapsto (x_1, x_2),$$

and we define  $P$  by  $P = \phi \circ \tilde{P} \circ \phi^{-1}$ .

Let  $X = [6.5, 8.5] \times [0, 2] \subset \mathbf{R}^2$ . Our goal is to prove the following

**Theorem 1.**

- (i) For all  $x \in X$ , the positive orbit of  $\phi^{-1}(x) \in S$  returns to  $S$  at least once, whence the Poincaré map is well-defined in  $X$ . Moreover, there exists a unique fixed point of  $P|_X$ . Accordingly, the corresponding periodic orbit for (1) exists.
- (ii) The fixed point of  $P$  in  $X$  is contained in the open ball centred at

$$(8.0430011130, 0.5)$$

with radius  $2.0 \times 10^{-10}$ , where the radius is measured in the norm  $\max\{|x_1|, |x_2|\}$ .

- (iii) The fixed point of  $P$  in  $X$  is a saddle.

Before explaining the numerical verification method, we remark that our system (1) is a conservative system and, in addition, most of the orbits are unbounded. Some theories on computation of dynamical systems assume dissipativity and/or existence of a compact absorbing set. Accordingly the results of numerical verification

such as [4] are not directly applicable to the present problem, although some of ideas may be used. As for the numerical verification for existence of a periodic orbit in a conservative system, there are several works. For example, [5] and [6] study the many-body problem in celestial mechanics. The periodic orbit in their system is not isolated, since the system has first integrals and each hypersurface corresponding to the first integral possesses a periodic orbit. This seems to be a difficulty for their problem. They overcame it by deriving a reduced system in which a periodic orbit is isolated, and they proved existence of a periodic orbit by a method of numerical verification. However, proving the uniqueness in as large region as possible does not seem to be their prime objective. On the other hand, it should be noted that our theorem above guarantees that the periodic orbit is unique in a relatively large region.

**3. Interval arithmetic.** We briefly recall terminology of the interval arithmetic. What follows is standard one, and the reader can find details in [10] or [12, Part 1].

In this paper, a closed interval is simply called an interval. An interval is denoted by  $[x]$  with a square bracket. For two intervals, binary operations such as addition, subtraction, multiplication, and division are defined. For example, multiplication of  $[x]$  and  $[y]$  is defined by  $[x] \cdot [y] := \{\tilde{x} \cdot \tilde{y} \mid \tilde{x} \in [x], \tilde{y} \in [y]\}$ , where the right hand side is also an interval.

A vector and a matrix whose entries are intervals are called an interval vector and an interval matrix, respectively. Operations for interval vectors and matrices are defined in an obvious way. We denote the set of all  $n$ -dimensional interval vectors by  $\mathbf{IR}^n$ . Similarly, we denote the set of all  $m \times n$  interval matrices by  $\mathbf{IR}^{m \times n}$ .

An interval vector and an interval matrix are identified with a direct product of intervals. For example, we may identify a rectangular set  $[a, b] \times [c, d]$  with an interval vector  $([a, b], [c, d])$ . Inclusion of interval vectors and matrices is defined in the same way as a set operation.

Many computer softwares and libraries for the interval arithmetic are available now, and one can easily implement the algorithms for numerical verification below. We use CAPD library [13], which is a C++ library designed for computer-assisted analysis for dynamical systems. We employ it since it offers subroutines for solving ordinary differential equations with verification.

Suppose that we are given  $[A] = ([A_{ij}])_{i,j=1}^2 \in \mathbf{IR}^{2 \times 2}$  and  $[b] = ([b_1], [b_2]) \in \mathbf{IR}^2$ , where  $[A_{ij}]$  and  $[b_j]$  ( $i, j = 1, 2$ ) are intervals. We define an interval  $[d]$  by

$$(5) \quad [d] = [A_{11}] \cdot [A_{22}] - [A_{12}] \cdot [A_{21}],$$

which contains  $\det \tilde{A}$  for all  $\tilde{A} \in [A]$ . If  $[d]$  does not contain 0, then all the matrices  $\tilde{A} \in [A]$  are invertible. Let  $[y] \in \mathbf{IR}^2$  be defined by

$$(6) \quad [y] = \begin{pmatrix} ([A_{22}] \cdot [b_1] - [A_{12}] \cdot [b_2]) / [d] \\ (-[A_{21}] \cdot [b_1] + [A_{11}] \cdot [b_2]) / [d] \end{pmatrix},$$

which is the interval version of Cramer's rule for  $2 \times 2$  matrix. Hence  $[y]$  contains the set

$$(7) \quad \{y = \tilde{A}^{-1} \tilde{b} \mid \tilde{A} \in [A], \tilde{b} \in [b]\},$$

provided that  $0 \notin [d]$ .

There are more sophisticated methods for solving a general linear system with verification, e.g., the Krawczyk method and the interval Gauss–Seidel method: see [10, Chap. 7] and [12, Sec. 10] for details. Nevertheless, it is enough for us to compute (6) by interval arithmetic since our problem is two-dimensional and the coefficient matrix is well-conditioned. As far as we have tested for our problem, the diameter of  $[y]$  obtained by (6) does not differ much from that obtained by the Krawczyk method.

**4. Existence of a fixed point.** In this section, we recall the interval Newton method by which we prove the existence of a fixed point of  $P$ . The method was originally introduced by Moore [9]. As far as we know, [3] is the first paper in which the interval Newton method is applied to the Poincaré map, and it is now a standard method for proving the existence of a periodic orbit. Let us briefly summarise the method to clarify how we have implemented our computer programme.

We use the interval Newton method in the following form. Let  $[X] \in \mathbf{IR}^2$  be an interval vector, and let  $f : [X] \rightarrow \mathbf{R}^2$  be a continuously differentiable map. Here  $[X]$  is identified with a rectangle in  $\mathbf{R}^2$ , as was explained above. Let  $\tilde{x} \in [X]$  be the centre of  $[X]$  viewed as a rectangle. Suppose that every matrix  $\tilde{A} \in \text{hull}(f'([X]))$  is invertible, where  $f'(x)$  is the Jacobian matrix of  $f$  evaluated at  $x$ , and  $\text{hull}(f'([X]))$  is the smallest interval matrix which contains  $f'([X])$ . Define

$$(8) \quad N([X]) = \{\tilde{x} - \tilde{A}^{-1} f(\tilde{x}) \mid \tilde{A} \in \text{hull}(f'([X]))\}.$$

**Lemma 2** ([12]). *Under the setting above,  $[X]$  contains a unique zero  $x^*$  of  $f$  in  $[X]$  if  $N([X]) \subseteq [X]$ . Moreover,  $x^* \in N([X])$ . If  $N([X]) \cap [X] = \emptyset$ , then there is no zero of  $f$  in  $[X]$ .*

This lemma is a special case of Theorem 13.2 in [12], and we may omit the proof.

Lemma 2 is used in the following way. We first define  $f : \phi(S) \rightarrow \mathbf{R}^2$  by  $f(x) = P(x) - x$ . We apply Zgliczynski's method [14], which is implemented in CAPD library [13], to compute for a given  $[x] \in \phi(S)$  and a given  $[X] \in \phi(S)$  an interval vector  $[Q]$  and an interval matrix  $[R]$  satisfying  $P([x]) \subseteq [Q]$  and  $P'([X]) \subseteq [R]$ . Note that

$$\begin{aligned} f([x]) &\subseteq P([x]) - [x] \subseteq [Q] - [x], \\ f'([X]) &\subseteq P'([X]) - I_2 \subseteq [R] - I_2, \end{aligned}$$

where  $I_2$  is the identity matrix of order 2 and is identified with an interval vector  $\text{diag}([1, 1], [1, 1])$ . Then we have

$$(9) \quad \begin{aligned} f([x]) &\subseteq [b] := [Q] - [x], \\ f'([X]) &\subseteq [A] := [R] - I_2. \end{aligned}$$

Suppose now that  $[X] = ([\underline{X}_1, \overline{X}_1], [\underline{X}_2, \overline{X}_2]) \in \mathbf{IR}^2$  is given, where  $\underline{X}_1, \overline{X}_1, \underline{X}_2, \overline{X}_2 \in \mathbf{R}$  with  $\underline{X}_1 \leq \overline{X}_1$  and  $\underline{X}_2 \leq \overline{X}_2$ . Although  $N([X])$  itself is not computable, an interval vector which contains  $N([X])$  is computed in the following way.

1 Compute  $[x] \in \mathbf{IR}^2$  by

$$[x] = \frac{1}{2}(\underline{X}_1 + \overline{X}_1, \underline{X}_2 + \overline{X}_2)$$

via interval arithmetic. Note that  $[x]$  contains the centre of  $[X]$ .

- 2 Compute  $[b] \in \mathbf{IR}^2$  and  $[A] \in \mathbf{IR}^{2 \times 2}$  with  $f([x]) \subseteq [b]$  and  $f'([X]) \subseteq [A]$ , as is shown in (9).
- 3 Compute an interval  $[d]$  by (5). If  $0 \notin [d]$ , then go to the next step. Otherwise, stop.
- 4 Compute  $[y] \in \mathbf{IR}^2$  by (6), and compute  $[Y] = [x] - [y]$  by interval arithmetic.

As a result,  $[Y]$  contains  $N([X])$ . Let us refer the above procedure as  $\text{INO}([X])$ , that is, we regard  $\text{INO}$  as a function which returns  $[Y]$  for an input  $[X]$ .

Since  $N([X]) \subseteq [Y]$ , we see that  $[Y] \subseteq [X]$  implies  $N([X]) \subseteq [X]$  and that  $[Y] \cap [X] = \emptyset$  implies  $N([X]) \cap [X] = \emptyset$ . This criterion serves as a sufficient condition for the assumption of Lemma 2 to hold. If neither  $[Y] \subseteq [X]$  nor  $[Y] \cap [X] = \emptyset$ , one cannot conclude anything about existence by the lemma. In this case we say that the verification failed. In addition, if  $\text{INO}$  returns “stop” at the third step above, there is the possibility that  $[A]$  contains a singular matrix and one cannot apply Lemma 2. In those cases we re-define the input  $[X]$  and try again.

**5. Stability of a fixed point.** We now explain how to test the stability of the fixed point of  $P$ . Suppose that we have proved that there exists a fixed point  $x^*$  of  $P$  in  $[Y] \in \mathbf{IR}^{2 \times 2}$ . We denote the eigenvalues of  $P'(x^*)$  by  $\mu_1$  and  $\mu_2$ . Since (4) is a divergence-free vector field in  $\mathbf{R}^3$  and  $P$  is a two-dimensional map, we have

$$\mu_1 \mu_2 = \det(P'(x^*)) = 1.$$

Therefore  $\mu_1$  and  $\mu_2$  are given by

$$\mu_1 = \frac{1}{2}(\tau + \sqrt{\tau^2 - 4}), \quad \mu_2 = 1/\mu_1,$$

where  $\tau$  is the trace of  $P'(x^*)$ . Note that  $\tau > 2$  implies that both  $\mu_1$  and  $\mu_2$  are real and  $0 < \mu_2 < 1 < \mu_1$ . Later we show that this is indeed the case.

**6. Uniqueness of a fixed point.** The interval vector  $[X]$  of Lemma 2 must be small enough. Otherwise, the computation breaks down due to inflation of upper bounds of errors. As far as we have tested for our problem, the computation of the

Poincaré map often breaks down, if the diameter of  $[X]$  is greater than  $O(10^{-2})$ . Moreover, one may fail to compute the Jacobian matrix of the Poincaré map within desired accuracy if the diameter is greater than  $O(10^{-5})$ . Therefore the diameter of the uniqueness region which is guaranteed by a single use of Lemma 2 is quite small.

We therefore wish to enlarge, by another method, the region in which the uniqueness of the fixed point of  $P$  is guaranteed. Our method is as follows. Let  $d$  and  $l_{\max}$  be positive integers. An interval vector  $[X] \in \mathbf{IR}^2$  and a non-negative integer  $l$  are the inputs of the following procedure.

- 1 If  $l \geq l_{\max}$ , then output  $[X]$  and exit from this procedure. If  $l < l_{\max}$ , go to the next step.
- 2 Divide  $[X]$  into  $m = 2^d \times 2^d$  small sub-interval vectors  $[X_1], [X_2], \dots, [X_m]$ , and go to the next step.
- 3 For each  $j = 1, 2, \dots, m$ , compute a  $[P_j] \in \mathbf{IR}^2$  such that  $P([X_j]) \subseteq [P_j]$  by Zgliczynski's method and test whether  $[P_j] \cap [X_j] = \emptyset$  or not. If the intersection is nonempty, go to step 1 with inputs  $[P_j] \cap [X_j]$  and  $l + 1$ , that is, call this procedure recursively.

When the procedure above finishes, a considerable number of intervals are judged to contain no fixed point and we have only a small number of interval vectors which intersects its image of  $P$ , in which a fixed point may possibly exist. We expect each of the outputs of the procedure above is small enough to be verified by Lemma 2. They actually are and we can prove uniqueness of the fixed point in a considerably large region. Though it may be a brute-force method, it actually worked well.

We have not excluded a periodic point of  $P$ , e.g., an  $x$  such that  $P(x) \neq x$  but  $P(P(x)) = x$ . We need a different idea to prove non-existence of a periodic orbit corresponding to  $n$ -periodic orbit of  $P$  for  $n > 1$ . Galias [4] has proposed a systematic way to find all the periodic orbits with period  $n$  and has applied it to the Rössler system. It is, however, not straightforward to apply Galias's method to our problem since the method uses a positively invariant bounded set of  $P$  but (4) does not seem to have one. Therefore we have focused ourselves on the fixed point of  $P$ .

**7. Computational detail.** In this section, we show the computational details. Our computer environment is as follows: Hewlett-Packard Z420 Workstation, Intel Xeon E5-1660 3.30GHz, Ubuntu

12.04 (Precise) 64bit (Linux 3.2.0-60-generic), the GNU Compiler Collection version 4.6.3. (We made another experiment in Linux machines to obtain the same result.) We implemented our algorithms by C++ using CAPD library version 3.0 [13]. We used `IVector` and `IPoincareMap` classes, which are provided by the CAPD library for manipulating interval vectors and Poincaré maps, respectively. They are based on intervals whose endpoints are double-precision floating point numbers.

We examined existence and uniqueness of a fixed point in  $[X] = ([6.5, 8.5], [0, 2])$ . Computation was carried out in the following way. First, we divided  $[X]$  into  $2^9 \times 2^9$  subintervals  $[X_{ij}] (1 \leq i, j \leq 2^9)$ . For each  $[X_{ij}]$ , we applied the algorithm in Section 6 with

$$l = 0, \quad l_{\max} = 5, \quad d = 2,$$

to get a list of intervals which may possibly contain a fixed point of  $P$ . (Under this setting, the diameter of an output interval vector will be at most  $2^{-18} \approx 3.8 \times 10^{-6}$ , and it is sufficiently small for the interval Newton method to be applied.) Then 9144 intervals were listed. We next applied the algorithm in Section 4 to each of these 9144 intervals. As a result, non-existence in 9142 intervals was proved, while the verification failed for the remaining two interval vectors. They are approximately

$$\left( \begin{array}{c} [8.0429992, 8.0430031] \\ [0.4999961, 0.5] \end{array} \right)$$

and

$$\left( \begin{array}{c} [8.0429992, 8.0430031] \\ [0.5, 0.5000038] \end{array} \right),$$

which are adjacent to each other. We applied the algorithm in Section 4 to the union of these interval vectors, which is approximately

$$\left( \begin{array}{c} [8.0429992, 8.0430031] \\ [0.4999961, 0.5000038] \end{array} \right),$$

and the unique existence of a fixed point in this interval was proved. In particular, the fixed point of  $P|_{[X]}$  is unique.

Next, we explain how we prove (ii) of Theorem 1. We apply an iterative technique proposed by Caprani and Madsen [1]. For easy implementation we employ the method labelled (viii) in the survey by Mayer [7, p. 149]. First, we compute an approx-

imate value as accurately as possible, say,

$$x_a = (8.04300111305391, 0.5).$$

We obtained this value by applying the Newton method to  $P(x) - x = 0$  with the initial guess  $(8.043, 0.5)$ . Let  $[x^{(0)}] = [x_a, x_a]$  and  $[y^{(0)}] = [x^{(0)}]$ . With a prescribed  $\varepsilon > 0$ , we generate sequences of interval vectors by

$$(10) \quad [x^{(j+1)}] := (1 + \varepsilon)[y^{(j)}] - \varepsilon[y^{(j)}],$$

$$(11) \quad [y^{(j+1)}] := \text{INO}([x^{(j+1)}]),$$

for  $j = 0, 1, \dots$ , until  $[y^{(j+1)}] \subset [x^{(j+1)}]$  is satisfied, where INO is the procedure in Section 4 to compute an interval vector containing the set defined by (8). If  $[y^{(j+1)}] \subset [x^{(j+1)}]$  holds for some  $j$ , then Lemma 2 implies the existence of a fixed point of  $P$  in  $[y^{(j+1)}]$ . We applied the above method with  $\varepsilon = 0.1$ . Then in our computer environment and programme, INO did not return “stop”, and the iteration successfully ended at  $j = 2$ . Therefore there exists a fixed point of  $P$  in the interval vector  $[y^{(3)}]$  which is approximately given by

$$\left( \begin{array}{c} [8.04300111293052, 8.04300111317316] \\ [0.49999999999337, 0.50000000000689] \end{array} \right).$$

Finally, we consider the stability of the fixed point. We computed an interval matrix  $[A] = ([A_{ij}])_{i,j=1}^2$  which contains  $P'([y^{(3)}])$ , and we computed the trace  $[A_{11}] + [A_{22}]$  by Zgliczynski's method [14]. The result is: it is contained in

$$[2.010569, 2.010571],$$

showing that it is greater than 2. Therefore the eigenvalues  $\mu_1$  and  $\mu_2$  of the fixed point satisfy  $0 < \mu_2 < 1 < \mu_1$ , and the fixed point is a saddle. Moreover, it follows that

$$\mu_1 \in [1.108229, 1.108236],$$

$$\mu_2 \in [0.902334, 0.902340].$$

Thus Theorem 1 is proved.

**8. Concluding remarks.** We demonstrated that the numerical verification method is powerful enough to prove existence of a periodic orbit in a conservative dynamical system. A dynamical system related to (1) was proposed by Pehlivan [11]. The system possesses chaotic orbits generated by period-doubling cascades. In [8] we will use our method to prove period-doubling bifurcations. Also, we will analyse (1) and a related system in [2] in our companion paper.

**Acknowledgments.** The authors are very grateful to Prof. Alexander Craik, whose advice and encouragement has been a driving force of writing the present paper. This work was completed while T.M. was a JSPS Research Fellow at RIMS, Kyoto University. He thanks JSPS and RIMS for the support. He is supported by the Grant-in-Aid for JSPS Fellows No. 24-5312. H.O. is partially supported by JSPS KAKENHI 24244007. Finally, we thank the anonymous reviewers for their valuable comments.

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