Dirichlet finite harmonic functions and points at infinity of graphs and manifolds

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Abstract: In this paper, we consider the Royden compactifications relative to p-Dirichlet integrals of infinite graphs and noncompact Riemannian manifolds, and study the behavior of rough isometries in the compactifications, proving bijective correspondence of the spaces of p-harmonic functions with finite p-energy.

Key words: graph; Riemannian manifold; p-energy; rough isometry; Royden's compactification.

1. Introduction. Let M be a connected Riemannian manifold. The Riemannian distance and the Riemannian measure will be denoted by d_M and μ_M, respectively. The Dirichlet space $L^{1,p}(M)$ of exponent p, $1 < p < \infty$, is the class of all functions $f \in L^1_{loc}(M)$ whose distributional gradients ∇f belong to $L^p(M)$, and it is provided with a semi-norm $D_p^{1/p}$, where

$$D_p(f) = \int_M |\nabla f|^p d\mu_M, \qquad f \in L^{1,p}(M).$$

Let us denote by $L_0^{1,p}(M)$(resp. $HL^{1,p}(M)$) the space of functions f in $L^{1,p}(M)$ to which a sequence of Lipschitz continuous functions f_n of compact supports converge almost everywhere in such a way that $D_p(f_n - f) \to 0$ as $n \to \infty$ (resp. the space of p-harmonic functions in $L^{1,p}(M)$, that is, functions $h \in L^{1,p}(M)$ satisfying $\int_M |\nabla h|^{p-2}\langle \nabla h, \nabla g\rangle d\mu_M = 0$ for all $g \in L_0^{1,p}(M)$). In the case where $1 \in L_0^{1,p}(M)$, M is called p-parabolic; otherwise it is called p-hyperbolic. In the latter case, the Royden decomposition says that a function $f \in L^{1,p}(M)$ is uniquely expressed as follows: $f = h + g$, where $h \in HL^{1,p}(M)$ and $g \in L_0^{1,p}(M)$. The Royden p-algebra $A^p(M)$ consists of all functions of $BL^{1,p}(M) \cap C(M)$ with multiplication and addition defined pointwise on M, where $BL^{1,p}(M)$ stands for $L^{1,p}(M) \cap L^\infty(M)$ and this convention will be kept in this paper. Given the norm $\|f\| = \|f\|_{L^\infty} + D_p(f)^{1/p}$, $A^p(M)$ is a commutative Banach algebra with unit element 1 and separates points in M. The maximal ideal space $\Re_p(M)$ of $A^p(M)$ is referred to as the Royden p-compactification, which can also be characterized as the compact Hausdorff space containing M as its open and dense subspace such that every function of $A^p(M)$ is continuously extended to $\Re_p(M)$ and $A^p(M)$, viewed as a subspace of $C(\Re_p(M))$ by this continuous extension, is dense in $C(\Re_p(M))$ with respect to its supremum norm. We call the boundary of M in $\Re_p(M)$ the Royden p-boundary of M, which will be denoted by $\partial\Re_p(M)$. For $f \in A_p(M)$, we denote by $\bar{f}$ the continuous extension of f to $\Re_p(M)$ and by $tr(f)$ the restriction of $\bar{f}$ to $\partial\Re_p(M)$. We distinguish the following important part of the Royden p-boundary: $\Delta_p(M) = \{x \in \partial\Re_p(M) \mid tr(f)(x) = 0,\ \forall f \in A_p(M) \cap L_0^{1,p}(M)\}$. The set $\Delta_p(M)$ is called the p-harmonic boundary of M. It is known that $\Delta_p(M)$ is empty if and only if M is p-parabolic, and in the case where M is p-hyperbolic, we have the following duality: $A_p(M) \cap L_0^{1,p}(M) = \{f \in A_p(M) \mid tr(f)(x) = 0, \forall x \in \Delta_p(M)\}$. Moreover the maximum principle says that for all $h \in BHL^{1,p}(M)$, $\min_{\Delta_p(M)} tr(h) \le h \le \max_{\Delta_p(M)} tr(h)$, and for $h_1, h_2 \in BHL^{1,p}(M)$, $h_1 = h_2$ if $tr(h_1) = tr(h_2)$. (See e.g., [14,16,17] for the facts mentioned above.) In this paper, unless otherwise is stated, connected Riemannian manifolds M are assumed to be complete and noncompact, and further to satisfy the following conditions:

(PI): *M supports a weak locally $(1,p)$-Poincaré inequality $(1 < p < +\infty)$ with a constant $C_p > 0$, that is, for all points $x \in M$, all $r \in (0,1]$ and all functions $f \in L^{1,p}_{loc}(M)$,*

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$$\fint_{B(x,r)} |f - f_{B(x,r)}| d\mu_M \leq$$
$$C_p r \left(\fint_{B(x,2r)} |\nabla f|^p d\mu_M \right)^{1/p},$$

where, if u is a measurable function on M and B is a bounded measurable subset of M, then we set

$$u_B := \frac{1}{\mu_M(B)} \int_B u \, d\mu_M := \fint_B u \, d\mu_M,$$

and $B(x,r)$ stands for the metric ball of M around a point x with radius r.

(VD): *A locally volume doubling condition with a constant $C_v \geq 0$ holds on M, that is, for all $x \in M$ and all $r \in (0,1]$,*

$$\mu_M(B(x,2r)) \leq C_v \mu_M(B(x,r)).$$

(Va): *There exists a constant $v_0 > 0$ such that for all $x \in M$,*

$$\frac{1}{v_0} \leq \mu_M(B(x,1)) \leq v_0.$$

It is known that a p-harmonic function h on M is Hölder continuous and in fact the following estimate holds;

$$(1) \qquad |h(x) - h(y)| \leq$$
$$C_1 d_M(x,y)^\alpha \left(\fint_{B(z,10)} |\nabla h|^p d\mu_M \right)^{1/p}$$

for all x, $y \in B(z,1)$, where $\alpha \in (0,1)$ and $C_1 > 0$ are constants depending only on p, C_p and C_v (cf. [2]). Now a main result of this paper is stated in the following

Theorem 1. *Let M and N be connected noncompact complete Riemannian manifolds. Suppose that conditions (PI), (VD) and (Va) hold for M and N and that there exists a map $\Phi : M \longrightarrow N$ satisfying the following condition*:

(RI): *There exist constants $a \geq 1$ and $b > 0$ such that*

$$\frac{1}{a} d_M(x,y) - b \leq d_N(\Phi(x), \Phi(y)) \leq a d_M(x,y) + b$$

for all x, $y \in M$; for some $\epsilon > 0$, the ϵ-neighborhood of the image $\Phi(M)$ covers N. Then there is a bijective map $\eta : BHL^{1,p}(N) \longrightarrow BHL^{1,p}(M)$ such that

$$\frac{1}{C} D_p(h) \leq D_p(\eta(h)) \leq C D_p(h)$$

for all $h \in BHL^{1,p}(N)$, where $C > 0$ is a constant depending only on the given p, C_p, C_v, a and b; moreover in the case where $p = 2$, η is a linear isomorphism.

Following [7], we say that a map Φ of a metric space (X, d_X) to another one (Y, d_Y) is a rough isometry if it satisfies condition (RI) as in Theorem 1 and (X, d_X) is roughly isometric to (Y, d_Y) if there exists a rough isometry between them: being roughly isometric is an equivalent relation.

Now a connected noncompact complete Riemannian manifold M under consideration will be assumed to satisfy, in addition to conditions (PI), (VD) and (Va), the following

(Vb): *for some constants $s > 0$, $C_b > 0$, it holds that*

$$\mu_M(B) \geq C_b r^s \mu_M(B_o)$$

whenever B_o is an arbitrary ball of radius 1 and $B = B(x,r)$, $x \in B_o$, $r \leq 1$.

Notice that the volume doubling condition (VD) always implies (Vb) for some exponent $s \leq \log_2 C_v$; moreover it is known that if $p > s$, then

$$(2) \qquad |f(x) - f(y)| \leq$$
$$C_2 d_M(x,y)^{1-s/p} \left(\fint_{B(z,10)} |\nabla f|^p d\mu_M \right)^{1/p}$$

for all $f \in L^{1,p}_{loc}(M)$, all x, $y \in B(z,1)$ and $z \in M$, where C_2 is a positive constant depending only on p, C_p, C_v and C_b (cf. [4; Theorem 5.1]). This estimate (2) is crucial for proving the following

Theorem 2. *Let M and N be connected noncompact complete Riemannian manifolds satisfying conditions (PI), (VD), (Va) and (Vb). Suppose that M and N are roughly isometric. Then for $p > s$, where s is as in (Vb), a rough isometry $\Phi : M \to N$ induces a homeomorphism $tr(\Phi)$ between the Royden p-boundaries $\partial\Re_p(M)$ and $\partial\Re_p(N)$ which sends $\Delta_p(M)$ onto $\Delta_p(N)$.*

Theorem 1 (resp. Theorem 2) follows from Theorems 3 and 7 (resp. Theorems 3 and 8) in the next two sections.

We remark that a connected noncompact complete Riemannian manifold of dimension n such that the Ricci curvature is bounded from below and the injectivity radius is positive, for example, a noncompact co-compact Riemannian regular cover, satisfies conditions (PI), (VD), (Va) and (Vb) (; in this case, we can take $p = 1$ in (PI) and $s = n$ in (Vb)).

2. Graphs and rough isometries. In this section, we consider a graph $G = (V, E)$ with vertex set V and edge set E. The set V is assumed to be countably infinite. For a pair of vertices x, y, we say that x and y are adjacent if x is joined to y by an edge of E, that is, $\{x, y\} \in E$; in this case, we write $x \sim y$. For $x \in V$, we set $V_x = \{y \in V | \ y \sim x\}$ and the cardinality of the set V_x is called the degree of x and denoted by $\deg(x)$. In what follows, we assume that G is locally finite, that is, $\deg(x)$ is finite for any $x \in V$. The measure on the graph is the counting measure on V and the distance $d_G(x, y)$ between a pair of vertices x, y is the smallest number of vertices $x = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = y$ needed so that $x_i \sim x_{i+1}$. The graph G is connected if and only if $d_G(x, y)$ is finite for any $x, y \in V$. For a function f on V, the p-energy of f is defined by

$$D_p(f) = \frac{1}{2} \sum_{x \in V} |\nabla f|^p(x),$$

where $|\nabla f|(x) = (\sum_{y \in V_x} |f(x) - f(y)|^2)^{1/2}$. The p-Dirichlet space of G, $L^{1,p}(G)$, consists of functions on V with finite p-energy, and the closure of the subspace of finitely supported functions in $L^{1,p}(G)$ will be denoted by $L_0^{1,p}(G)$. A function f of $L^{1,p}(G)$ is said to be p-harmonic if $\sum_{x \in V} \sum_{y \in V_x} |\nabla f|^{p-2}(x)(f(y) - f(x))(g(y) - g(x)) = 0$ for all $g \in L_0^{1,p}(G)$; p-harmonic functions are characterized as local minimizers of p-energy. The space of p-harmonic functions will be denoted by $HL^{1,p}(G)$. As in the case of Riemannian manifolds, we have the Royden decomposition, the Royden p-algebra, the Royden compactification $\Re_p(G)$, the Royden p-boundary $\partial\Re_p(G)$ and the p-harmonic boundary $\Delta_p(G)$ of G (cf. e.g., [18], Chap. VI); for a function $f \in BL^{1,p}(G)$, $\bar{f}$ (resp. $tr(f)$) stands for the continuous extension of f to $\Re_p(G)$ (resp. the restriction of $\bar{f}$ to $\partial\Re_p(G)$).

Theorem 3. *Let $G = (V, E)$ and $G' = (V', E')$ be two infinite graphs with bounded degrees. Suppose that G and G' are roughly isometric, that is, there exists a rough isometry $\Phi : (V, d_G) \longrightarrow (V', d_{G'})$. Then Φ extends to a continuous map $\bar{\Phi}$ of $\Re_p(G)$ to $\Re_p(G')$ whose restriction to $\partial\Re_p(G)$ induces a homeomorphism $tr(\Phi)$ between $\partial\Re_p(G)$ and $\partial\Re_p(G')$ such that $tr(\Phi)(\Delta_p(G)) = \Delta_p(G')$. Moreover assigning a function h of $BHL^{1,p}(G')$ the unique function $\eta(h)$ of $BHL^{1,p}(G)$ whose trace on $\Delta_p(G)$ coincides with $tr(h) \circ tr(\Phi)$ is bijective, and moreover it holds that for some constant $C > 0$,*

$$C^{-1} D_p(h) \le D_p(\eta(h)) \le C D_p(h), \quad h \in BHL^{1,p}(G').$$

Proof. By the definition of a rough isometry and the assumption of the graphs having bounded degrees, we first notice that there exists a constant C_3 such that $D_p(f \circ \Phi) \le C_3 D_p(f)$ for all $f \in L^{1,p}(G')$, and that $f \circ \Phi \in L_0^{1,p}(G)$ whenever $f \in L_0^{1,p}(G')$. For a sequence $\{v_i\}$ of V converging to a point $v_\infty \in \partial\Re_p(G)$ and a function $f \in BL^{1,p}(G')$, if a subsequence $\{\Phi(v_j)\}$ of $\{\Phi(v_i)\}$ tends to a point $w \in \partial\Re_p(G')$, then we have $tr(f)(w) = \lim_{j \to \infty} f(\Phi(v_j)) = tr(f \circ \Phi)(v_\infty)$. This shows that the sequence $\{\Phi(v_i)\}$ converges to a point of $\partial\Re_p(G')$, denoted by $\bar{\Phi}(v_\infty)$, and $tr(f)(\bar{\Phi}(v_\infty)) = tr(f \circ \Phi)(v_\infty)$. In particular, $tr(f)(\bar{\Phi}(v_\infty)) = 0$ whenever $v_\infty \in \Delta_p(G)$ and $f \in L_0^{1,p}(G')$. Hence we see that $\bar{\Phi}(\Delta_p(G)) \subset \Delta_p(G')$. Evidently the map $\bar{\Phi} : \Re_p(G) \to \Re_p(G')$ just obtained is continuous and the restriction of $\bar{\Phi}$ to $\partial\Re_p(G)$ will be denoted by $tr(\Phi)$. Let $\Psi : V \to V'$ be another rough isometry such that $d_{G'}(\Phi(v), \Psi(v)) \le d$ for some constant d and all $v \in V$. Then for any $f \in BL^{1,p}(G')$ and every $v \in V$, we have

$$|f(\Phi(v)) - f(\Psi(v))| \le \left(\sum_{\{w \in V' | d_{G'}(\Phi(v), w) \le d\}} |\nabla f|^p(w) \right)^{1/p};$$

the right side tends to 0 as v goes to $\partial\Re_p(G)$. This implies that $tr(f) \circ tr(\Phi) = tr(f \circ \Phi) = tr(f \circ \Psi) = tr(f) \circ tr(\Psi)$ for all $f \in BL^{1,p}(G')$, and thus we can conclude that $tr(\Phi) = tr(\Psi)$. Now we observe that if we have a rough isometry $\Theta : G' \to G$, then $tr(\Theta \circ \Phi) = tr(\Theta) \circ tr(\Phi)$. In particular, if, in addition, $d_G(\Theta \circ \Phi(v), v)$ is bounded from above as $v \in V$ goes to $\partial\Re_p(G)$, then $tr(\Theta) \circ tr(\Phi)$ coincides with the identity map of $\partial\Re_p(G)$. Thus taking such a rough isometry $\Theta : G' \to G$, we see that $tr(\Phi)$ induces a homeomorphism from $\partial\Re_p(G)$ onto $\partial\Re_p(G')$. Moreover given a function $h \in BHL^{1,p}(G')$, let $\eta(h)$ be the p-harmonic part of $h \circ \Phi$. Then h is the p-harmonic part of $\eta(h) \circ \Theta$, since $tr(\eta(h) \circ \Theta) = tr(\eta(h)) \circ tr(\Theta) = tr(h) \circ tr(\Phi) \circ tr(\Theta) = tr(h)$. Therefore we have $D_p(\eta(h)) \le D_p(h \circ \Phi) \le C_3 D_p(h)$ and also $D_p(h) \le D_p(\eta(h) \circ \Theta) \le C_4 D_p(\eta(h))$ for some constant $C_4 > 0$. Thus we arrive at the inequality of the theorem. This completes the proof of Theorem 3. □

3. Discrete approximation of Riemannian manifolds. Let M be a connected noncompact complete Riemannian manifold satisfying conditions (PI), (VD) and (Va). Fix a positive number κ and take a maximal κ-separated subset V of M, where we mean by a κ-separated subset that $d_M(v,w) \geq \kappa$ for all pairs of distinct two points v, w of the subset. Define a graph $G = (V, E)$ as follows: a point v in V is adjacent to another point w, that is, $\{v, w\}$ belongs to the set of edges E if $0 < d_M(v,w) \leq 3\kappa$. Then the graph G obtained in this way is roughly isometric to M. Moreover by virtue of (VD), we see that G has uniformly bounded degrees and more precisely the numbers of points in a metric r-ball $B_G(v,r)$ of G and also the intersection of a metric r-ball $B_M(x,r)$ of M with V are bounded by a constant $n(r)$ depending only on C_v, κ and r. Given a locally summable function f on M, we define a function $\mu(f)$ on V by

$$\mu(f)(v) = f_{B(v,4\kappa)}, \quad v \in V.$$

Define a sort of "inverse" of μ, we choose a partition of unity $\{\chi_v\}_{v\in V}$ associated to V in such a way that $\operatorname{supp} \chi_v \subset B_M(v, 2\kappa)$ and $\sup|\nabla\chi_v| \leq C_5$ for some positive constant C_5 depending only on C_v and κ. Then given a function f on V, we define a function $\nu(f)$ on M by

$$\nu(f)(x) = \sum_{v\in V} f(v)\chi_v(x), \quad x \in M.$$

In what follows, C_i's denote some positive constants depending only on the given p, C_p, C_v, v_0 and κ.

Then the following results are due to Kanai [8] (see also [5]).

Lemma 4. (*i*) *For any f of $L^{1,p}(M)$, $\mu(f)$ belongs to $L^{1,p}(G)$ and satisfies*

$$D_p(\mu(f)) \leq C_6 D_p(f);$$

in addition, $\mu(f) \in L_0^{1,p}(G)$ whenever $f \in L_0^{1,p}(M)$.

(*ii*) *For every f of $L^{1,p}(G)$, $\nu(f)$ belongs to $L^{1,p}(M)$ and satisfies*

$$D_p(\nu(f)) \leq C_7 D_p(f);$$

in addition, $\nu(f) \in L_0^{1,p}(M)$ whenever $f \in L_0^{1,p}(G)$.

Now we prove the following

Lemma 5. *There exists a constant C_8 such that for any $f \in BL^{1,p}(G)$ and every $v \in V$,*

$$|f(v) - \mu(\nu(f))(v)| \leq C_8 \left(\sum_{w\in B_G(v,8)} |\nabla f|^p(w) \right)^{1/p};$$

in particular one has

$$tr(\mu(\nu(f))) = tr(f) \quad on\ \partial\Re_p(G).$$

Proof. Observe first that $(\chi_w)_{B_M(v,4\kappa)} = 0$ if $d_G(v,w) > 8$. Therefore we have

$$\begin{aligned}
&|f(v) - \mu(\nu(f))(v)| \\
&= \left| \sum_{d_G(v,w)\leq 8} (f(v) - f(w))(\chi_w)_{B_M(v,4\kappa)} \right| \\
&\leq \sum_{d_G(v,w)\leq 8} |f(v) - f(w)| \\
&\leq n(8)^{1-1/p} \left(\sum_{d_G(v,w)\leq 8} |\nabla f|^p(w) \right)^{1/p}.
\end{aligned}$$

This shows that $\lim_{r\to\infty} \sup_{V\setminus B_G(o,r)} |f - \mu(\nu(f))| = 0$ for any fixed $o \in V$. Thus the proof of Lemma 5 is completed. □

Lemma 6. *There exists a constant C_9 such that for any $h \in BHL^{1,p}(M)$ and every $x \in M$,*

$$|h(x) - \nu(\mu(h))(x)| \leq C_9 \left(\int_{B_M(x,8\kappa)} |\nabla h|^p d\mu_M \right)^{1/p};$$

in particular, one has

$$tr(\nu(\mu(h))) = tr(h) \quad on\ \partial\Re_p(M).$$

Proof. In view of (1), we see that for any $x \in M$ and every $v \in V$ such that $\chi_v(x) \neq 0$,

$$|h(x) - h_{B_M(v,4\kappa)}| \leq C_{10} \left(\int_{B_M(x,8\kappa)} |\nabla h|^p d\mu_M \right)^{1/p}.$$

Using this, we obtain

$$\begin{aligned}
&|h(x) - \nu(\mu(h))(x)| \\
&= \left| \sum_{v\in V} (h(x) - h_{B(v,4\kappa)})\chi_v(x) \right| \\
&\leq \sum_{v\in V, \chi_v(x)\neq 0} |h(x) - h_{B(v,4\kappa)}| \\
&\leq n(2\kappa) C_{10} \left(\int_{B_M(x,8\kappa)} |\nabla h|^p d\mu_M \right)^{1/p}.
\end{aligned}$$

This implies that $\lim_{r\to\infty} \sup_{M\setminus B_M(o,r)} |h - \nu(\mu(h))| = 0$ for any fixed $o \in M$. This completes the proof of Lemma 6. □

Now given a function h of $BHL^{1,p}(M)$, we denote by $\sigma(h)$ the p-harmonic part of the function $\mu(h)$ in the Royden decomposition. Then σ induces a map of $BHL^{1,p}(M)$ to $BHL^{1,p}(G)$. In the same way, for a function h of $BHL^{1,p}(G)$, $\rho(h)$ denotes the p-harmonic part of $\nu(h)$. Then ρ defines a map of $BHL^{1,p}(G)$ to $BHL^{1,p}(M)$.

Theorem 7. *Let $\sigma : BHL^{1,p}(M) \to BHL^{1,p}(G)$ and $\rho : BHL^{1,p}(G) \to BHL^{1,p}(M)$ be as above. Then $\sigma \circ \rho(h) = h$ for all $h \in BHL^{1,p}(G)$ and $\rho \circ \sigma(h) = h$ for all $h \in BHL^{1,p}(M)$. Moreover there exists a constant C depending only on p, C_p, C_v, v_0 and κ such that*

$$C^{-1} D_p(h) \le D_p(\sigma(h)) \le C D_p(h)$$

for all $h \in BHL^{1,p}(M)$, and it holds that $\|\sigma(h)\|_{L^\infty} = \|h\|_{L^\infty}$ for all $h \in BHL^{1,p}(M)$.

Proof. Given $h \in BHL^{1,p}(M)$, $\mu(h) - \sigma(h)$ belongs to $L_0^{1,p}(G)$ and $\nu(\mu(h)) - \rho(\sigma(h))$ is a function in $L_0^{1,p}(M)$, since $\nu(\mu(h)) - \rho(\sigma(h)) = \nu(\mu(h) - \sigma(h)) + \nu(\sigma(h)) - \rho(\sigma(h))$. This, together with Lemma 5, implies that $tr(\rho(\sigma(h))) = tr(\nu(\mu(h))) = tr(h)$. Therefore we conclude that $\rho(\sigma(h)) = h$. In the same way, using Lemma 6, we see that $\sigma(\rho(h)) = h$ for all $h \in BHL^{1,p}(G)$. Finally the inequalities in Theorem 7 follow from Lemma 4. This completes the proof of the theorem. $\square$

In what follows, we consider the case where a manifold M under consideration satisfies further condition (Vb) and $p > s$, where s is the exponent in (Vb). Then in view of (2), we see that for any $f \in L^{1,p}(M)$, the restriction of f to V, $r_V(f)$, belongs to $L^{1,p}(G)$ and for all $x \in M$, we have

$$|f(x) - \nu(r_V(f))(x)| \le C_{11} \left(\int_{B_M(x,8\kappa)} |\nabla f|^p d\mu_M \right)^{1/p},$$

where $C_{11} > 0$ is a constant depending only on p, s C_p, C_v, v_0, C_b and κ; in addition, $r_V(f) \in L_0^{1,p}(G)$ if $f \in L_0^{1,p}(M)$. Now applying the same arguments as in the proof of Theorem 3, we arrive at the following

Theorem 8. *Let M and $G = (V, E)$ be as above and suppose that $p > s$. Then the inclusion map ι of V into M extends to a continuous map $\bar{\iota}$ of $\mathfrak{R}_p(G)$ to $\mathfrak{R}_p(M)$ whose restriction to $\partial\mathfrak{R}_p(G)$ induces a homeomorphism $tr(\iota)$ between $\partial\mathfrak{R}_p(G)$ and $\partial\mathfrak{R}_p(M)$ such that $tr(\iota)(\Delta_p(G)) = \Delta_p(M)$.*

Relevantly to Theorem 8, we would like to show another implication of estimate (2). Let K be a nonempty closed subset of a Riemmanian manifold M of dimension n. We fix an exponent $p > n$, so that any function in $L^{1,p}(M)$ may be assumed to be continuous in M. In fact, estimate (2) holds for relatively compact open subsets of M, although the constants there may vary upon the subsets. For a continuous function u on K, we set $\mathcal{A}_u = \{f \in L^{1,p}(M) \mid f_{|K} = u\}$ and $D_p(u|K) = \inf\{D_p(f) \mid f \in \mathcal{A}_u\}$. Here we understand $D_p(u|K) = +\infty$ if $\mathcal{A}_u$ is empty. Let $L^{1,p}(K|M) = \{u \in C(K) \mid D_p(u|K) < +\infty\}$. Then we have the following

Proposition 9. *Let M be a Riemannian manifold of dimension n and consider the Dirichlet space $L^{1,p}(M)$ with exponent $p > n$. Then the following assertions hold*:

(i) *Given a nonempty closed subset K of M and a function $u \in L^{1,p}(K|M)$, there exists a unique p-energy minimizer H_u in $\mathcal{A}_u$ that is p-harmonic outside of K.*

(ii) *For nonempty closed subsets K, L with $K \subset L$ and a function $u \in L^{1,p}(L|M)$, one has $D_p(u_{|K}|K) \le D_p(u|L)$.*

(iii) *For an increasing sequence $\{K_i\}$ of finite subsets of M whose union is dense in M, one has*

$$L^{1,p}(M) = \{f \in C(M) \mid \lim_{i\to\infty} D_p(f_{|K_i}|K_i) < +\infty\};$$
$$D_p(f) = \lim_{i\to\infty} D_p(f_{|K_i}|K_i).$$

Proof. To prove (i), we take a minimizing sequence $\{f_i\}$ in $\mathcal{A}_u$. Then it follows from the uniform convexity of the semi-norm $D_p^{1/p}$ and estimate (2) that $\{f_i\}$ is a Cauchy sequence relative to $D_p^{1/p}$ and converges to a function $f \in L^{1,p}(M)$ uniformly on each compact subset of M. Since $f_i = u$ on K, f also equals u there and hence belongs to $\mathcal{A}_u$. Thus f is a minimizer in $\mathcal{A}_u$; the uniqueness is also implied by the uniform convexity of $D_p^{1/p}$. Thus assertion (i) is verified. The second one is obvious. To prove assertion (iii), we first notice that $\lim_{i\to\infty} D_p(f_{|K_i}|K_i) \le D_p(f)$ by the definition of $D_p(f|K)$. For the opposite direction, we take a unique minimizer H_i in $\mathcal{A}_{f_i}$ for each K_i, where $f_i = f_{|K_i}$. Since $D_p(H_i) \le \lim_{i\to\infty} D_p(f_i|K_i) < +\infty$ and $H_i = H_j = f$ on K_i for all $j \ge i$, in view of (2), $\{H_i\}$ converges to f uniformly on each compact subset of M. Hence we obtain $D_p(f) \le \liminf_{i\to\infty} D_p(H_i) \le \lim_{i\to\infty} D_p(f_i|K_i)$. Thus assertion (iii) is verified. This completes the proof of Proposition 9. $\square$

We conclude this paper with the following

Remarks. (i) In the presence of conditions (PI), (VD), (Va) and (Vb), our results are not restricted to the case of the p-Dirichlet integrals on Riemannian manifolds and applicable to the cases of the p-energy associated to subelliptic operators on manifolds studied in [2], the p-Dirichlet integrals on admissible Riemannian polyhedra investigated in [3], those on Alexandrov spaces taken up in [11], and so on. (ii) We refer the reader to [1,5,6,12,15,18] and the references therein for some related results to ours. In [13], similar claims to Theorems 1 and 2 are found; the proofs, however, have certain gaps in using Lemmas 3 and 4 there. (iii) We say that a (locally finite) graph $G = (V, E)$ satisfies a strong isoperimetric inequality if there exists a constant $C_I > 0$ such that $\sharp(\partial_V A) \geq C_I \sharp A$ for every finite subset A, where, for a finite subset A, we denote by $\sharp A$ its cardinality and by $\partial_V A$ the combinatorial boundary of A defined as the subset of all vertices in A which are adjacent to a vertex not in A. Then it is known that (a) for two rough isometric graphs G_1 and G_2 of bounded degrees, if G_1 satisfies a strong isoperimetric inequality, then so does G_2; (b) a graph $G = (V, E)$ of bounded degree satisfies a strong isoperimetric inequality if and only if there is a constant $C > 0$ such that $\sum_{x\in V} \deg(x)|f(x)|^p \leq CD_p(f)$ for every finitely supported f on V (cf. [1,9,18] and the references therein for details and related topics). As a result of this inequality, we see that if a graph $G = (V, E)$ of bounded degree satisfies a strong isoperimetric inequality, then $\partial\Re_p(G) = \Delta_p(G)$ for any $p \in (1, +\infty)$. Using this, we can deduce for instance that for a homogeneous tree T of degree greater than two, any singleton of $\Delta_p(T)$ is not a G_δ set and further $\Delta_p(T)$ has no isolated points. This is observed in [18], Chap. VI when $p = 2$; but the arguments there are valid for any $p \in (1, \infty)$. On the other hand, it is proved in [10] that for a graph $G = (V, E)$, $\partial\Re_2(G) = \Delta_2(G)$ and $\Re_2(G)$ is metrizable if G is of bounded effective resistance, that is, $\sup_{x,y\in V} R(x, y) < +\infty$, where we put $R(x, y) = \sup\{|f(x) - f(y)|^2/D_2(f) \mid f \in L^{1,2}(G)\}$ for $x, y \in V$.

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