PAPERS COMMUNICATED

40. *On a Condition of Stability for a Differential Equation.*

By Masuo FUKUHARA and Mitio NAGUMO.
Mathematical Institute, Tokyo Imperial University.
(Rec. Mar. 24, 1930. Comm. by T. YOSIE, M.I.A., April 12, 1930.)

Recently O. Perron[1] has pointed out the inaccuracy of Fatou's criterion for stability in relation to the differential equation

$$(1) \qquad \frac{d^2x}{dt^2}+\Phi(t)x=0\,,$$

where $\Phi(t)$ denotes a continuous real function lying between the positive boundaries $a^2 \leqq \Phi(t) \leqq b^2$ for all values of t.

Fatou asserted that the integrals of the differential equation (1) and their derivatives are bounded, while Perron gave an example having an integral not bounded even when $\lim_{t\to\infty}\Phi(t)=1$.

Fatou's assertion may however be amended in the following manner:

If the improper integral $\int_{t_0}^{\infty}|\Phi(t)-c^2|\,dt$ *converges, where* c *is a positive constant, then the integrals of the differential equation* (1) *and their first derivatives are bounded for* $t > t_0$.

Proof: Consider the integral $x(t)$ of (1) and the integral $y(t)$ of the differential equation

$$(2) \qquad \frac{d^2y}{dt^2}+c^2y=0\,,$$

with the same initial values for $t=t_1\,(>t_0)$. From (1) and (2) we obtain the identity

$$\frac{d^2}{dt^2}(x-y)+c^2(x-y)=(c^2-\Phi)\,x\,,$$

and hence

$$(3) \qquad x-y=\frac{1}{c}\left\{\sin ct\int_{t_1}^{t}(c^2-\Phi)x\cos ct\,dt-\cos ct\int_{t_1}^{t}(c^2-\Phi)x\sin ct\,dt\right\}.$$

As it is always possible from our assumption, let us now take t_1 so large that

1) O. Perron, Über ein vermeintliches Stabilitätskriterium, Gött. Nachr. (1930), 1.

$$\int_{t_1}^{t} |c^2 - \Phi|\, dt < \frac{c\alpha}{2} \qquad (0 < \alpha < 1)$$

for $t > t_1$. If $x(t)$ were not bounded for $t > t_0$, then there exists a positive number l for any large constant M such that

$$|x(t)| < M \quad \text{for} \quad t_1 \leqq t < t_1 + l, \quad \text{and}$$

$$|x(t_1 + l)| = M. \tag{4}$$

Then we obtain from (3)

$$|x(t)| \leqq N + \alpha M \quad \text{for} \quad t_1 \leqq t \leqq t_1 + l,$$

if the constant N is taken such that $|y(t)| \leqq N$ for $|t| < \infty$.

If therefore we take $M > \dfrac{N}{1-\alpha}$, then it follows

$$|x(t)| < M \quad \text{for} \quad t_1 \leqq t \leqq t_1 + l.$$

This contradicts (4), hence $x(t)$ must be bounded for $t > t_0$.

It can also be easily proved, that the first derivatives are bounded.

Remark: If $\Phi(t)$ is a positive non-decreasing function, it is not difficult to prove that every solution of (1) is bounded for $t > 0$. Particularly, when $\Phi(t)$ tends to a finite value for $t \to \infty$, its derivatives are also bounded.