40. Contribution to the Theory of Semi-groups. I

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Following S. Schwarz [5], a semi-group*) S is called a *periodic semi-group* if, for every element a of S, the semi-group $(a)=\{a \mid a, a^2, \cdots, a^n \cdots\}$ generated by a contains a finite number of different elements.

It is known that the commutative finite semi-group (a) for every $a \in S$ contains only one idempotent (for detail, see K. Iséki [3]). Let e be an idempotent, and $K^{(e)}$ the set of all elements a such that (a) contains the idempotent e, i.e. $K^{(e)}=\{a \mid a^\rho = e$ for some $\rho\}$.

S. Schwarz [5] proved that if S is *commutative* or *totally non-commutative*, $K^{(e)}$ is a maximal semi-group belonging to the idempotent e of S and S is the sum of disjoint semi-groups $K^{(e)}$, each containing only one idempotent.

G. Thierrin [7] defined a new class of semi-groups as follows: A semi-group is said to be *strongly reversible*, if, for any two elements a, b of S, there are three positive integers r, s, and t such that

$$(ab)^r = a^s b^t = b^t a^s.$$

It is clear that any commutative semi-group is strongly reversible. We shall show Theorem 1 which is a generalisation of S. Schwarz result.

Theorem 1. If a periodic semi-group S is strongly reversible, then $K^{(e)}$ is a semi-group.

Proof. Let a, b be any two elements of $K^{(e)}$, then there are integers ρ, τ such that $a^\rho = e = b^\tau$. Since S is strongly reversible, there are three integers r, s, and t such that

$$(ab)^r = a^s b^t = b^t a^s.$$

Therefore, we have

$$(ab)^{r\rho\tau} = ((ab)^r)^{\rho\tau} = (a^s b^t)^{\rho\tau}$$
$$= a^{s\rho\tau} b^{t\rho\tau} = e \cdot e = e.$$

Hence $ab \in K^{(e)}$. This completes the proof.

Theorem 1 and a result of S. Schwarz [5] imply the following

Theorem 2. Any strongly reversible periodic semi-group is the sum of disjoint semi-groups, each containing only one idempotent.

Let S be a strongly reversible or a totally non-commutative periodic semi-group. Then we shall remark that *each $K^{(e)}$ does not*

*) For general theory of semi-groups, see P. Dubreil [1].

contain proper prime ideal.

Suppose that some $K^{(e)}$ contains a proper prime ideal P. Let $e \in P$, then, for any element a of $K^{(e)}$, we have $a^{\rho}=e$ for some ρ. Hence $a^{\rho} \in P$. Since P is prime ideal, $a \in P$. This shows that $P=K^{(e)}$, which is a contradiction. Therefore $e \bar{\in} P$. Suppose that P is non-empty, then take one element a of P. There is an integer ρ such that $a^{\rho}=e$. Since P is an ideal, $a^{\rho} \in P$. This shows that $e \in P$, which is a contradiction. Hence $K^{(e)}$ does not contain proper prime ideal.

Next, we consider a semi-group containing only one idempotent.

Suppose that a periodic semi-group S contains only one idempotent e. Then, by a result of S. Schwarz [5], $K^{(e)}$ is a semi-group and $S=K^{(e)}$. On the other hand, for any periodic semi-group, he has proved that every set $K^{(e)}$ contains a maximal subgroup $G^{(e)}$ and $eK^{(e)}=K^{(e)}\cdot e=G^{(e)}$. Hence, by $S=K^{(e)}$, we have $eSe=G^{(e)}$. Therefore eSe is a maximal subgroup of S.

By a theorem of R. J. Koch [4], we obtain that SeS is the minimal ideal. From $eSe=Se=eS$, eSe is minimal left and right ideals, i.e. two-sided ideal. Therefore $eSe=SeS$. Hence we have the following

Theorem 3. If a semi-group S is periodic, and S contains only one idempotent e, then we have

1) $eSe=SeS$.
2) *eSe is a maximal subgroup and the minimal ideal.*

References

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