32. On Banach Theorem on Contraction Mappings

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In this short note, we shall generalize the well known theorem on a contraction mapping by S. Banach to general metric spaces. As was proved in [1], any topological semifield contains a topological field isomorphic with the real line. These elements are denoted by the Greek letter α and follow the rules of operations on real numbers.

Let X be a sequential complete metric space over a topological semifield R, $f(x)$ a mapping on X such that

$$\rho(f(x), f(y)) \ll \alpha\rho(x, y),$$

where α is a positive number less than 1, and $\ll$ denotes the order in R. Then there is a fixed element x' of the mapping f, i.e. $f(x')=x'$.

The result is a slight generalization of the theorem of S. Banach.

To prove it, take an element x_0 of X, then by a recursive way, we define a sequence $\{x_n\}$ by $x_{n+1}=f(x_n)(n=0, 1, 2, \cdots)$. For the sequence $\{x_n\}$, we have

$$\begin{aligned}\rho(x_1, x_2)&=\rho(f(x_0), f(x_1)) \ll \alpha\rho(x_0, x_1),\\ \rho(x_2, x_3)&=\rho(f(x_1), f(x_2)) \ll \alpha\rho(x_1, x_2)\\ &\ll \alpha^2\rho(x_0, f(x_0)),\end{aligned}$$

and, in general

$$\rho(x_n, x_{n+1}) \ll \alpha^n \rho(x_0, f(x_0)).$$

Hence, we have

$$\begin{aligned}\rho(x_n, x_{n+m}) &\ll \rho(x_n, x_{n+1})+ \cdots +\rho(x_{n+m-1}, x_{n+m})\\ &\ll (\alpha^n+\alpha^{n+1}+ \cdots +\alpha^{n+m-1})\rho(x_0, f(x_0))\\ &=\frac{\alpha^n-\alpha^{n+m}}{1-\alpha}\rho(x_0, f(x_0)).\end{aligned}$$

By the hypothesis $\alpha<1$, we have

$$\rho(x_n, x_{n+m}) \ll \frac{\alpha^n}{1-\alpha}\rho(x_0, f(x_0)).$$

Therefore $\{x_n\}$ is a Cauchy sequence. X is sequential complete, so $\{x_n\}$ has a limit x' in X. To prove $f(x')=x'$, consider the following inequality,

$$\begin{aligned}\rho(x', f(x')) &\ll \rho(x', x_n)+\rho(x_n, f(x'))\\ &=\rho(x', x_n)+\rho(f(x_{n-1}), f(x'))\\ &\ll \rho(x', x_n)+\alpha\rho(x', x_{n-1}).\end{aligned}$$

This shows $\rho(x', f(x'))=0$. Hence x' is a fixed element of $f(x)$. The element x' is uniquely determined. Suppose that x'' is also a fixed element of $f(x)$. Then we have

$$\begin{aligned}\rho(x', x'') &= \rho(f(x'), f(x'')) \\ &\ll \alpha\rho(x', x'').\end{aligned}$$

α is a positive number less than 1, so we have $\rho(x', x'')=0$.

As easily seen, we can not replace a positive number α into an element e such that $e\ll 1$.

Reference

[1] M. Antonovski, V. Boltjanski, and T. Sarymsakov: Topological semifields (in Russian). Tashkent (1960).