13. *Logics without Craig's Interpolation Property*

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It has been shown by Craig, Schütte and Gabbay that Craig's interpolation theorem holds for predicate logics LK, LJ, $LJ+(a)$, $LJ+(b)$ and $LJ+(a)+(b)$ (cf. [1]), where

(a) $\forall x \neg\neg A(x) \supset \neg\neg \forall x A(x)$, (b) $\neg P \vee \neg\neg P$.

In this paper, we shall show that there exist some superclassical predicate logics where Craig's interpolation theorem does not hold and that Craig's interpolation theorem does not hold for any intermediate propositional logic on slice $\mathcal{S}_n$ for $3 \leqq n < \omega$.

§ 1. **Superclassical predicate logics.** The definition of *predicate logics* is obtained by excluding the condition 1) $LJ \subset L \subset LK$ from Definition 1.1 in Ono [4]. By *a superclassical predicate logic* (*SCPL*), we mean a predicate logic including the predicate logic LK. We denot the set of formulas valid in any model whose domain has n elements by $L(n)$. We can easily see that $L(n)$ is a $SCPL$ and

$$LK \subsetneqq \bigcap_{n<\omega}(n) \subsetneqq \cdots \subsetneqq L(n) \subsetneqq \cdots \subsetneqq L(2) \subsetneqq L(1) \subsetneqq W.$$

Here W denotes the set of all formulas. We write $A_1, A_2, \cdots, A_n \to B_1, B_2, \cdots, B_m \in L$ if the formula $A_1 \wedge A_2 \wedge \cdots \wedge A_n \supset B_1 \vee B_2 \cdots \vee B_m \in L$.

Theorem 1.1. *Craig's interpolation theorem does not hold for* $L(n)$ $(n \geqq 2)$.

Proof. We define the formulas Q_n^i $(1 \leqq i \leqq n)$ as follows:

$$Q_n^i = \left(\bigwedge_{\substack{j=1 \\ j \neq i}}^{n} q_i(x_j) \right) \wedge \neg q_i(x_i).$$

Then, we have that $Q_n^1, Q_n^2, \cdots, Q_n^n, p(x_1), p(x_2), \cdots, p(x_n) \to p(y) \in L(n)$. Therefore, we have

$$Q_n^1, Q_n^2, \cdots, Q_n^n \to p(y), \neg p(x_1), \neg p(x_2), \cdots, \neg p(x_n) \in L(n).$$

In the above sequent, the antecedent and the succedent contain no predicate symbol in common. And we can show that *the antecedent* $\to \notin L(n)$ and $\to$ *the succedent* $\notin L(n)$. Q.E.D.

We don't know whether Craig's interpolation theorem holds for the $SCPL$ $\bigcap_{n<\omega} L(n)$.

§ 2. **Intermediate propositional logics.** In the proof of Theorem 1.1, we constructed the sequent $\Gamma \to \Theta$ such that Γ and Θ contain no predicate symbol in common and $\Gamma \to \Theta \in L(n)$ and $\Gamma \to \notin L(n)$ and $\to \Theta$

$\notin L(n)$. By the following theorem, it is known that we can not construct such a sequent in the intermediate propositional logics.

Theorem 2.1. *For any intermediate propositional logic L, if $A\supset B\in L$ and A and B contain no propositional variable in common, then $\neg A\in L$ or $B\in L$.*

Proof. Suppose that $\neg A\notin L$ and $B\notin L$. By $B\notin L$, there exist a pseudo-Boolean algebra P and an assignment f of P such that $L\subseteq L(P)$ and $f(B)\neq 1$. Here $L(P)$ denotes the set of formulas valid in P, and 1 and 0 are the largest and the smallest elements of P, respectively. Since $LJ\subseteq L$, $\neg A\notin LJ$. By Glivenko's theorem, $\neg A\notin LK$. So there exists an assignment g of P such that $g(\neg A)=0$ and $g(p)=1$ or $g(p)=0$ for any propositional variable p. Let h be an assignment of P such that $h(p)=g(p)$ if p appears in A and $h(p)=f(p)$ if p appears in B. Then, we have that $h(A\supset B)=g(A)\supset f(B)=1\supset f(B)=f(B)\neq 1$. Therefore, $A\supset B\notin L$. Q.E.D.

We define the formula P_n by $P_0=p_0$ and $P_{n+1}=[(p_{n+1}\supset P_n)\supset p_{n+1}]\supset p_{n+1}$. LP_n denotes $LJ+P_n$. S_n denotes the logic determined by the linear model with $n+1$ values. LP_n and S_n are the minimum and the maximum elements of $\mathcal{S}_n$, respectively.

Hosoi [2] gave a cut free Gentzen-type system for LP_n. Hence, we have conjectured that Craig's theorem holds for LP_n. But it is not true. Γ_n $(n\geqq 1)$ denotes the sequence

$$(p_n\supset p_{n-1})\supset p_{n-1},\cdots,(p_2\supset p_1)\supset p_1,\ \neg\neg p_1,\ p_{n-1}\supset p_n,\cdots,p_1\supset p_2.$$

Lemma 2.2. $\Gamma_n\to p_n\in LP_n$ $(n=1,2,3,\cdots)$.

Proof. We prove this by the induction on n. If $n=1$, then this sequent is $\neg\neg p_1\to p_1$ and $\neg\neg p_1\to p_1\in LP_1=LK$. By the inductive hypothesis, $\Gamma_{n-1}\to p_{n-1}\in LP_{n-1}$. We use the Gentzen-type system in [2].

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p_n\to p_n}{p_n,\Gamma_{n-1}\to p_n,p_{n-1}}\qquad[\Gamma_{n-1}\to p_{n-1}]}{\Gamma_{n-1}\to p_n,\ p_n\supset p_{n-1}}\qquad p_{n-1}\to p_{n-1}}{(p_n\supset p_{n-1})\supset p_{n-1},\Gamma_{n-1}\to p_n,p_{n-1}}\qquad p_n\to p_n}{p_{n-1}\supset p_n,(p_n\supset p_{n-1})\supset p_{n-1},\Gamma_{n-1}\to p_n,p_n}}{\Gamma_n\to p_n.}$$

Q.E.D.

R_n denotes the formula $[(p_n\supset p_{n-1})\supset p_{n-1}]\wedge(p_{n-1}\supset p_n)\supset p_n$.

Lemma 2.3. *There exists no formula P of one propositional variable p_{n-1} such that $\Gamma_{n-1}\to P\in S_n$ and $P\to R_n\in S_n$ $(3\leqq n<\omega)$.*

Proof. By Nishimura [3], there exists no formula of one variable other than Nishimura's basic formulas if we identify equivalent formulas in LJ. In Figure 1, we illustrate Fig. 1 in [3], but the order is the reverse of the original. We have (1), (2) and (3) if we consider the following models in Fig. 2, respectively.

(1) $\neg p_{n-1} \to R_n \notin S_2$.

(2) $\neg\neg p_{n-1} \to R_n \notin S_3$.

(3) $\Gamma_{n-1} \to p_{n-1} \notin S_n$.

By (1), any formula P such that $P \geqq \neg p_{n-1}$ in Fig. 1 can not be an interpolant. By (3), any formula P such that $P \leqq p_{n-1}$ in Fig. 1 can not be an interpolant. Hence, by (1), (2) and (3), we have this lemma.

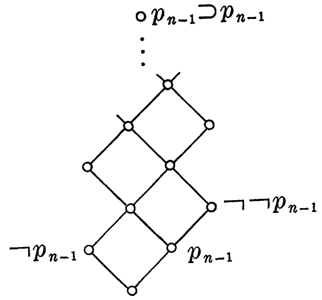

Fig. 1

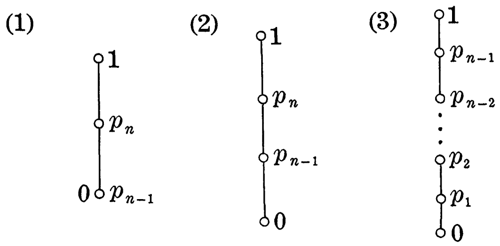

Fig. 2

Q.E.D.

Theorem 2.4. *Craig's interpolation theorem does not hold for any logic on $\mathcal{S}_n$ if $3 \leqq n < \omega$.*

Proof. Suppose $L \in \mathcal{S}_n$. Because $LP_n \subseteq L$, $\Gamma_{n-1} \to R_n \in L$ by Lemma 2.2. Γ_{n-1} and R_n have no common propositional variable other than p_{n-1}. As $L \subseteq S_n$, the above sequent has no interpolant by Lemma 2.3.

Q.E.D.

References

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