9. A Note on a Generalization of Prime Ideals

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1. Generalizing the concepts of prime ideals and primary ideals in rings, Murata *et al.* [1] introduced the notions of f-prime ideals and f-primary ideals in rings, and they obtained the uniqueness of f-primary decomposition of ideals under the assumptions that

(α) each ideal is f-related to itself,

(β) for any ideal A and any ideal B not contained in $r(A)$, $A:B \neq \emptyset$,

(γ) if S is an f-system with kernel S^*, and if for any ideal A, $S\cap A\neq\emptyset$, then $S^*\cap A\neq\emptyset$,

(δ) for any f-primary ideal Q, $Q:Q=R$.

Y. Kurata and S. Kurata [2] also discussed the isolated component and the isolated set of ideals having f-primary decompositions.

It is then natural to ask whether these assumptions are independent or there are some relations between them. The purpose of this note is to examine these questions. For all undefined notions see [1].

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2. Throughout this note R is an associative ring and assume that for each element $a\in R$ we associate an ideal $f(a)$ in R which is uniquely determined by a and satisfies the following conditions:

(I) $a\in f(a)$,

(II) $x\in f(a)+A \Rightarrow f(x)\leqq f(a)+A$ for any ideal A.

The principal ideal (a) generated by a is an example of the $f(a)$ and we call simply f *principal* if $f(a)=(a)$ for all $a\in R$. As is pointed out in [1], if f is principal, then the assumptions (α) to (δ) are all satisfied. However we can point out that there is a ring R and a non-principal f for which all of (α) to (δ) are satisfied.

Let R be as in [1, Example 2.3]. Then the ideals in R are R, M, K and (0). We define $f(a)=(a, M)$ for each $a\in R$. This f is not principal and 0 is f-related to each ideal, i.e. (α) is satisfied.

Ideals R and M are f-prime, K and (0) are not f-prime and so we have

$$r(R)=R,\ r(K)=R,\ r(M)=M \quad\text{and}\quad r((0))=M,$$

from which we see that K is f-primary and (0) is not f-primary. To show that (β) is satisfied, we may note that each of $M:R$, $M:K$, $(0):R$

and $(0):K$ contains 0.

Let $S(S^*)$ be any f-system with kernel S^* and A an ideal in R. We want to show that $S\cap A\neq\emptyset$ implies $S^*\cap A\neq\emptyset$. In case $A=R$ or $A=M$ this is clear. Let $A=K$, $S\cap K\neq\emptyset$ and take $s\in S\cap K$. Then $f(s)\cap S^*\neq\emptyset$ by definition. For each $s'\in f(s)\cap S^*$ we can find $a\in R$ such that $s'as'\in S^*$. By the definition of the multiplication of R, $s'as'\in K$ and thus we see that $S^*\cap K\neq\emptyset$. Likewise $S\cap(0)\neq\emptyset$ implies $S^*\cap(0)\neq\emptyset$. This shows that (γ) is satisfied.

Finally to show that (δ) is satisfied, we have to check that $R:R=K:K=M:M=R$, but this is easy and we will omit it.

3. On the other hand, there is an example of R and f which does not satisfy any of (α) to (δ). Let R be a field. Then we have either

$$f(0)=(0) \quad\text{and}\quad f(a)=R \quad \text{for } a\ (\neq 0)\in R$$

or

$$f(a)=R \qquad \text{for all } a\in R.$$

The former is principal and satisfies all of (α) to (δ). However the latter does not satisfy any of (α) to (δ). Each non-zero element of $f(0)$ is not a zero-divisor mod (0). So 0 is not f-related to (0) and hence (α) is not satisfied.

Since $C((0))=R^*$ is multiplicatively closed, (0) is f-prime and $r((0))=(0)$. The ideal R is not contained in $r((0))$ and certainly $(0):R$ is empty. Thus (β) is not satisfied.

$R(R^*)$ is also an f-system and $R\cap(0)\neq\emptyset$ and $R^*\cap(0)=\emptyset$, from which we see that (γ) is not satisfied.

To show that (δ) is not satisfied, we may note that (0) is f-primary and $(0):(0)$ is empty.

As is pointed out in [1] whenever R has no right zero-divisors, (α) is equivalent to the fact that f is principal and hence in this case

$$(\alpha) \text{ implies } (\beta),\ (\gamma), \quad\text{and}\quad (\delta).$$

This however can be strengthened at once when R is a field from the consideration above.

Proposition 1. *For a field R, each of (α) to (δ) is equivalent.*

4. Returning to the general case, we now assume that R has an identity 1. For an ideal A in R, we set

$$S_A^*=\{1-a : a\in A\}$$

and

$$S_A=\{a\in R : f(a)\cap S_A^*\neq\emptyset\}.$$

Then S_A^* is a multiplicatively closed subset of R and S_A becomes an f-system with kernel S_A^*. As is easily seen, $a\in S_A$ if and only if $f(a)+A=R$, and $S_A^*\cap A\neq\emptyset$ if and only if $A=R$, or equivalently, $0\in S_A^*$.

If there exists an ideal $A\neq R$ such that $S_A\cap A\neq\emptyset$, then (γ) is not satisfied for the f-system $S_A(S_A^*)$. Consequently we have

Proposition 2. *If R has an identity and if (γ) is satisfied, then*

$S_A \cap A$ must be empty for all proper ideals A in R.

For example, if there exists a maximal ideal A and an element $a \in A$ such that $f(a) \nleqq A$, then (γ) is never satisfied, and [1, Example 3.1] is a special case of this.

We also have

Proposition 3. *If R has an identity 1, then (α) implies that $S_A \cap A$ is empty for all proper ideals A in R.*

Proof. Suppose that (α) is satisfied and that $S_A \cap A \neq \emptyset$ for some ideal A $(\neq R)$. Take $a \in S_A \cap A$. Then $f(a)+A=R$ and so we can find $a' \in f(a)$ and $a'' \in A$ such that $1=a'+a''$. Since a is f-related to A, there exists $c \notin A$ such that $a'c \in A$. Hence we have $c=a'c+a''c \in A$, a contradiction. Thus $S_A \cap A$ must be empty for all proper ideals A in R.

The following proposition shows that the converses of these propositions are also true under an additional assumption on R.

Proposition 4. *Suppose that R has an identity and that the intersection of all the maximal ideals in R is zero. Then the following conditions are equivalent:*

(1) *f is principal.*

(2) (α).

(3) (γ).

(4) *$S_A \cap A=\emptyset$ for all maximal ideals A in R.*

Proof. It is enough to show that (4) implies (1). So assume (4), and let A be any maximal ideal and take any $a \in A$. Then by the assumption $a \notin S_A$ and so $f(a)+A \lneqq R$. It follows that $A=f(a)+A$ and thus we have $f(a) \leqq A$. From this we claim that f is principal. If $f(0) \neq (0)$, then we can find a non-zero element $a \in f(0)$ and a maximal ideal A such that $a \notin A$. Since $0 \in A$, $f(0) \leqq A$ and hence a must be in A, a contradiction. Thus we have $f(0)=(0)$ and f is principal.

5. Assume again that R has an identity 1. In this case, since $f(1)=R$, (δ) means that for any f-primary ideal Q in R $f(b) \leqq Q$ for all $b \in Q$, or equivalently, for any ideal A which is represented as an intersection of f-primary ideals, $f(a) \leqq A$ for all $a \in A$. From this we have

Proposition 5. *If R has an identity and if any ideal in R can be represented as an intersection of f-primary ideals, then (δ) is equivalent to saying that f is principal. So we have the implications*

$$(\delta) \Rightarrow (\alpha),\ (\beta) \quad \textit{and} \quad (\gamma).$$

References

[1] K. Murata, Y. Kurata, and H. Marubayashi: A generalization of prime ideals in rings. Osaka J. Math., **6**, 291–301 (1969).

[2] Y. Kurata and S. Kurata: A generalization of prime ideals in rings. II. Proc. Japan Acad., **45**, 75–78 (1969).