BERNSTEIN THEOREMS FOR HARMONIC MORPHISMS

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0. INTRODUCTION

Let $\phi : M \to N$ be a continuous mapping between connected smooth Riemannian manifolds. Then ϕ is called a *harmonic morphism* if for every function f harmonic on an open set $V \subset N$, the composition $f \circ \phi$ is harmonic on $\phi^{-1}(V) \subset M$. It follows by choosing smooth harmonic local coordinates on N [11], that any harmonic morphism is necessarily smooth.

The harmonic morphisms are precisely the harmonic maps which are *horizontally weakly conformal* (see [10], [14]). For a map $\phi : \mathbf{R}^3 \to \mathbf{C}$ this is equivalent to ϕ satisfying the equations

$$\sum_{i=1}^{3} \frac{\partial^2 \phi}{\partial x_i^2} = 0 \qquad (0.1)$$

$$\sum_{i=1}^{3} \left(\frac{\partial \phi}{\partial x_i}\right)^2 = 0 \ . \qquad (0.2)$$

Harmonic morphisms are a subject of considerable interest. Their history goes back to Jacobi [15] in 1847, who considered the problem of finding compex-valued (harmonic) functions ϕ on $\mathbf{R}^3$ satisfying (0.1) and (0.2) above. More recently they have been studied in the context of stochastic processes, where they are found to be the *Brownian path preserving mappings* (see [5]). In fact our main Theorem (Theorem 1) solves a problem first posed by Bernard, Cambell and Davie in [5].

The study of harmonic morphisms from domains in $\mathbf{R}^m$ to a Riemann surface has striking analogies with the study of minimal surfaces in $\mathbf{R}^m$. For example, the fibres of such a harmonic morphism are minimal and the associated Gauss map (see [4] for definition) obeys a holomorphicity condition similar to that for a minimal immersion (see[6]). In fact the analogies are so striking that on expects to find corresponding results more generally. This turns out to be true for the well known Bernstein Theorems for complete minimal surfaces in $\mathbf{R}^3$ (see [16]). We show that the only non-constant harmonic morphism ϕ defined on the whole of $\mathbf{R}^3$, taking values in a Riemann surface N, is the simplest possible, namely an orthogonal projection $\mathbf{R}^3 \to \mathbf{R}^2$ followed by a weakly conformal mapping $\mathbf{R}^2 \to N$.

Similarly for S^3, where the only non-constant harmonic morphism $\phi : S^3 \to N$ taking values in a Riemann surface N is the Hopf map $S^3 \to S^2$ follwed by a weakly conformal map $S^2 \to N$.

Below we state our main Theorems precisely and outline the proofs. The reader is referred to [4] for a detailed exposition. This is joint work with J.C. Wood, and the authors would like to express their thanks to J. Eells and T. Ransford for helpful comments.

1. MAIN THEOREMS

Theorem 1 *Let $\phi : \mathbf{R}^3 \to N$ be a harmonic morphism. Either (i) ϕ is constant, (ii) $N = \mathbf{R}$ or S^1 and ϕ is a harmonic map, (iii) $\dim N = 2$ and ϕ is the composition ; $\phi = \sigma \circ \pi$ where $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ is orthogonal projection and $\sigma : \mathbf{R}^2 \to N$ is weakly conformal or (iv) $N = \mathbf{R}^3$ and ϕ is an affine transformation.*

Remarks 1) By orthogonal projection we mean that after a suitable choice of axes is made in $\mathbf{R}^3$, then π can be considered as the projection $\pi(x_1, x_2, x_3) = (x_1, x_2)$.
2) An affine transformation $\phi : \mathbf{R}^3 \to \mathbf{R}^3$ has the form $\phi(x) = \lambda Ax + b$, where $A \in O(\mathbf{R}^3)$ is orthogonal, $b \in \mathbf{R}^3$ is a fixed translation and $\lambda \in (0, \infty)$. These are precisely the homothetic transformations.

Corollary *Let $\phi : \mathbf{R}^3 \to N$ be a non-constant harmonic morphism. Then $N = \mathbf{R}, S^1, \mathbf{R}^3$ or N equivalent to the plane $\mathbf{C}$, the punctured plane $\mathbf{C} \setminus \{\text{point}\}$, the torus T^2 or the sphere S^2.*

Proof of Corollary By composing with complex conjugation if necessary we may assume that $\sigma : \mathbf{C} \to N$ is holomorphic. But then it lifts to a map into the universal cover of N. This must be $\mathbf{C}$ or S^2 (the disc being excluded by Liouville's Theorem - since otherwise $\sigma \circ \pi$ would be a bounded harmonic function on $\mathbf{R}^3$).

Theorem 2 *Let $\phi : S^3 \to N$ be a non-constant harmonic morphism. Then either (i) $\dim N = 2$ and ϕ is the composition $\phi = \sigma \circ \pi$, where $\pi : S^3 \to S^2$ is a Hopf map and $\sigma : S^2 \to N$ is a weakly conformal map, or (ii) $N = cS^3$, $c \in (0, \infty)$, and ϕ is an isometry followed by a scaling.*

Remark By a Hopf map, we mean that after a suitable choice of axes in $\mathbf{R}^4$, we may assume π is the Hopf map $\pi(z_1, z_2) = -z_1/z_2 \in \mathbf{C} \cup \{\infty\} = S^2$, for $(z_1, z_2) \in S^3$.

Corollary *If ϕ is a non-constant harmonic morphism on S^3 then $N = cS^3$, $c \in (0, \infty)$ or N is conformally equivalent to S^2 and ϕ is surjective.*

Proof of Corollary Similar to the previous Corollary. Surjectivity follows since any non-constant conformal map from S^2 to a Riemann surface must be surjective.

2. SKETCH PROOF OF THEOREM 1

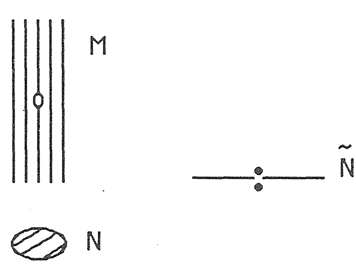
Let $\phi : \mathbf{R}^3 \to N$ be a non-constant harmonic morphism onto a Riemann surface N. Without loss of generality we can assume ϕ is surjective. Let K denote the critical set of ϕ, $K = \{x \in \mathbf{R}^3 \mid d\phi_x = 0\}$. The fibres of $\phi|_{\mathbf{R}^3\backslash K}$ are minimal [3], and so are straight lines in $\mathbf{R}^3$. We define a Gauss map $\Psi : \mathbf{R}^3\backslash K \to S^2$, which at each point $x \in \mathbf{R}^3\backslash K$ gives the direction $\Psi(x) \in S^2$ of the fibre through x.

<u>Step 1</u> *the Gauss map* $\Psi : \mathbf{R}^3\backslash K \to S^2$ *extends across* K *to a smooth harmonic map, which we also denote by* $\Psi : \mathbf{R}^3 \to S^2$, *such that the fibre of* ϕ *through a point* $x \in K$ *is the line through* x *with direction determined by* $\Psi(x)$.

That Ψ extends continuously across K is a result of Bernard, Cambell and Davie [5]. Furthermore Ψ turns out to be a harmonic morphism and is hence smooth. From this we see that the fibre components of ϕ define a smooth foliation on $\mathbf{R}^3$ by complete straight lines.

<u>Step 2</u> We define $\tilde{N}$, the space of fibre components of ϕ. It is simply the leaf space of the foliation associated to ϕ and we give $\tilde{N}$ the quotient topology induced from $\mathbf{R}^3$. Then $\tilde{N}$ is connected and Hausdorff.

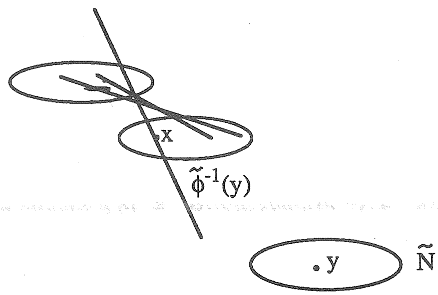
For more general domains, $\tilde{N}$ may not be Hausdorff. For example, let $\phi : \mathbf{R}^3\backslash\{0\} \to \mathbf{R}^2$ be the projection mapping $\phi(x,y,z) = (x,y)$.



Then the fibre over the origin in $\mathbf{R}^2$ consists of two disjoint connected components which correspond to two distinct points of $\tilde{N}$. No two neighbourhoods of these points in $\tilde{N}$ can be disjoint.

The map ϕ now factors as $\phi = \zeta \circ \tilde{\phi}$, where $\tilde{\phi} : \mathbf{R}^3 \to N$ is the natural projection and $\zeta(\tilde{\phi}(x)) = \phi(x)$. Then $\tilde{N}$ can be given the structure of a smooth Riemann surface with respect to which $\tilde{\phi}$ is a submersive harmonic morphism.

We do this by using local slices in $\mathbb{R}^3$ as coordinate neighbouhoods about a point $y \in \tilde{N}$. The transition functions are obtained by passing from one slice to another along the fibres. The smoothness of this structure follows from the smoothness of Ψ.



The conformal structure on $\tilde{N}$ is determined by requiring $\tilde{\phi}$ to be horizontally conformal. That is, at $y \in \tilde{N}$, choose some x belonging to the fibre determined by y. Then we may define a conformal structure at y by requiring $d\tilde{\phi}|_{H_xM} : H_xM \to T_y\tilde{N}$ be conformal. That this is independant of x follows from the horizontal conformality of ϕ. With respect to this complex structure, $\zeta : \tilde{N} \to N$ is a weakly conformal mapping and $x \in \mathbb{R}^3$ is a critical point of ϕ if and only if $\phi(x)$ is a branch point of ζ.

By construction, $\tilde{\phi} : \mathbb{R}^3 \to \tilde{N}$ is a fibre bundle. By the homotopy exact sequence of such [21], all the homotopy groups $\pi_i(\tilde{N})$ vanish. Thus by the Riemann Mapping Theorem (see e.g. [7]), $\tilde{N}$ is conformally equivalent to either the complex plane $\mathbb{C}$, or the disc D. However, if $\tilde{N}$ is conformally eqivalent to D, then ϕ is a bounded harmonic function. By Liouville's Theorem any such must be constant. Hence $\tilde{N}$ *is conformally equivalent to* $\mathbb{C}$.

<u>Step 3</u> We study the submersive harmonic morphism $\tilde{\phi} : \mathbb{R}^3 \to \tilde{N}$. We may associate to $\tilde{\phi}$ a meromorphic mapping $\xi : \tilde{N} \to (\mathbb{C} \cup \{\infty\})^3$ such that, writing $\xi = (\xi_1, \xi_2, \xi_3)$, $\sum \xi_i^2 = 0$. In fact the fibre of ϕ over $z \in \tilde{N}$ is determined by the equation

$$\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 = 1 \quad . \qquad (*)$$

The meromorphic mapping ξ may be written uniquely as

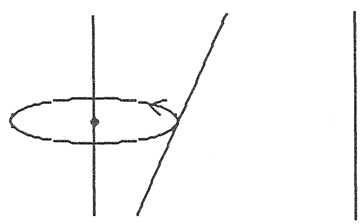
$$\xi = \frac{1}{2h}\left(1 - g^2, \; i(1 + g^2), \; -2g\right),$$

where h is a holomorphic and g a meromorphic function on $\tilde{N}$ (see [16]). Furthermore g(z) represents the direction of the fibre over z and h(z) represents the postion of the fibre (in charts given by stereographic projection). In particular g *is constant if and only if the Gauss map* Ψ *of* ϕ *is constant*. We study the properties of h and g which follows from the fact that ϕ is defined on the <u>whole</u> of $\mathbf{R}^3$.

In fact g *is injective*. For suppose not, then there exist $z_1, z_2 \varepsilon N$, $z_1 \neq z_2$, with $g(z_1) = g(z_2)$. Furthermore we may assume z_1 is not a branch point of g. We choose a complex coordinate z on $\tilde{N}$ such that z_1 is the origin.

<u>Claim</u>: As we circle the origin in $\tilde{N}$, some fibre must hit the fibre at z_2, contradicting the fact that the fibres of ϕ cannot intersect.

This claim is intuitively obvious if we think of the fibre at a point z as a searchlight beam emanating from a point $c(z) \varepsilon \mathbf{R}^3$ (which corresponds to h(z)), which is, to a first approximation $(az, 0) \varepsilon \mathbf{R}^3$, where a is a non-zero constant (see [4]).



As z rotates, this point rotates once. Because the searchlight beam has direction whose horizontal component is governed by g(z) and so rotates in the <u>same sense</u>, there must be a point where the searchlight beam hits the fibre at z_2.

By the little Picard Theorem (see e.g. [6]) and injectivity, it follows that g(C) is biholomorphically equivalent to $S^2\backslash\{\text{point}\}$. By a rotation in $\mathbf{R}^3$ we may choose this point to be ∞, so g has no poles and $g : \tilde{N} \to \mathbf{C}$ *is a holomorphic diffeomorphism*. Thus we may choose a global coordinate z on $\tilde{N}$ such that $g(z) = z$.

We now consider fibres over points of the circle $|z| = 1$ in $\tilde{N} = \mathbf{C}$. Under inverse stereographic projection, the circle $C : |z| = 1$ is mapped to the equator in S^2. Fibres of $\tilde{\phi}$ over points of C are all parallel to the (x_1,x_2)-plane containing the equator. Consider the *height function* h_C, defined on C, which at each point $z \varepsilon C$, is defined to be the height of the fibre $\tilde{\phi}^{-1}(z)$ over the (x_1,x_2)-plane. By continuity there is some $z_0 \varepsilon C$ which is a maximum (or minimum) for h_C. Then, close to z_0, there are points z_1, z_2, $z_1 \neq z_2$ with $h_C(z_1) = h_C(z_2)$.

Since g is injective the fibres over z_1, z_2 are not parallel and hence must intersect.

Step4 From the arguments of Step 3 we conclude that g *is constant*. Without loss of generality we may assume $g(z) = 0$, for all $z \in N$. Then all fibres are parallel to the x_3-axis. We may then identify N with the (x_1, x_2)-plane with its usual conformal structure and $\tilde{\phi}$ is just the projection $\tilde{\phi}(x_1, x_2, x_3) = (x_1, x_2)$. Since $\phi = \zeta \circ \tilde{\phi}$ where $\zeta : \tilde{N} \to N$ is weakly conformal, Theorem 1 is proven.

Similar arguments apply to the S^3 case to establish Theorem 2.

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