Elhamdadi, M. and Hajij, M.
Osaka J. Math.
54 (2017), 363–381

PRETZEL KNOTS AND q-SERIES

Mohamed ELHAMDADI and Mustafa HAJIJ

(Received December 4, 2015, revised May 6, 2016)

Abstract

The tail of the colored Jones polynomial of an alternating link is a q-series invariant whose first n terms coincide with the first n terms of the n-th colored Jones polynomial. Recently, it has been shown that the tail of the colored Jones polynomial of torus knots give rise to Ramanujan type identities. In this paper, we study q-series identities coming from the colored Jones polynomial of pretzel knots. We prove a false theta function identity that goes back to Ramanujan and we give a natural generalization of this identity using the tail of the colored Jones polynomial of Pretzel knots. Furthermore, we compute the tail for an infinite family of Pretzel knots and relate it to false theta function-type identities.

Contents

1. Introduction

The discovery of the Jones polynomial using Von Neumann algebras [14, 13] and its generalizations [8] and [23] lead to quantum invariants of knots and 3-manifolds. The Kauffman bracket polynomial [16] is the simplest interpretation of the Jones polynomial using knot diagrams. Reshetikhin and Turaev [24] gave the first rigorous construction of quantum invariants as linear sums of quantum invariants of framed links. Soon after, various approaches of constructing quantum invariants were developed using different methods such as using surgery along links [5, 18, 26] and simplicial complexes [25].

The colored Jones polynomial $J_{n,L}(q)$ of a link L can be understood as a sequence of polynomials with integer coefficients that take values in $\mathbb{Z}[q, q^{-1}]$. The label n stands for the coloring. The polynomial $J_{2,L}(q)$ is the original Jones polynomial. Recently, there has been a growing interests in the coefficient of the colored Jones polynomial. Dasbach and Lin [6] used the definition of the colored Jones polynomial coming from Kauffman bracket skein

2010 Mathematics Subject Classification. Primary 57M27; Secondary 11P84.

theory to show that for an alternating link L the absolute value of the first and the last three leading coefficients of $J_{n,L}(q)$ are independent of the color n, for large values of n. As a consequence, they obtained lower and upper bounds for the volume of the knot complement for an alternating prime non-torus knot K in terms of the leading two and last two coefficients of $J_{2,K}(q)$ extending their previous result from [7]. In [6] it was conjectured that the first n coefficients of $J_{n,L}(q)$ agree with the first n coefficients of $J_{n+1,L}(q)$ for any alternating link L. This gives rise to a q-power series called the tail of the colored Jones polynomial of the alternating link L with many interesting properties. Using skein theory, Armond gave a proof in [2] for the existence of the tail of the colored Jones polynomial of adequate links, hence alternating links and also for closures of positive braids in [3]. Garoufalidis and Lê [9] used R-matrices to prove the existence of the tail of the colored Jones polynomial of alternating links and proved that higher order stabilization also occur. An alternative proof for the stability was also given in [10]. In [12], the second author investigated certain skein element in the relative Kauffman bracket skein module of the disk with some marked points in order to compute the head and the tail of the colored Jones polynomial obtaining a simple q-series for the tail of the knot 8_5, the first knot in the knot table that is not directly obtained from the work in [4]. This investigation was generalized to the study of tail of quantum spin networks in [11].

One of the earliest connection between the colored Jones polynomial and Ramanujan type q-series was made in [15] in which the author investigated the asymptotic behaviors of the colored Jones polynomials of torus knots. However, the point of view in [15] is different from the point of view of [12, 11] that we shall adopt here. This point of view allows us to prove more q-series identities in a structured manner. Among many interesting properties that the tail of the colored Jones polynomial enjoys as q-series is that it is equal to theta functions or false theta functions for many knots with small crossing numbers. For instance all knots in the knots table up to 8_4, the tail of their colored Jones polynomial are Ramanujan theta, false theta functions or a product of these functions as demonstrated in [4]. This does not seem to be the case of knot 8_5 whose tail is computed in [12]. More interestingly, the study of the tail has been used to prove Andrews-Gordon identities for the two variable Ramanujan theta function in [4] and a corresponding identities for the false theta function in [11]. These two families of q-series identities were obtained from investigating $(2, p)$-torus knots. For q-series techniques proving these identities refer to [17].

In this paper we show that similar observations hold for other natural family of knots, namely Pretzel knots. In particular, we show that pretzel knots give rise to a natural family of q-series identities. The paper is organized as follows. In section 2 we review the basics of skein theory, some number theory relevant to our work, and some review of the colored Jones polynomial. In section 3 we list the main results of this paper. Section 4 is devoted to Ramanujan type identities that were recovered in the literature using the tail of the colored Jones polynomial and we show how our contribution here fits in this literature. In section 5 we give an explicit formula for the tail of colored Jones polynomial of the Pretzel knots $P(2u + 1, 2, 2k + 1)$ where $k, u \geq 1$. In section 6 we use two skein theoretic techniques to compute the tail of the colored Jones polynomial of a certain family of pretzel knots and we show that these computations give rise to a Ramanujan type identities.

2. Background

Let $\tilde{\mathbb{Z}}[A, A^{-1}]$ denotes the set of rational functions $\frac{P}{Q}$ where $P, Q \in \mathbb{Z}[A, A^{-1}]$. Let M be an orientable 3-manifold. We will denote the Kauffman bracket skein module of the 3-manifold M and the ring $\tilde{\mathbb{Z}}[A, A^{-1}]$ by $\mathcal{S}(M)$. When $M = I \times F$ where F is a surface we will denote the Kauffman bracket of M by $\mathcal{S}(F)$. We will assume that the reader is familiar with linear skein theory associated with the Kauffman bracket skein module [22, 18] and quantum spin networks [19]. In particular we assume that the reader is familiar with the graphical definition of the Jones-Wenzl projector, its properties and its connection with trivalent graphs [19]. We will follow the notations and definitions of [18, 19].

Recall that, for any integers l, i such that $0 \leq i \leq l$, the quantum binomial coefficients are defined by :

$$\begin{bmatrix} l \\ i \end{bmatrix}_q = \frac{(q;q)_l}{(q;q)_i (q;q)_{l-i}}.$$

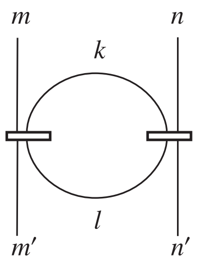
where $(a;q)_n$ is q-Pochhammer symbol which is defined as

$$(a;q)_n = \prod_{j=0}^{n-1} (1 - aq^j).$$

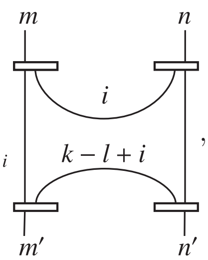
We will need the following identity [12].

Theorem 2.1. *(The bubble expansion formula) Let $m, n, m', n' \geq 0$, and $k \geq l$; $k, l \geq 1$. Then*

(2.1)



$$= \sum_{i=0}^{\min(m,n,l)} \begin{bmatrix} m & n \\ k & l \end{bmatrix}_i$$



,

where

$$\begin{bmatrix} m & n \\ k & l \end{bmatrix}_i := (-A^2)^{i(i-l)} \frac{\prod_{j=0}^{l-i-1} \Delta_{k-j-1} \prod_{s=0}^{i-1} \Delta_{n-s-1} \Delta_{m-s-1}}{\prod_{t=0}^{l-1} \Delta_{n+k-t-1} \Delta_{m+k-t-1}} \begin{bmatrix} l \\ i \end{bmatrix}_{A^4} \prod_{j=0}^{l-i-1} \Delta_{m+n+k-i-j}.$$

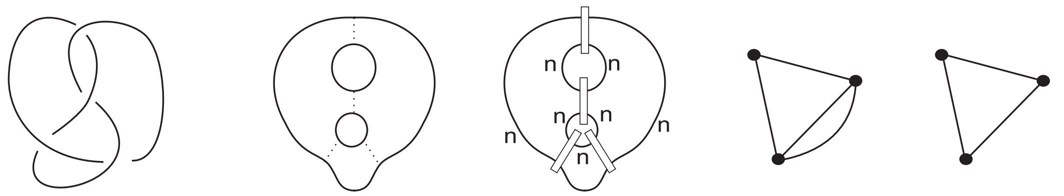
We will denote the skein element on the right handside of (2.1) by $\mathcal{B}^{m,n}_{m',n'}(k, l)$ and we will call it the *bubble skein element*.

2.1. The Tail of The Colored Jones Polynomial. We briefly review the basics of the head and the tail of the colored Jones polynomial. For more details see [12, 11].

Let L be a framed link in S^3. Decorate every component of L, according to its framing, by the n^{th} Jones-Wenzl idempotent and consider the evaluation of the decorated framed link as an element of $\mathcal{S}(S^3)$. Up to a power of $\pm A$, that depends on the framing of L, the value of this element is defined to be the n^{th} (unreduced) colored Jones polynomial $\tilde{J}_{n,L}(A)$. Recovering the reduced Jones polynomial is a matter of changing a variable and dividing by Δ_n. Namely,

$$J_{n+1,L}(q) = \frac{\tilde{J}_{n,L}(A)}{\Delta_n}\bigg|_{A=q^{1/4}} \tag{2.2}$$

If $P_1(q)$ and $P_2(q)$ are elements in $\mathbb{Z}[q^{-1}][[q]]$, we write $P_1(q) \doteq_n P_2(q)$ if their first n coefficients agree up to a sign. It was proven in [4] that the coefficients of the colored Jones polynomial of an alternating link L stabilize in the following sense: For every $n \geq 2$, we have $J_{n+1,L}(q) \doteq_n J_{n,L}(q)$. This motivated the authors of [4] to define the tail of the colored Jones polynomial of a link. More precisely, define the q-series series associated with the colored Jones polynomial of an alternating link L whose n^{th} coefficient is the n^{th} coefficient of $J_{n,L}(q)$. Stated differently, the tail of the colored Jones polynomial of a link L is defined to be a series $T_L(q)$, that satisfies $T_L(q) \doteq_n J_{n,L}(q)$ for all $n \geq 1$. In the same way, the head of the colored Jones polynomial of a link L is defined to be the tail of $J_{n,L}(q^{-1})$. The head and the tail of the colored Jones polynomial of an alternating link L can be recovered from a sequence of skein elements in $\mathcal{S}(S^2)$. The study of this sequence of skein elements is relatively easier than the study of the entire colored Jones polynomial. For more details see [4] and [11]. We recall this fact here. Let L be a link in S^3 and D be an alternating knot diagram of L. Consider the all B-smoothings state of D, the state obtained by replacing each crossing by a B-smoothing. We record the places of this smoothing by a dashed line as can be seen in Figure 1 for an example. Write $S_B^{(n)}(D)$ for the all B-smoothing state and consider the skein element obtained from $S_B(D)$ by decorating each circle in $S_B(D)$ with the n^{th} Jones-Wenzl idempotent and replacing each dashed line in $S_B(D)$ with the $(2n)^{th}$ Jones-Wenzl idempotent. See Figure 1.



D $\quad S_B(D)$ $\quad S_B^{(n)}(D)$ $\quad \mathbb{G}_B$ $\quad \mathbb{G}'_B$

Fig. 1. A link diagram D, its all-B state $S_B(D)$, the skein element $S_B^{(n)}(D)$, the B-graph $\mathbb{G}_B(D)$, and the reduced all B-graph $\mathbb{G}'_B(D)$.

The following theorem from [4] relates the tail of the colored Jones polynomial of an alternating link D to the skein element $S_B^{(n)}(D)$.

Theorem 2.2. *Let L be an alternating link in S^3 and let D be an alternating diagram of L. Then*

$$\tilde{J}_{n,L}(A) \doteq_{4(n+1)} S_B^{(n)}(D)$$

This theorem states basically that the study of the tail of the colored Jones polynomial of the alternating knot D can be reduced to the study of the tail of the sequence of skein elements $S_B^{(n)}(D)$. This theorem also implies that the tail of the colored Jones polynomial depends on the so called the *reduced B-graph* of the diagram D. The B-graph of the diagram

D, denoted $\mathbb{G}_B(D)$ is the graph whose vertices are the circles of $S_B(D)$ and whose edges are the dashed lines. The reduced B-graph of D, denoted by $\mathbb{G}'_B(D)$, is obtained from $\mathbb{G}_B(D)$ by replacing parallel edges by a single edge. See the most right two drawings in Figure 1.

Remark 2.3. Since the colored Jones polynomial of a diagram D depends only on its reduced B-graph, we will sometimes use the term *the tail a graph* G to refer to the tail of colored Jones polynomial of an alternating knot diagram D such that $\mathbb{G}'_B(D) = G$. Conversely, Given a planar graph G, we can obtain an alternating knot diagram D such that $\mathbb{G}'_B(D) = G$ by replacing every edge in G by a crossing as illustrated in Figure 2. For this reason, if G is a planar graph then the tail of G will be denoted by T_G. Furthermore, the notation $S_B^{(n)}(G)$ will refer to the skein element obtained from the reduced graph G by replacing each vertex with a circle and each edge with the $2n^{th}$ Jones-Wenzl projector.

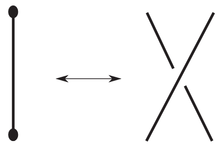


Fig. 2. Obtaining an alternating knot from a graph.

Remark 2.4. In general the computations of the tail of the colored Jones polynomial is done for the reduced case. In order to use Theorem 2.2 one needs to do a change of variable and normalize by Δ as can be seen from the relation (2.2).

The tail of the colored Jones polynomial has been computed for all knots in the knot table up to the knot 8_4 by Armond and Dasbach in [4]. In [11], the second author gave a formula for 8_5.

3. Main Results

In this section we list the main results of the paper. Let $a_1, \dots, a_n$ be positive integers. Denote by $P(a_1, \dots, a_n)$ the pretzel knot with n crossing regions given in Figure 3.

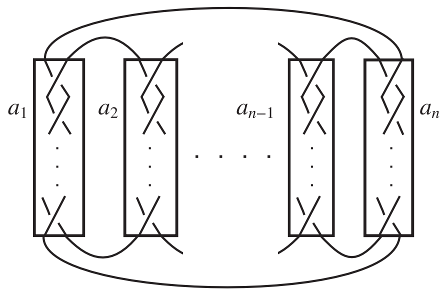


Fig. 3. Pretzel knot $P(a_1, \dots, a_n)$

In the following theorem, we give a formula for the tail of the colored Jones polynomial of the pretzel knot $P(2k+1, 2, 2u+1)$ for $u, k \geq 1$.

Theorem 3.1. *The tail of the pretzel knot* $P(2k+1, 2, 2u+1)$ *is given by*

$$T_{P(2k+1,2,2u+1)}(q) = (q;q)_\infty^2 \sum_{l_1=0}^{\infty} ... \sum_{l_k=0}^{\infty} \sum_{p_1=0}^{\infty} ... \sum_{p_u=0}^{\infty} g(q;l_1,...,l_k)g(q;p_1,...,p_u)(q;q)_{i+j}$$

where

$$g(q;l_1,...,l_k) = \frac{q^{\sum\limits_{j=1}^{k}(i_j(i_j+1))}}{(q;q)_{l_k}^2 \prod\limits_{j=1}^{k-1}(q;q)_{l_j}}$$

with $i_j = \sum\limits_{s=j}^{k} l_s$.

This formula generalizes the one of the tail of colored Jones polynomial of the knot 8_5 given in [12]. Furthermore, we give a formula for the tail of the colored Jones polynomial of the pretzel knot $P(2,\ldots,2)$ with $k+1$ crossing regions.

Proposition 3.2. *Let* $k \geq 1$ *and let* P_k *denotes* $P(2,\ldots,2)$ *with* $k+1$ *crossing regions. Then*

$$T_{P_k}(q) = (q;q)_\infty^k \sum_{i=0}^{\infty} \frac{q^i}{(q;q)_i^k}.$$

We use skein theoretic techniques to give another method to compute $T_{P_k}(q)$ and we obtain the following identity.

Corollary 3.3. *For* $k \geq 1$ *we have*

$$(q;q)_\infty \sum_{i=0}^{\infty} \frac{q^i}{(q;q)_i^{k+1}} = \sum_{i_1=0}^{\infty} ... \sum_{i_k=0}^{\infty} \frac{q^{\sum_{j=1}^k i_j + i_j^2 + \sum_{s=2}^k \sum_{j=s}^k i_{s-1} i_j}}{\prod_{j=1}^k (q;q)_{i_j} (q;q)_{\sum_{s=1}^j i_s}}.$$

This gives a natural generalization of the following well-known false theta function identity (by letting $a = 1$ in Entry 6.7.1 on page 169 of [1] or see page 200 in [27]):

$$(q;q)_\infty^2 \sum_{i=0}^{\infty} \frac{q^i}{(q;q)_i^2} = (q;q)_\infty \sum_{i=0}^{\infty} \frac{q^{i^2+i}}{(q;q)_i^2}.$$

4. Alternating Knots and Rogers-Ramanujan Type Identities

In this section, we review the Rogers-Ramanujan type identities that were recovered in the literature using techniques related to the tail of the colored Jones polynomial of alternating links. Furthermore, we show the false theta function type identities that we recover in this paper from Pretzel knots.

The general two variable Ramanujan false theta function is given by (e.g. [1]):

$$\Psi(a,b) = \sum_{i=0}^{\infty} a^{\frac{i(i+1)}{2}} b^{\frac{i(i-1)}{2}} - \sum_{i=1}^{\infty} a^{\frac{i(i-1)}{2}} b^{\frac{i(i+1)}{2}}. \tag{4.1}$$

When $a = q^3$ and $b = q$, we obtain the following well-known identities:

$$\Psi(q^3,q)=\sum_{k=0}^{\infty}(-1)^k q^{\frac{k^2+k}{2}}=(q;q)_\infty\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_k^2}=(q;q)_\infty^2\sum_{k=0}^{\infty}\frac{q^k}{(q;q)_k^2}. \tag{4.2}$$

In [12], the second author recovered the second identity in 4.2 using the tail of the $(2,4)$-torus link. Furthermore, the tail of the colored Jones polynomial of $(2,2k)$-torus links, where $k\geq 2$, to give a natural extension of the same identity 4.2. For all $k\geq 2$, this identity is given by:

$$\Psi(q^{2k-1},q)=(q;q)_\infty\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\cdots\sum_{l_{k-1}=0}^{\infty}\frac{q^{\sum_{j=1}^{k-1}(i_j(i_j+1))}}{(q;q)_{l_{k-1}}^2\prod_{j=1}^{k-2}(q;q)_{l_j}}, \tag{4.3}$$

where $i_j=\sum_{s=j}^{k-1}l_s$. On the other hand, a similar identity for the theta function, known as Roger-Ramanujan identity for the two-variable theta function, can be recovered from the tail of the colored Jones polynomial of $(2,2k+1)$-torus knots. Recall that the general two variable Ramanujan theta function is defined by:

$$f(a,b)=\sum_{i=0}^{\infty}a^{i(i+1)/2}b^{i(i-1)/2}+\sum_{i=1}^{\infty}a^{i(i-1)/2}b^{i(i+1)/2}. \tag{4.4}$$

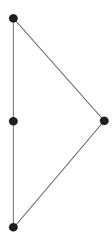
The function $f(a,b)$ specializes to:

$$f(-q^{2k},-q)=\sum_{i=0}^{\infty}(-1)^i q^{k(i^2+i)}q^{i(i-1)/2}+\sum_{i=1}^{\infty}(-1)^i q^{k(i^2-i)}q^{i(i+1)/2}. \tag{4.5}$$

For $k\geq 1$ the Roger-Ramanujan identity for the theta function is given by:

$$f(-q^{2k},-q)=(q;q)_\infty\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\cdots\sum_{l_{k-1}=0}^{\infty}\frac{q^{\sum_{j=1}^{k-1}(i_j(i_j+1))}}{\prod_{j=1}^{k-1}(q;q)_{l_j}}, \tag{4.6}$$

where $i_j=\sum_{s=j}^{k-1}l_s$. The identities 4.3 and 4.6 were recovered using a unified skein theoretic method in [12]. Note that the identities (4.6) and (4.3) are coming from cyclic graphs with odd and even number of vertices respectively. It is plausible to think that a natural family of knots, or graphs, gives rise to a natural family of q-series identities. In this paper we recover the third identity (4.2) using the tail of the colored Jones polynomial. This q-series correspond to the graph given in the following Figure.



Furthermore, we give a natural generalization of this identity using the tail of the graph L_k, where $k \geq 1$, given in Figure 8. Note that the graph L_{k-1} corresponds to the the pretzel knot P_k in Proposition 6.2.

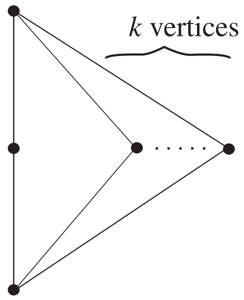


FIG. 4. The graph L_k

We show that this generalization is given by:

$$(q;q)_\infty \sum_{i=0}^{\infty} \frac{q^i}{(q;q)_i^{k+1}} = \sum_{i_1=0}^{\infty} \cdots \sum_{i_k=0}^{\infty} \frac{q^{\sum_{j=1}^{k} i_j + i_j^2 + \sum_{s=2}^{k} \sum_{j=s}^{k} i_{s-1} i_j}}{\prod_{j=1}^{k} (q;q)_{i_j} (q;q)_{\sum_{s=1}^{j} i_s}}. \tag{4.7}$$

5. The Tail of the Colored Jones Polynomial of the Pretzel Knots $P(2k+1, 2, 2u+1)$

In [12], the second author computed the tail of the knot 8_5. The tail of this knot is given by:

$$T_{8_5}(q) = (q;q)_\infty^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{(i+i^2+j+j^2)} (q;q)_{i+j}}{(q;q)_i^2 (q;q)_j^2}. \tag{5.1}$$

The series T_{8_5} is similar to the following q-series:

$$T_\Gamma = (\Psi(q^3, q))^2 = (q;q)_\infty^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{(i+i^2+j+j^2)}}{(q;q)_i^2 (q;q)_j^2}, \tag{5.2}$$

where Γ is the graph shown on the right handside of Figure 5. This similarity is not surprising since the graph associated to the knot 8_5 is given in left handside of the Figure 5.

FIG. 5. The reduced B-graph for 8_5 on the left and the graph Γ on the right.

Motivated by this observation, in this section we will study the tail of the family of graphs given in Figure 6 and show the relation between this q-series and the false theta function. Note that this graph corresponds to pretzel knots $(2k+1, 2, 2u+1)$ where $u, k \geq 1$.

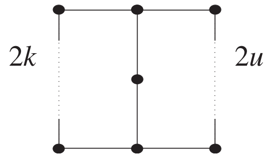
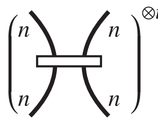
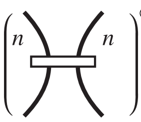


FIG. 6. The graph $\Phi_{k,u}$

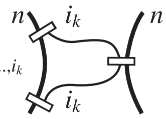
For our tail computations, we will study the element  $^{\otimes t}$ for $t \geq 1$. Note that when $t = 2$ we obtain the bubble element $\mathcal{B}^{n,n}_{n,n}(n, n)$.

Lemma 5.1. *Let $n \geq 1$, then we have*

(1) *For $k \geq 1$, we have*



$$^{\otimes(2k+1)} = \sum_{i_1=0}^{n}\sum_{i_2=0}^{i_1}\cdots\sum_{i_k=0}^{i_{k-1}} E_{n,i_1,\ldots,i_k}$$

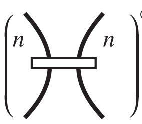


,

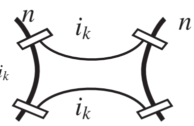
where

$$E_{n,i_1,\ldots,i_k} = \begin{bmatrix} n & n \\ n & n \end{bmatrix}_{i_1} \frac{\Delta_{2n}}{\Delta_{n+i_1}} \prod_{j=2}^{k} \begin{bmatrix} n & i_{j-1} \\ n & n \end{bmatrix}_{i_j} \frac{\Delta_{2n}}{\Delta_{n+i_j}} \tag{5.3}$$

(2) *For $k \geq 2$, we have*



$$^{\otimes(2k)} = \sum_{i_1=0}^{n}\sum_{i_2=0}^{i_1}\cdots\sum_{i_k=0}^{i_{k-1}} P_{n,i_1,\ldots,i_k}$$

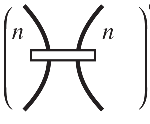


,

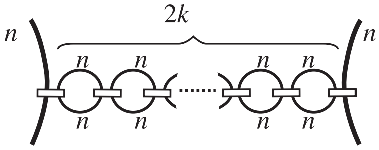
where

$$P_{n,i_1,\ldots,i_k} = \begin{bmatrix} n & n \\ n & n \end{bmatrix}_{i_1} \frac{\Delta_{2n}}{\Delta_{n+i_1}} \prod_{j=2}^{k-1} \begin{bmatrix} n & i_{j-1} \\ n & n \end{bmatrix}_{i_j} \frac{\Delta_{2n}}{\Delta_{n+i_j}} \begin{bmatrix} n & i_{k-1} \\ n & n \end{bmatrix}_{i_k}. \tag{5.4}$$

Proof. (1) Note first that

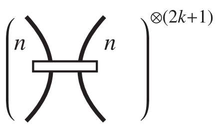


$$^{\otimes(2k+1)} =$$

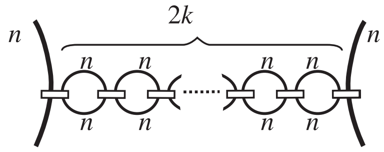


.

We apply the bubble expansion formula k times on the previous equation to obtain:

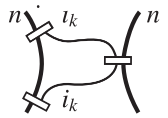


$=$



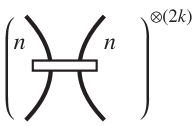
$$= \sum_{i_1=0}^{n}\sum_{i_2=0}^{i_1}\cdots\sum_{i_k=0}^{i_{k-1}} \begin{bmatrix} n & n \\ n & n \end{bmatrix}_{i_1} \frac{\Delta_{2n}}{\Delta_{n+i_1}} \prod_{j=2}^{k} \begin{bmatrix} n & i_{j-1} \\ n & n \end{bmatrix}_{i_j}$$

$$\times \quad \frac{\Delta_{2n}}{\Delta_{n+i_j}}$$

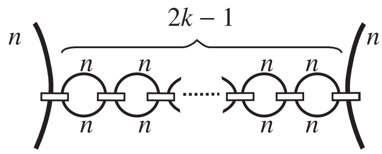


the result then follows.

(2) We apply the bubble expansion formula $k-1$ times and we obtain:

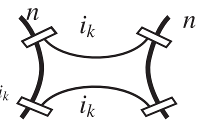


$=$



$$= \sum_{i_1=0}^{n}\sum_{i_2=0}^{i_1}\cdots\sum_{i_k=0}^{i_{k-1}} \begin{bmatrix} n & n \\ n & n \end{bmatrix}_{i_1} \frac{\Delta_{2n}}{\Delta_{n+i_1}} \prod_{j=2}^{k-1} \begin{bmatrix} n & i_{j-1} \\ n & n \end{bmatrix}_{i_j}$$

$$\times \quad \frac{\Delta_{2n}}{\Delta_{n+i_j}} \begin{bmatrix} n & i_{k-1} \\ n & n \end{bmatrix}_{i_k}$$

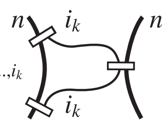


.

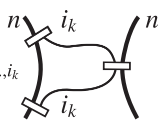
□

Lemma 5.2. (1) *Let $k \geq 1$. Then,*

$$\sum_{i_1=0}^{n}\sum_{i_2=0}^{i_1}\cdots\sum_{i_k=0}^{i_{k-1}} E_{n,i_1,\ldots,i_k}$$



$$= \sum_{i_k=0}^{n}\sum_{i_{k-1}=i_k}^{n}\cdots\sum_{i_1=i_2}^{n} E_{n,i_1,\ldots,i_k}$$



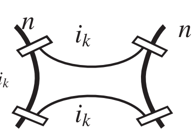
.

Moreover,

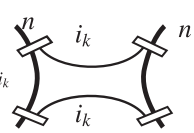
$$\begin{aligned} E_{n,i_1,\ldots,i_k} &= (-1)^{kn+\sum_{j=1}^{k} i_j} q^{kn/2+\sum_{j=1}^{k}(i_j(i_j/2+1))} \\ &\times \frac{(q;q)_n^{4k+2}(q;q)_{3n-i_1+1}}{(q;q)_{2n}^{k+1}(q;q)_{2n+1}(q;q)_{n-i_1}(q;q)_{n-i_k}^2(q;q)_{i_k}^2} \\ &\prod_{j=2}^{k} \frac{(q;q)_{i_{j-1}-i_j+2n+1}}{(q;q)_{i_{j-1}-i_j}(q;q)_{n+i_{j-1}}(q;q)_{n-i_{j-1}}^2(q;q)_{n+i_{j-1}+1}} \prod_{j=1}^{k} \frac{\Delta_{2n}}{\Delta_{n+i_j}} \end{aligned}$$

(2) *For $k \geq 1$, we have*

$$\sum_{i_1=0}^{n}\sum_{i_2=0}^{i_1}\cdots\sum_{i_k=0}^{i_{k-1}} P_{n,i_1,\ldots,i_k}$$



$$= \sum_{i_k=0}^{n}\sum_{i_{k-1}=i_k}^{n}\cdots\sum_{i_1=i_2}^{n} P_{n,i_1,\ldots,i_k}$$



.

Moreover,

$$P_{n,i_1,\ldots,i_k} = (-1)^{kn+\sum_{j=1}^{k} i_j} q^{kn/2+\sum_{j=1}^{k}(i_j(i_j/2+1))} \times \frac{(q;q)_n^{4k+2}(q;q)_{3n-i_1+1}}{(q;q)_{2n}^{k+1}(q;q)_{2n+1}(q;q)_{n-i_1}(q;q)_{n-i_k}^2(q;q)_{i_k}^2} \prod_{j=2}^{k} \frac{(q;q)_{i_{j-1}-i_j+2n+1}}{(q;q)_{i_{j-1}-i_j}(q;q)_{n+i_{j-1}}(q;q)_{n-i_{j-1}}^2(q;q)_{n+i_{j-1}+1}} \prod_{j=1}^{k-1} \frac{\Delta_{2n}}{\Delta_{n+i_j}}.$$

Proof. (1) Using the fact that

$$\prod_{i=0}^{j}[n-i] = q^{(2+3j+j^2-2n-2jn)/4}(1-q)^{-1-j}\frac{(q;q)_n}{(q;q)_{n-j-1}}$$

one obtains :

$$\begin{bmatrix} n & a \\ n & n \end{bmatrix}_b = (-1)^{n+b} q^{b/2+b^2-n/2} \frac{(q;q)_a^2(q;q)_n^4(q;q)_{1+a-b+2n}}{(q;q)_{a-b}(q;q)_b^2(q;q)_{2n}(q;q)_{a+n}(q;q)_{1+a+n}(q;q)_{-b+n}^2}.$$

This implies,

$$\begin{bmatrix} n & n \\ n & n \end{bmatrix}_{i_1} \prod_{j=2}^{k} \begin{bmatrix} n & i_{j-1} \\ n & n \end{bmatrix}_{i_j} = (-1)^{kn+\sum_{j=1}^{k} i_j} q^{kn/2+\sum_{j=1}^{k}(i_j(i_j/2+1))} \times \frac{(q;q)_n^{4k+2}(q;q)_{3n-i_1+1}}{(q;q)_{2n}^{k+1}(q;q)_{2n+1}(q;q)_{n-i_1}(q;q)_{n-i_k}^2(q;q)_{i_k}^2} \prod_{j=2}^{k} \frac{(q;q)_{i_{j-1}-i_j+2n+1}}{(q;q)_{i_{j-1}-i_j}(q;q)_{n+i_{j-1}}(q;q)_{n-i_{j-1}}^2(q;q)_{n+i_{j-1}+1}}.$$

On the other hand, one has

$$\sum_{i_1=0}^{n}\sum_{i_2=0}^{i_1}\cdots\sum_{i_k=0}^{i_{k-1}} F(i_1,\ldots,i_k) = \sum_{i_k=0}^{n}\sum_{i_{k-1}=i_k}^{n}\cdots\sum_{i_1=i_2}^{n} F(i_1,\ldots,i_k)$$

The result then follows.

(2) The proof is similar to (1). □

Theorem 5.3. *The tail of the graph $\Phi_{k,u}$ is given by*

$$T_{\Phi_{k,u}}(q) = (q;q)_\infty^2 \sum_{l_1=0}^{\infty}\cdots\sum_{l_k=0}^{\infty}\sum_{p_1=0}^{\infty}\cdots\sum_{p_u=0}^{\infty} g(q;l_1,\ldots,l_k)g(q;p_1,\ldots,p_u)(q;q)_{i+j},$$

where

$$g(q;l_1,\ldots,l_k) = \frac{q^{\sum_{j=1}^{k}(i_j(i_j+1))}}{(q;q)_{l_k}^2 \prod_{j=1}^{k-1}(q;q)_{l_j}}$$

with $i_j = \sum_{s=j}^{k} l_s$.

Proof. Using Theorem 2.2, we have

$$T_{\Phi_{k,u}}(q) \doteq_n \frac{S_B^{(n)}(\Phi_{k,u})}{\Delta_n}\bigg|_{A=q^{1/4}} \tag{5.5}$$

where $S_B^{(n)}(\Phi_{k,u})$ is the skein element given in Figure 7.

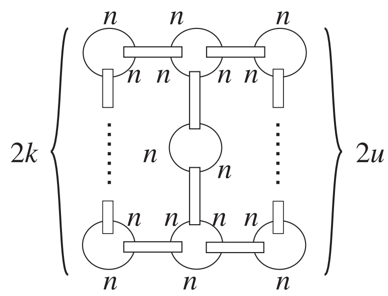
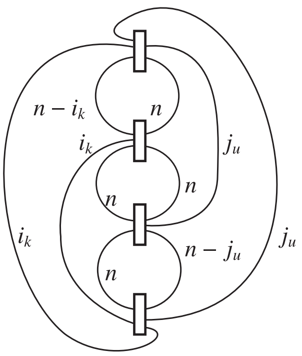


FIG. 7. The skein element $S_B^{(n)}(\Phi_{k,u})$.

Using Lemma 5.1, we can write

(5.6)

$$T_{\Phi_{k,u}}(q) \doteq_n \frac{1}{\Delta_n}\sum_{i_1=0}^{n}\sum_{i_2=0}^{i_1}\cdots\sum_{i_k=0}^{i_{k-1}}\sum_{j_1=0}^{n}\sum_{j_2=0}^{j_1}\cdots\sum_{j_u=0}^{j_{u-1}} P_{n,i_1,\dots,i_k}P_{n,j_1,\dots,j_u}\frac{\Delta_{2n}}{\Delta n+i_k}\frac{\Delta_{2n}}{\Delta_{n+j_u}}$$



Denote the element on the right handside of (5.6) by Γ_{n,i_k,j_u}. Now, Lemma 6.6 in [12] implies:

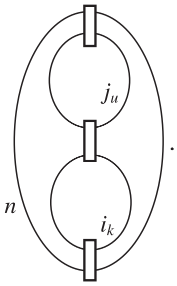
$$\Gamma_{n,i_k,j_u} = \begin{bmatrix} i_k & n \\ n & n-j_u \end{bmatrix}_0 \begin{bmatrix} j_u & n \\ n & n-i_k \end{bmatrix}_0 \begin{bmatrix} j_u & i_k \\ n & n \end{bmatrix}_0 \Delta_{i_k+j_u}. \tag{5.7}$$

Here,

$$\begin{bmatrix} j_u & n \\ n & n-i_k \end{bmatrix}_0 = (-1)^{n-i_k}q^{(i_k-n)/2}\frac{(q;q)_{i_k+j_u}(q;q)_n(q;q)_{n+i_k}(q;q)_{2n+j_u+1}}{(q;q)_{i_k}(q;q)_{2n}(q;q)_{j_u+n}(q;q)_{n+j_u+i_k+1}}. \tag{5.8}$$

Moreover Lemma 6.1 in [12] gives,

$$\left[\begin{array}{cc} j_u & i_k \\ n & n \end{array}\right]_0 \Delta_{i_k+j_u} =$$



A formula for the skein element on the right hand of the previous equation can be found in [19]. Using this allows us to obtain:

$$\left[\begin{array}{cc} j_u & i_k \\ n & n \end{array}\right]_0 \Delta_{i+j} = (-1)^{i_k+j_u+n} q^{-(i_k+j_u+n)/2} \frac{(q;q)_n (q;q)_{j_u} (q;q)_{i_k} (q;q)_{n+j_u+i_k+1}}{(1-q)(q;q)_{i_k+n}(q;q)_{j_u+n}(q;q)_{j_u+i_k}}. \tag{5.9}$$

Using (5.9) and (5.8) in (5.7) we obtain :

$$\Gamma_{n,i_k,j_u} = (-1)^n q^{-3n/2} \frac{(q;q)_{i_k+j_u}(q;q)_n^3 (q;q)_{1+i_k+2n}(q;q)_{1+j_u+2n}}{(1-q)(q;q)_{2n}^2 (q;q)_{i_k+n}(q;q)_{j_u+n}(q;q)_{1+i_k+j_u+n}}. \tag{5.10}$$

One the other hand, Lemma 5.2 implies

$$\begin{aligned} &\sum_{i_1=0}^{n} \cdots \sum_{i_k=0}^{i_{k-1}} \sum_{j_1=0}^{n} \cdots \sum_{j_u=0}^{j_{u-1}} P_{n,i_k} P_{n,j_u} \frac{\Delta_{2n}^2}{\Delta_{n+i_k}\Delta_{n+j_u}} \Gamma_{n,i_k,j_u} \\ &= \sum_{i_k=0}^{n} \cdots \sum_{i_1=i_2}^{n} \sum_{j_u=0}^{n} \cdots \sum_{j_1=j_2}^{n} P_{n,i_k} P_{n,j_u} \frac{\Delta_{2n}^2}{\Delta_{n+i_k}\Delta_{n+j_u}} \Gamma_{n,i_k,j_u}. \end{aligned} \tag{5.11}$$

Now

$$\frac{(q;q)_n}{(q;q)_{2n}} = \frac{\prod_{i=0}^{n-1}(1-q^{i+1})}{\prod_{i=0}^{2n-1}(1-q^{i+1})} = \frac{1}{\prod_{i=n}^{2n-1}(1-q^{i+1})} = \prod_{i=0}^{n-1} \frac{1}{(1-q^{i+n+1})} \doteq_n 1. \tag{5.12}$$

Moreover,

$$\frac{(q;q)_{3n-i+1}}{(q;q)_{2n+1}} = 1 - q^{2n+2} + O(2n+3) =_n 1 \tag{5.13}$$

and

$$\frac{(q;q)_{2n+i+1}}{(q;q)_{n+i}} = \frac{\prod_{k=0}^{3n+i}(1-q^{k+1})}{\prod_{i=0}^{n+i-1}(1-q^{k+1})} = \prod_{i=n+i}^{3n+i}(1-q^{k+1}) \doteq_n 1. \tag{5.14}$$

Hence, using Lemma 5.2, the equation (5.10) and the facts (5.12), (5.13) and (5.14) in 5.11

yield the equation:

$$\Phi_{k,u}(q) \doteq_n (q;q)_n^2 \sum_{i_k=0}^{n} \cdots \sum_{i_1=i_2}^{n} \sum_{j_u=0}^{n} \cdots \sum_{j_1=j_2}^{n} \frac{q^{\sum_{p=1}^{k}(i_p(i_p+1))}}{(q;q)_{i_k}^2 \prod_{p=2}^{k}(q;q)_{i_{p-1}-i_p}} \frac{q^{\sum_{l=1}^{u}(j_l(j_l+1))}}{(q;q)_{j_u}^2 \prod_{l=2}^{u}(q;q)_{j_{l-1}-j_l}} (q;q)_{i_k+j_u}.$$

Now set $s_p = i_p - i_{p+1}$ for $p = 1, ..., k-1$ and $s_k = i_k$, we obtain $i_p = \sum_{m=p}^{k} s_m$. Similarly, set $h_l = j_l - j_{l+1}$ for $l = 1, ..., u-1$ and $h_u = j_u$, we obtain $j_l = \sum_{r=l}^{u} h_d$. Changing the indexes in the previous equation yield the result. □

6. A Family of Pretzel Knots and Rogers-Ramanunjan Type Identities

In this section, the tail of the graph L_k, for $k \geq 1$, given in Figure 8 below is computed in two methods. Note that this graph correponds to the pretzel knot P_{k+1} defined in section 4.

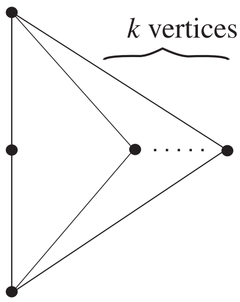
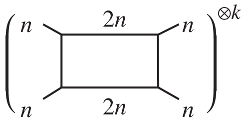


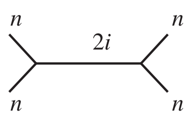
FIG. 8. The graph L_k

The first method utilizes the algorithm given by Masbaum and Vogel in [19] to compute the evaluation of a quantum spin network in $S(S^2)$. The second method uses the bubble skein element (2.1). Each method give rise to one of side of the q-series identities given in (4.7) generalizing the false theta identity given in (4.2). We start first by computing the tail of the graph in Figure 8 using the techniques given in [19].

Lemma 6.1. *Let* $k, n \geq 1$. *Then,*



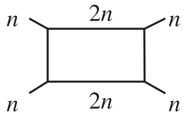
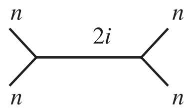
$$= \sum_{i=0}^{n} C_{n,i}$$



where

$$C_{n,i} = \frac{\theta(2n,2n,2i)^k}{\theta(n,n,2i)^{k+1}} \Delta_{2i}. \tag{6.1}$$

Proof. Note that


$$= \sum_{i=0}^{n} B_{n,i}$$


where

$$(6.2) \qquad B_{n,i} = \frac{Tet\begin{bmatrix} 2i & n & n \\ n & 2n & 2n \end{bmatrix} Tet\begin{bmatrix} 2i & 2n & 2n \\ n & n & n \end{bmatrix}}{\theta(2n,2n,2i)(\theta(n,n,2i))^2}\Delta_{2i}.$$

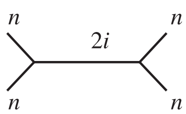
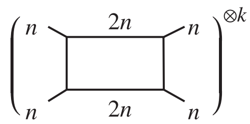
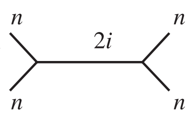
However,

$$(6.3) \qquad Tet\begin{bmatrix} 2i & n & n \\ n & 2n & 2n \end{bmatrix} = Tet\begin{bmatrix} 2i & 2n & 2n \\ n & n & n \end{bmatrix} = \theta(2n,2n,2i).$$

Hence

$$B_{n,i} = \frac{\theta(2n,2n,2i)}{\theta(n,n,2i)^2}\Delta_{2i}.$$

Moreover,


$$\otimes$$

$$= (P_{n,i})^k$$


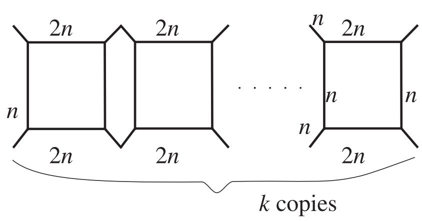
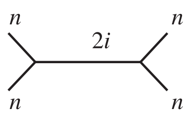
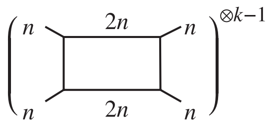
where

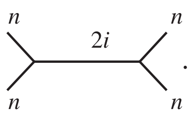
$$P_{n,i} = \frac{Tet\begin{bmatrix} 2i & n & n \\ n & 2n & 2n \end{bmatrix} Tet\begin{bmatrix} 2i & 2n & 2n \\ n & n & n \end{bmatrix}}{\theta(2n,2n,2i)\theta(n,n,2i)}.$$

However, equation (6.3) implies:

$$P_{n,i} = \frac{\theta(2n,2n,2i)}{\theta(n,n,2i)}.$$

Thus,


$$= \sum_{i=0}^{n} B_{n,i}$$

$$\otimes$$


$$= \sum_{i=0}^{n} B_{n,i}(P_{n,i})^{k-1}$$


The result follows. $\square$

Proposition 6.2. *For $k \geq 1$ we have*

$$T_{L_k}(q) = (q;q)_\infty^{k+1} \sum_{i=0}^{\infty} \frac{q^i}{(q;q)_i^{k+1}}.$$

Proof. By theorem 2.2 we know that the tail of the graph L_k is determined by the skein element $S_B^{(n)}(L_k)$. This element is equivalent to the quantum spin network in Figure 9. Note that there are $k+1$ copies of the box graph labeled by $2n$ and n in Figure 9.

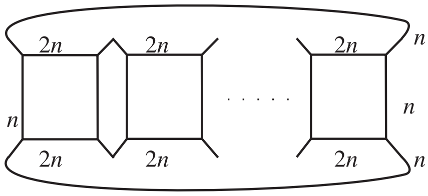


FIG. 9. The quantum spin network corresponding to graph L_k.

$\square$

By Lemma 6.1 we have

$$S_B^{(n)}(L_k) = \sum_{i=0}^{n} \frac{\theta(2n,2n,2i)^{k+1}}{\theta(n,n,2i)^{k+1}} \Delta_{2i} \tag{6.4}$$

However,

$$\theta(n,n,2i) = \frac{(-1)^{i+n} q^{-\frac{1}{2}(n+i)} (q,q)_i^2 (q,q)_{-i+n} (q,q)_{1+i+n}}{(1-q)(q,q)_{2i}(q;q)_n^2}. \tag{6.5}$$

Putting (6.5) in (6.4) and using Theorem 2.2 we obtain

$$T_{L_k}(q) \doteq_n S_B^{(n)}(L_k) = \frac{1}{\Delta_n} \sum_{i=0}^{n} \left(\frac{(-1)^{-n} q^{n/2} (q;q)_{2n}^2 (q;q)_{n-i} (q;q)_{n+i+1}}{(q;q)_n^2 (q;q)_{2n-i} (q;q)_{2n+i+1}} \right)^{k+1} \Delta_{2i}. \tag{6.6}$$

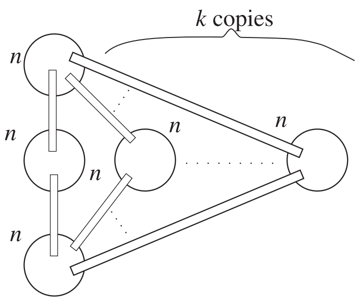
Similar techniques to the ones used in Theorem 5.3 imply:

$$T_{L_k}(q) = (q;q)_\infty^{k+1} \sum_{i=0}^{\infty} \frac{q^i}{(q;q)_i^{k+1}}. \tag{6.7}$$

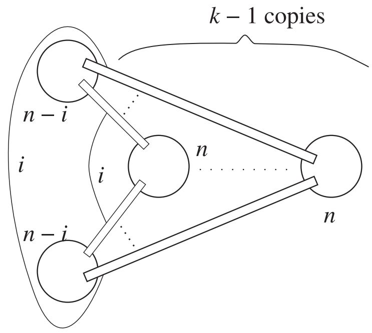
Proposition 6.3. *For $k \geq 1$ we have*

$$T_{L_k} = (q;q)_\infty^{k} \sum_{i_1=0}^{\infty} \dots \sum_{i_k=0}^{\infty} \frac{q^{\sum_{j=1}^{k} i_j + i_j^2 + \sum_{s=2}^{k} \sum_{j=s}^{k} i_{s-1} i_j}}{\prod_{j=1}^{k} (q;q)_{i_j} (q;q)_{\sum_{s=1}^{j} i_s}}. \tag{6.8}$$

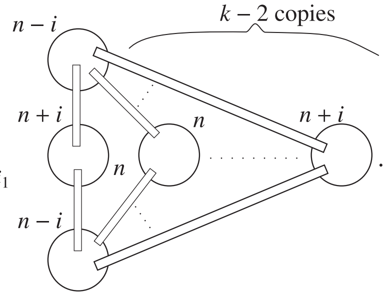
Proof. We apply the bubble skein formula to obtain:



$$= \sum_{i=0}^{n} \left[\begin{matrix} n & n \\ n & n \end{matrix} \right]_{i_1}$$

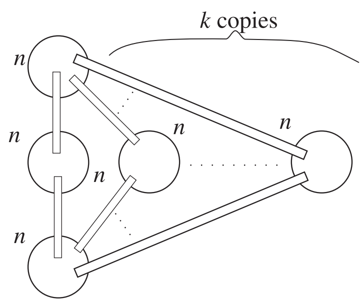


$$= \sum_{i=0}^{n} \left[\begin{matrix} n & n \\ n & n \end{matrix} \right]_{i_1}$$

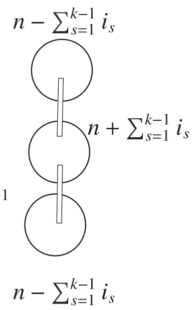


The skein element in the last equation is obtained from the skein element in the first equation by isotopy of the strands and the properties of the Jones-Wenzl idempotent. Similarly, we apply the bubble skein relation $(k-1)$ times on the skein element showing on the right handside of the previous equation to obtain

(6.9)



$$= \sum_{i_1=0}^{n} \sum_{i_2=0}^{n-i_1} \cdots \sum_{i_k=0}^{n-\sum_{l=1}^{k-1} i_l} \left[\begin{matrix} n & n \\ n & n \end{matrix} \right]_{i_1} \times \prod_{j=1}^{k-1} \left[\begin{matrix} n-\sum_{s=1}^{j} i_s & n-\sum_{s=1}^{j} i_s \\ n+\sum_{s=1}^{j} i_s & n \end{matrix} \right]_{i_{j+1}}$$



$$= \sum_{i_1=0}^{n} \sum_{i_2=0}^{n-i_1} \cdots \sum_{i_k=0}^{n-\sum_{l=1}^{k-1} i_l} \left[\begin{matrix} n & n \\ n & n \end{matrix} \right]_{i_1} \times \prod_{j=1}^{k-1} \left[\begin{matrix} n-\sum_{s=1}^{j} i_s & n-\sum_{s=1}^{j} i_s \\ n+\sum_{s=1}^{j} i_s & n \end{matrix} \right]_{i_{j+1}} \frac{(\Delta_{2n})^2}{\Delta_{n+\sum_{s=1}^{k-1} i_s}}.$$

However,

$$(6.10)\qquad \begin{bmatrix} n-a & n-a \\ n+a & n \end{bmatrix}_i = \frac{(-1)^{i+n} q^{i/2+ai+i^2-n/2}(q,q)_n^3(q,q)_{-a+n}^2(q,q)_{a+n}(q,q)_{1-a-i+3n}}{(q,q)_i(q,q)_{a+i}(q,q)_{2n}^2(q,q)_{-i+n}(q,q)_{-a-i+n}^2(q,q)_{1-a+2n}}.$$

Using equation (6.10) in 6.9 and using similar techniques to ones we used in Theorem 5.3 we obtain the result. □

Propositions 6.2 and 6.3 imply immediately the following

Corollary 6.4. *For $k \geq 1$ we have*

$$(q;q)_\infty \sum_{i=0}^{\infty} \frac{q^i}{(q;q)_i^{k+1}} = \sum_{i_1=0}^{\infty} \cdots \sum_{i_k=0}^{\infty} \frac{q^{\sum_{j=1}^k i_j + i_j^2 + \sum_{s=2}^k \sum_{j=s}^k i_{s-1} i_j}}{\prod_{j=1}^k (q;q)_{i_j} (q;q)_{\sum_{s=1}^j i_s}}.$$

References

[1] G.E. Andrews and B.C. Berndt: Ramanujan's lost notebook. Part I, Springer, New York, 2005, MR2135178.

[2] C.W. Armond: *The head and tail conjecture for alternating knots*, Algebr. Geom. Topol. **13** (2013), 2809–2826. MR3116304

[3] C.W. Armond: *Walks along braids and the colored Jones polynomial*, J. Knot Theory Ramifications **23** (2014), 1450007, 15. MR3197051

[4] C.W. Armond and O.T. Dasbach: *Rogers-Ramanujan type identities and the head and tail of the colored Jones polynomial*, arXiv:1106.3948 (2011).

[5] C. Blanchet, N. Habegger, G. Masbaum and P. Vogel: *Three-manifold invariants derived from the Kauffman bracket*, Topology **31** (1992), 685–699. MR1191373

[6] O.T. Dasbach and Xiao-Song Lin: *On the head and the tail of the colored Jones polynomial*, Compos. Math. **142** (2006), 1332–1342. MR2264669

[7] O.T. Dasbach and Xiao-Song Lin: *A volumish theorem for the Jones polynomial of alternating knots*, Pacific J. Math. **231** (2007), 279–291. MR2346497

[8] P. Freyd, D. Yetter, J. Hoste, W.B.R. Lickorish, K. Millett and A. Ocneanu: *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), 239–246. MR776477

[9] S. Garoufalidis and Thang T.Q. Lê: *Nahm sums, stability and the colored Jones polynomial*, Res. Math. Sci. **2** (2015), 2:1. MR3375651

[10] M. Hajij: *The colored kauffman skein relation and the head and tail of the colored jones polynomial*, arXiv preprint arXiv:1401.4537 (2014).

[11] M. Hajij: *The tail of a quantum spin network*, Ramanujan J. **40** (2016), 135–136.

[12] M. Hajij: *The Bubble skein element and applications*, J. Knot Theory Ramifications **23** (2014), 1450076, 30. MR3312619

[13] V.F.R. Jones: *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. (2) **126** (1987), 335–388. MR908150

[14] V.F.R. Jones: *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), 103–111. MR766964

[15] K. Hikami: *Volume conjecture and asymptotic expansion of q-series*, Experiment. Math. **12** (2003), 319–337. MR2034396

[16] L.H. Kauffman: *State models and the Jones polynomial*, Topology **26** (1987), 395–407. MR899057

[17] A. Keilthy and R. Osburn: *Rogers-Ramanujan identities for alternating knots*, Journal of Number Theory (2015).

[18] W.B.R. Lickorish: *Calculations with the Temperley-Lieb algebra*, Comment. Math. Helv. **67** (1992), 571–591. MR1185809

[19] G. Masbaum and P. Vogel: 3-*valent graphs and the Kauffman bracket*, Pacific J. Math. **164** (1994), 361–381. MR1272656

[20] H.R. Morton: *Invariants of links and* 3*-manifolds from skein theory and from quantum groups*, Topics in knot theory (Erzurum 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 399, Kluwer Acad. Publ., Dordrecht, 1993, pp. 107–155. MR1257908

[21] H. Murakami and J. Murakami: *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math. **186** (2001), 85–104. MR1828373

[22] J.H. Przytycki: *Fundamentals of Kauffman bracket skein modules*, Kobe J. Math. **16** (1999), 45–66, MR1723531

[23] J.H. Przytycki and P. Traczyk: *Invariants of links of Conway type*, Kobe J. Math. **4** (1988), 115–139. MR945888

[24] N. Reshetikhin and V.G. Turaev: *Invariants of* 3*-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991), 547–597. MR1091619

[25] V.G. Turaev and O. Ya. Viro: *State sum invariants of* 3*-manifolds and quantum* 6*j-symbols*, Topology **31** (1992), 865–902. MR1191386

[26] V. Turaev and H. Wenzl: *Quantum invariants of* 3*-manifolds associated with classical simple Lie algebras*, Internat. J. Math. **4** (1993), 323–358. MR1217386

[27] S. Ramanujan: The lost notebook and other unpublished papers, Springer-Verlag, Berlin; Narosa Publishing House, New Delhi 1988. With an introduction by George E. Andrews. MR947735

[28] H. Wenzl: *On sequences of projections*, C.R. Math. Rep. Acad. Sci. Canada **9** (1987), 5–9. MR873400

Mohamed Elhamdadi
Department of Mathematics
University of South Florida
Tampa, FL 33647
USA
e-mail: emohamed@math.usf.edu

Mustafa Hajij
Department of Mathematics
University of South Florida
Tampa, FL 33647
USA
e-mail: mhajij@usf.edu