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QUADRATIC APPROXIMATION IN $\mathbb{F}_q((T^{-1}))$

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Abstract

In this paper, we study Diophantine exponents w_n and w_n^* for Laurent series over a finite field. Especially, we deal with the case $n = 2$, that is, quadratic approximation. We first show that the range of the function $w_2 - w_2^*$ is exactly the closed interval $[0, 1]$. Next, we estimate an upper bound of the exponent w_2 of continued fractions with low complexity partial quotients.

Contents

1. Introduction

Let p be a prime and q be a power of p. We denote by $\mathbb{F}_q$ the finite field of elements q, $\mathbb{F}_q[T]$ the ring of polynomials over $\mathbb{F}_q$, $\mathbb{F}_q(T)$ the field of rational functions over $\mathbb{F}_q$, and $\mathbb{F}_q((T^{-1}))$ the field of Laurent series over $\mathbb{F}_q$. For $\xi \in \mathbb{F}_q((T^{-1})) \setminus \{0\}$, we can write

$$\xi = \sum_{n=N}^{\infty} a_n T^{-n},$$

where $N \in \mathbb{Z}$, $a_n \in \mathbb{F}_q$, and $a_N \neq 0$. We define the absolute value on $\mathbb{F}_q((T^{-1}))$ by $|0| := 0$ and $|\xi| := q^{-N}$. The absolute value can uniquely extend on the algebraic closure of $\mathbb{F}_q((T^{-1}))$ and we continue to write $|\cdot|$ for the extended absolute value.

Throughout this paper, we regard elements of $(\mathbb{F}_q[T])[X]$ as polynomials in X. For $P(X) \in (\mathbb{F}_q[T])[X]$, the *height* of $P(X)$, denoted by $H(P)$, is defined to be the maximal of absolute values of the coefficients of $P(X)$. We denote by $(\mathbb{F}_q[T])[X]_{\min}$ the set of non-constant irreducible primitive polynomials $P(X) \in (\mathbb{F}_q[T])[X]$ whose the leading coefficient

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is monic in T. For $\alpha \in \overline{\mathbb{F}_q(T)}$, there exists a unique polynomial $P(X) \in (\mathbb{F}_q[T])[X]_{\min}$ such that $P(\alpha) = 0$. We call $P(X)$ the *minimal polynomial* of α. The *height* (resp. the *degree*, the *inseparable degree*) of α, denoted by $H(\alpha)$ (resp. $\deg \alpha$, $\operatorname{insep} \alpha$), is defined to be the height of $P(X)$ (resp. the degree of $P(X)$, the inseparable degree of $P(X)$). For $\xi \in \mathbb{F}_q((T^{-1}))$ and integers $n, H \geq 1$, let $w_n(\xi, H)$ and $w_n^*(\xi, H)$ be given by

$$w_n(\xi, H) = \min\{|P(\xi)| \mid P(X) \in (\mathbb{F}_q[T])[X], H(P) \leq H, \deg_X P \leq n, P(\xi) \neq 0\},$$
$$w_n^*(\xi, H) = \min\{|\xi - \alpha| \mid \alpha \in \overline{\mathbb{F}_q(T)}, H(\alpha) \leq H, \deg \alpha \leq n, \alpha \neq \xi\}.$$

The Diophantine exponents w_n and w_n^* are defined by

$$w_n(\xi) = \limsup_{H\to\infty} \frac{-\log w_n(\xi, H)}{\log H}, \quad w_n^*(\xi) = \limsup_{H\to\infty} \frac{-\log H w_n^*(\xi, H)}{\log H}.$$

In other words, $w_n(\xi)$ (resp. $w_n^*(\xi)$) is the supremum of a real number w (resp. w^*) which is satisfied that

$$0 < |P(\xi)| \leq H(P)^{-w} \quad (\text{resp. } 0 < |\xi - \alpha| \leq H(\alpha)^{-w^*-1})$$

for infinitely many $P(X) \in (\mathbb{F}_q[T])[X]$ of degree at most n (resp. $\alpha \in \overline{\mathbb{F}_q(T)}$ of degree at most n).

As in classical continued fraction theory of real numbers, if $\xi \in \mathbb{F}_q((T^{-1}))$, then we can write

$$\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}},$$

where $a_0, a_n \in \mathbb{F}_q[T]$, $\deg a_n \geq 1$ for $n \geq 1$. For simplicity of notation, we write $\xi = [a_0, a_1, a_2, \ldots]$. The a_0 and a_n are called the *partial quotients* of ξ. We define p_n and q_n by

$$\begin{cases} p_{-1} = 1,\ p_0 = a_0,\ p_n = a_n p_{n-1} + p_{n-2},\ n \geq 1, \\ q_{-1} = 0,\ q_0 = 1,\ q_n = a_n q_{n-1} + q_{n-2},\ n \geq 1. \end{cases}$$

We call $(p_n/q_n)_{n\geq 0}$ *convergent sequence* of ξ and have $p_n/q_n = [a_0, a_1, \ldots, a_n]$ for $n \geq 0$ by induction on n.

In this paper, we study the difference of the Diophantine exponents w_2 and w_2^* using continued fractions. We denote by $\lfloor x \rfloor$ the integer part and $\lceil x \rceil$ the upper integer part of a real number x. We construct explicitly continued fractions $\xi \in \mathbb{F}_q((T^{-1}))$ for which $w_2(\xi) - w_2^*(\xi) = \delta$ for each $0 < \delta \leq 1$ as follows:

Theorem 1.1. *Let w be a real number which is greater than $(5 + \sqrt{17})/2$ when $p \neq 2$, and that of $(9 + \sqrt{65})/2$ when $p = 2$. Let $b, c \in \mathbb{F}_q[T]$ be distinct polynomials of degree at least one. We define a sequence $(a_{n,w})_{n\geq 1}$ by*

$$a_{n,w} = \begin{cases} c & \text{if } n = \lfloor w^i \rfloor \text{ for some integer } i \geq 0, \\ b & \text{otherwise.} \end{cases}$$

Set $\xi_w := [0, a_{1,w}, a_{2,w}, \ldots]$. *Then we have* $w_2^*(\xi_w) = w - 1$ *and* $w_2(\xi_w) = w$.

Theorem 1.2. *Let* $w \geq 25$ *be a real number and* $b, c, d \in \mathbb{F}_q[T]$ *be distinct polynomials of degree at least one. Let* $0 < \eta < \sqrt{w}/4$ *be a positive number and put*

$$m_i := \left\lfloor \frac{\lfloor w^{i+1} \rfloor - \lfloor w^i - 1 \rfloor}{\lfloor \eta w^i \rfloor} \right\rfloor$$

for all $i \geq 1$. *We define a sequence* $(a_{n,w,\eta})_{n\geq 1}$ *by*

$$a_{n,w,\eta} = \begin{cases} c & \text{if } n = \lfloor w^i \rfloor \text{ for some integer } i \geq 0, \\ d & \text{if } n \neq \lfloor w^i \rfloor \text{ for all integer } i \geq 0 \text{ and } n = \lfloor w^j \rfloor + m\lfloor \eta w^j \rfloor \text{ for some integer } 1 \leq m \leq m_j,\ j \geq 1, \\ b & \text{otherwise.} \end{cases}$$

Set $\xi_{w,\eta} := [0, a_{1,w,\eta}, a_{2,w,\eta}, \ldots]$. *Then we have*

$$w_2^*(\xi_{w,\eta}) = \frac{2w - 2 - \eta}{2 + \eta}, \quad w_2(\xi_{w,\eta}) = \frac{2w - \eta}{2 + \eta}.$$

Hence, we have

$$w_2(\xi_{w,\eta}) - w_2^*(\xi_{w,\eta}) = \frac{2}{2 + \eta}.$$

Theorems 1.1 and 1.2 are analogues of Theorems 4.1 and 4.2 in [8] and Theorems 1 and 2 in [9]. Theorems 1.1 and 1.2 are proved in a similar method of the proof of these analogue theorems.

In Section 5, we prove

$$0 \leq w_2(\xi) - w_2^*(\xi) \leq 1$$

for all $\xi \in \mathbb{F}_q((T^{-1}))$ (Proposition 5.6). We also prove $w_n(\xi) = w_n^*(\xi) = 0$ for all $n \geq 1$ and $\xi \in \mathbb{F}_q(T)$ (Theorem 5.2). Consequently, we determine the range of the function $w_2 - w_2^*$ from Theorems 1.1 and 1.2.

Corollary 1.3. *The range of the function* $w_2 - w_2^*$ *is exactly the closed interval* $[0, 1]$.

For $\xi \in \mathbb{F}_q((T^{-1}))$, we set

$$w(\xi) := \limsup_{n\to\infty} \frac{w_n(\xi)}{n}, \quad w^*(\xi) := \limsup_{n\to\infty} \frac{w_n^*(\xi)}{n}.$$

We say that ξ is an

A-number if $w(\xi) = 0$;

S-number if $0 < w(\xi) < +\infty$;

T-number if $w(\xi) = +\infty$ and $w_n(\xi) < +\infty$ for all n;

U-number if $w(\xi) = +\infty$ and $w_n(\xi) = +\infty$ for some n.

This classification of $\mathbb{F}_q((T^{-1}))$ was first introduced by Bundschuh [10] and is called *Mahler's classification*. Replacing w_n and w with w_n^* and w^*, we define A^*-, S^*-, T^*-, and

U^*-numbers as above. This classification of $\mathbb{F}_q((T^{-1}))$ was first introduced by Bugeaud [7, Section 9] and is called *Koksma's classification*. Let $n \geq 1$ be an integer, $\xi \in \mathbb{F}_q((T^{-1}))$ be a U-number, and $\zeta \in \mathbb{F}_q((T^{-1}))$ be a U^*-number. The number ξ (resp. the number ζ) is called a *U_n-number* (resp. *U_n^*-number*) if $w_n(\xi)$ is infinite and $w_m(\xi)$ is finite (resp. $w_n^*(\zeta)$ is infinite and $w_m^*(\zeta)$ is finite) for all $1 \leq m < n$.

Let $\mathcal{A}$ be a finite set. Let $\mathcal{A}^*$, $\mathcal{A}^+$, and $\mathcal{A}^{\mathbb{N}}$ denote the set of finite words over $\mathcal{A}$, the set of nonempty finite words over $\mathcal{A}$, and the set of infinite words over $\mathcal{A}$. We denote by $|W|$ the length of a finite word W over $\mathcal{A}$. For an integer $n \geq 0$, let $W^n = WW\cdots W$ (n times repeated concatenation of the word W) and $\overline{W} = WW\cdots W\cdots$ (infinitely many concatenation of the word W). Note that W^0 is equal to the empty word. More generally, for a real number $w \geq 0$, let $W^w = W^{\lfloor w \rfloor}W'$, where W' is the prefix of W of length $\lceil (w - \lfloor w \rfloor)|W| \rceil$. Let $\mathbf{a} = (a_n)_{n\geq 0}$ be a sequence over $\mathcal{A}$. We identify $\mathbf{a}$ with the infinite word $a_0a_1\cdots a_n\cdots$. Let ρ be a real number. We say that $\mathbf{a}$ satisfies *Condition* $(*)_\rho$ if there exist sequences of finite words $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and a sequence of nonnegative real numbers $(w_n)_{n\geq 1}$ such that

(i) the word $U_nV_n^{w_n}$ is the prefix of $\mathbf{a}$ for all $n \geq 1$,

(ii) $|U_nV_n^{w_n}|/|U_nV_n| \geq \rho$ for all $n \geq 1$,

(iii) the sequence $(|V_n^{w_n}|)_{n\geq 1}$ is strictly increasing.

The *Diophantine exponent* of $\mathbf{a}$, first introduced in [2], denoted by $\mathrm{Dio}(\mathbf{a})$, and is defined to be the supremum of a real number ρ such that $\mathbf{a}$ satisfy Condition $(*)_\rho$. It is obvious that

$$1 \leq \mathrm{Dio}(\mathbf{a}) \leq +\infty.$$

The infinite word $\mathbf{a}$ is called *ultimately periodic* if there exist finite words $U \in \mathcal{A}^*$ and $V \in \mathcal{A}^+$ such that $\mathbf{a} = U\overline{V}$. The *complexity function* of $\mathbf{a}$ is defined by

$$p(\mathbf{a}, n) = \mathrm{Card}\{a_ia_{i+1}\ldots a_{i+n-1} \mid i \geq 0\}, \quad \text{for } n \geq 1.$$

We state now the second main results.

Theorem 1.4. *Let $\kappa \geq 2, A \geq q$ be integers and $\mathbf{a} = (a_n)_{n\geq 1}$ be a sequence over $\mathbb{F}_q[T]$ with $q \leq |a_n| \leq A$ for all $n \geq 1$. Assume that there exists an integer $n_0 \geq 1$ such that*

$$p(\mathbf{a}, n) \leq \kappa n \text{ for all } n \geq n_0,$$

and the Diophantine exponent of $\mathbf{a}$ is finite. Set $\xi := [0, a_1, a_2, \ldots]$. Then we have

$$w_2(\xi) \leq 128(2\kappa + 1)^3\, \mathrm{Dio}(\mathbf{a}) \left(\frac{\log A}{\log q}\right)^4. \tag{1.1}$$

In particular, if the sequence $(|q_n|^{1/n})_{n\geq 1}$ converges, then we have

$$w_2(\xi) \leq 64(2\kappa + 1)^3\, \mathrm{Dio}(\mathbf{a}). \tag{1.2}$$

There are special sequences which are satisfied the assumption of Theorem 1.4, for example, automatic sequences, primitive morphic sequences, and Strumian sequence with some condition. The detail will appear in Section 2.

Theorem 1.5. *Let* $\mathbf{a} = (a_n)_{n\geq 1}$ *be a non-ultimately periodic sequence over* $\mathbb{F}_q[T]$ *with* $\deg a_n \geq 1$ *for all* $n \geq 1$. *Assume that* $(|q_n|^{1/n})_{n\geq 1}$ *is bounded. Put*

$$m := \liminf_{n\to\infty} |q_n|^{1/n}, \quad M := \limsup_{n\to\infty} |q_n|^{1/n}.$$

Set $\xi := [0, a_1, a_2, \ldots]$. *Then we have*

$$w_2(\xi) \geq w_2^*(\xi) \geq \max\left(2, \frac{\log m}{\log M}\mathrm{Dio}(\mathbf{a}) - 1\right). \tag{1.3}$$

In particular, if the sequence $(|q_n|^{1/n})_{n\geq 1}$ *converges, then we have*

$$w_2(\xi) \geq w_2^*(\xi) \geq \max(2, \mathrm{Dio}(\mathbf{a}) - 1).$$

Furthermore, assume that the sequence $(|a_n|)_{n\geq 1}$ *is bounded. Then we have*

$$w_2(\xi) \geq \max\left(2, \frac{\log m}{\log M}(\mathrm{Dio}(\mathbf{a}) + 1) - 1\right). \tag{1.4}$$

In particular, if the sequence $(|q_n|^{1/n})_{n\geq 1}$ *converges, then we have*

$$w_2(\xi) \geq \max(2, \mathrm{Dio}(\mathbf{a})).$$

Theorems 1.4 and 1.5 are analogues of Theorems 2.2 and 2.3 in [8]. Theorems 1.4 and 1.5 are proved in a similar method of the proof of these analogue theorems.

We state an immediately consequence of Theorems 1.4 and 1.5.

Corollary 1.6. *Let* $\mathbf{a} = (a_n)_{n\geq 1}$ *be a non-ultimately periodic sequence over* $\mathbb{F}_q[T]$ *with* $\deg a_n \geq 1$ *for* $n \geq 1$. *Assume that* $(|a_n|)_{n\geq 1}$ *is bounded and*

$$\limsup_{n\to\infty} \frac{p(\mathbf{a}, n)}{n} < +\infty.$$

Set $\xi := [0, a_1, a_2, \ldots]$. *Then the Diophantine exponent of* $\mathbf{a}$ *is finite if and only if* ξ *is not a* U_2*-number.*

We use the Vinogradov notation $A \ll B$ (resp. $A \ll_a B$) if $|A| \leq c|B|$ with some constant (resp. some constant depending at most on a) $c > 0$. We write $A \asymp B$ (resp. $A \asymp_a B$) if $A \ll B$ and $B \ll A$ (resp. $A \ll_a B$ and $B \ll_a A$) hold.

This paper is organized as follows. In Section 2, we define special sequences and apply Theorem 1.4 to these sequences. In Section 3, we prove Liouville inequalities, that is, a nontrivial lower bound of the absolute value of difference two algebraic numbers and that of polynomial at an algebraic point. In Section 4, we prove some lemmas with respect to continued fractions. In Section 5, we study the Diophantine exponents w_n and w_n^*. In Section 6, applying Liouville inequality, we prove lemmas to determine the value of w_2 and w_2^*. In Section 7, we prove combinational lemma to show Theorem 1.4. In Section 8, we prove Theorems 1.1, 1.2, 1.4, and 1.5. In Appendix A, we prove an analogue of Theorems 1.4 and 1.5 for Laurent series over a finite field.

2. Application of the main results

In this section, we first recall properties of special sequences. For a deeper discussion, we refer the reader to [6]. Let $k \geq 2$ be an integer. We denote by Σ_k the set $\{0, 1, \ldots, k-1\}$. A *k-automaton* is a sextuple

$$A = (Q, \Sigma_k, \delta, q_0, \Delta, \tau),$$

where Q is a finite set, $\delta : Q \times \Sigma_k \to Q$ is a map, $q_0 \in Q$, Δ is a set, and $\tau : Q \to \Delta$ is a map. For $q \in Q$ and a finite word $W = w_0 w_1 \cdots w_m$ over Σ_k, we define recursively $\delta(q, W)$ by $\delta(q, W) = \delta(\delta(q, w_0 w_1 \cdots w_{m-1}), w_m)$. Let $n \geq 0$ be an integer and $W_n = w_r w_{r-1} \cdots w_0$, where $\sum_{i=0}^{r} w_i k^i$ is the k-ary expansion of n. A sequence $\mathbf{a} = (a_n)_{n\geq 0}$ is said to be *k-automatic* if there exists a k-automaton $A = (Q, \Sigma_k, \delta, q_0, \Delta, \tau)$ such that $a_n = \tau(\delta(q_0, W_n))$ for all $n \geq 0$. The *k-kernel* of a sequence $\mathbf{a} = (a_n)_{n\geq 0}$ is the set of all sequences $(a_{k^i n + j})_{n\geq 0}$, where $i \geq 0$ and $0 \leq j < k^i$.

It is known about k-automatic sequences as follows:

Theorem 2.1 (Eilenberg [14])**.** *Let $k \geq 2$ be an integer. Then a sequence is k-automatic if and only if its k-kernel is finite.*

Lemma 2.2 (Adamczewski and Cassaigne [1])**.** *Let $k \geq 2$ be an integer. Let* $\mathbf{a}$ *be a non-ultimately periodic and k-automatic sequence. Let m be a cardinality of the k-kernel of* $\mathbf{a}$. *Then we have*

$$\mathrm{Dio}(\mathbf{a}) < k^m.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be finite sets. A map $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is a *morphism* if $\sigma(UV) = \sigma(U)\sigma(V)$ for all $U, V \in \mathcal{A}$. The *width* of σ is defined to be $\max_{a\in\mathcal{A}} |\sigma(a)|$. A morphism σ is said to be *k-uniform* if there exists an integer $k \geq 1$ such that $|\sigma(a)| = k$ for all $a \in \mathcal{A}$. In particular, we call a 1-uniform morphism a *coding*. A morphism $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ is *primitive* if there exists an integer $n \geq 1$ such that a occurs in $\sigma^n(b)$ for all $a, b \in \mathcal{A}$. A morphism $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ is *prolongable* on $a \in \mathcal{A}$ if $\sigma(a) = aW$, where $W \in \mathcal{A}^+$ and $\sigma^n(W)$ is not an empty word for all $n \geq 1$. A sequence $\mathbf{a} = (a_n)_{n\geq 0}$ is said to be *k-uniform morphic* (resp. *primitive morphic*) if there exist finite sets $\mathcal{A}, \mathcal{B}$, a k-uniform morphism (resp. a primitive morphism) $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ which is prolongable on some $a \in \mathcal{A}$, and a coding $\tau : \mathcal{A}^* \to \mathcal{B}^*$ such that $\mathbf{a} = \lim_{n\to\infty} \tau(\sigma^n(a))$. When $\mathbf{a}$ is a k-uniform morphic, we call $\mathcal{A}$ the *initial alphabet* associated with $\mathbf{a}$.

Theorem 2.3 (Cobham [12])**.** *Let $k \geq 2$ be an integer. Then a sequence is k-automatic if and only if it is k-uniform morphic.*

Mossé's result [20] implies the lemma below.

Lemma 2.4. *Let* $\mathbf{a}$ *be a non-ultimately periodic and primitive morphic sequence. Then the Diophantine exponent of* $\mathbf{a}$ *is finite.*

Let θ and ρ be real numbers with $0 < \theta < 1$ and θ is irrational. For $n \geq 1$, we put $s_{n,\theta,\rho} := \lfloor (n+1)\theta + \rho \rfloor - \lfloor n\theta + \rho \rfloor$ and $s'_{n,\theta,\rho} := \lceil (n+1)\theta + \rho \rceil - \lceil n\theta + \rho \rceil$. A sequence $\mathbf{a} = (a_n)_{n\geq 1}$ is called *Sturmian* if there exist an irrational number $0 < \theta < 1$, a real number ρ, a finite set $\mathcal{A}$, and a coding $\tau : \{0, 1\}^* \to \mathcal{A}^*$ with $\tau(0) \neq \tau(1)$ such that $(a_n)_{n\geq 1}$ is $(\tau(s_{n,\theta,\rho}))_{n\geq 1}$ or

$(\tau(s'_{n,\theta,\rho}))_{n\geq 1}$. Then we call the irrational number θ *slope* of **a** and the real number ρ *intercept* of **a**.

Lemma 2.5 (Adamczewski and Bugeaud [5]). *Let* **a** *be a Strumian sequence. Then the slope of* **a** *has bounded partial quotients if and only if the Diophantine exponent of* **a** *is finite.*

It is known that automatic sequences, primitive morphic sequences, and Strumian sequences have low complexity.

Lemma 2.6. *Let $k \geq 2$ be an integer and* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a k-automatic sequence. Let d be a cardinality of the internal alphabet associated with* **a**. *Then we have*

$$p(\mathbf{a}, n) \leq kd^2 n, \text{ for } n \geq 1.$$

Proof. See [6, Theorem 10.3.1] or [12]. □

Lemma 2.7. *Let* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a primitive morphic sequence over a finite set of cardinality of $b \geq 2$. Let v be the width of σ which generates the sequence* **a**. *Then we have*

$$p(\mathbf{a}, n) \leq 2v^{4b-2}b^3 n, \text{ for } n \geq 1.$$

Proof. See [6, Theorem 10.4.12]. □

Lemma 2.8. *Let* **a** *be a Sturmian sequence. Then we have*

$$p(\mathbf{a}, n) = n + 1, \text{ for } n \geq 1.$$

Proof. See [6, Theorem 10.5.8]. □

Consequently, by Theorems 1.4 and 1.5, we obtain the upper bound of w_2 of automatic, primitive morphic, or Strumian continued fractions as follows:

Theorem 2.9. *Let $k \geq 2$ be an integer. Let* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a non-ultimately periodic and k-automatic sequence over $\mathbb{F}_q[T]$ with $\deg a_n \geq 1$ for all $n \geq 0$. Let A be an upper bound of the sequence $(|a_n|)_{n\geq 0}$, m be a cardinality of k-kernel of* **a**, *and d be a cardinality of the initial alphabet associated with* **a**. *Set $\xi := [0, a_0, a_1, \ldots]$. Then we have*

$$w_2(\xi) \leq 128(2kd^2 + 1)^3 k^m \left(\frac{\log A}{\log q}\right)^4 .$$

Theorem 2.10. *Let* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a non-ultimately and primitive morphic sequence over $\mathbb{F}_q[T]$ with $\deg a_n \geq 1$ for all $n \geq 0$, which is generated by a primitive morphism σ over a finite set of cardinality $b \geq 2$. Let v be the width of σ. Set $\xi := [0, a_0, a_1, \ldots]$. Then we have*

$$w_2(\xi) \leq 128(4v^{4b-2}b^3 + 1)^3 \operatorname{Dio}(\mathbf{a}) \left(\frac{\log A}{\log q}\right)^4 .$$

Theorem 2.11. *Let* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a non-ultimately and Strumian sequence over* $\mathbb{F}_q[T]$ *with* $\deg a_n \geq 1$ *for all* $n \geq 0$. *Set* $\xi := [0, a_0, a_1, \ldots]$. *Then we have*

$$w_2(\xi) \leq 16000\, \mathrm{Dio}(\mathbf{a}) \left(\frac{\log A}{\log q} \right)^4$$

if the slope of $\mathbf{a}$ *has bounded partial quotients, and we have* $w_2(\xi) = +\infty$ *otherwise.*

3. Liouville inequalities

The following lemma is well-known and immediately seen.

Lemma 3.1. *Let* $P(X)$ *be in* $(\mathbb{F}_q[T])[X]$. *Assume that* $P(X)$ *can be factorized as*

$$P(X) = A \prod_{i=1}^{n} (X - \alpha_i),$$

where $A \in \mathbb{F}_q[T]$ *and* $\alpha_i \in \overline{\mathbb{F}_q(T)}$ *for* $1 \leq i \leq n$. *Then we have*

$$H(P) = |A| \prod_{i=1}^{n} \max(1, |\alpha_i|).$$

Furthermore, for $P(X), Q(X) \in (\mathbb{F}_q[T])[X]$, *we have*

$$H(PQ) = H(P)H(Q).$$

The lemma below is an analogue of Theorem A.1 in [7].

Proposition 3.2. *Let* $P(X), Q(X) \in (\mathbb{F}_q[T])[X]$ *be non-constant polynomials of degree* m, n, *respectively. Let* α *be a root of* $P(X)$ *of order* t *and* β *be a root of* $Q(X)$ *of order* u. *Assume that* $P(\beta) \neq 0$. *Then we have*

$$|P(\beta)| \geq \max(1, |\beta|)^m H(P)^{-n/u+1} H(Q)^{-m/u}. \tag{3.1}$$

Furthermore, we have

$$|\alpha - \beta| \geq \max(1, |\alpha|) \max(1, |\beta|) H(P)^{-n/tu} H(Q)^{-m/tu}. \tag{3.2}$$

Proof. Write $P(X) = A \prod_{i=1}^{r} (X - \alpha_i)^{t_i}$ and $Q(X) = B \prod_{i=1}^{s} (X - \beta_i)^{u_i}$, where $\alpha = \alpha_1, \beta = \beta_1, t = t_1, u = u_1$, and α's (resp. β's) are pairwise distinct. Let $Q_1(X) = B_1 \prod_{i=1}^{s_1} (X - \beta^{(i)})^g$ be the minimal polynomial of β, where $\beta^{(1)} = \beta, g = \mathrm{insep}\,\beta$, and $\beta^{(i)}$'s are pairwise distinct. Since P and Q_1 do not have common roots, the resultant $\mathrm{Res}(P, Q_1)$ is non-zero and is in $\mathbb{F}_q[T]$. Therefore, by $H(Q_1)^{u/g} \leq H(Q)$ and $s_1 u \leq n$, we obtain

$$\begin{aligned} 1 \quad &\leq \quad |\mathrm{Res}(P, Q_1)| = |B_1|^m \prod_{i=1}^{s_1} |P(\beta^{(i)})|^g \\ &\leq \quad |B_1|^m |P(\beta)|^g H(P)^{(s_1-1)g} \prod_{i=2}^{s_1} \max(1, |\beta^{(i)}|)^{mg} \end{aligned}$$

$$
\begin{aligned}
&= |P(\beta)|^g H(P)^{(s_1-1)g}\left(\frac{H(Q_1)}{\max(1,|\beta|)^g}\right)^m \\
&\le |P(\beta)|^g H(P)^{(n/u-1)g} H(Q)^{mg/u}\max(1,|\beta|)^{-mg}.
\end{aligned}
$$

As a result, we have (3.1). From Lemma 3.1, it follows that

$$
\begin{aligned}
|P(\beta)| &\le |\beta-\alpha|^t |A| \max(1,|\beta|)^{m-t}\prod_{i=2}^{r}\max(1,|\alpha_i|)^{t_i} \\
&= |\beta-\alpha|^t H(P)\max(1,|\alpha|)^{-t}\max(1,|\beta|)^{m-t}.
\end{aligned}
$$

Hence, we have (3.2) by (3.1). □

The lemma below is an analogue of Theorem A.3 in [7] and Lemma 2.3 in [22].

Lemma 3.3. *Let $P(X) \in (\mathbb{F}_q[T])[X]$ be an irreducible polynomial of degree $n \ge 2$. For any distinct roots α, β of $P(X)$, we have*

$$
|\alpha-\beta| \ge H(P)^{-n/f^2+1/f}, \tag{3.3}
$$

where f is inseparable degree of $P(X)$.

Proof. We can write $P(X) = A\prod_{i=1}^{m}(X-\alpha_i)^f$, where $\alpha_1 = \alpha, \alpha_2 = \beta$ and α's are pairwise distinct. Put $Q(X) := A\prod_{i=1}^{m}(X-\alpha_i^f)$. Since $Q(X)$ is separable, the discriminant $\mathrm{Disc}(Q)$ is non-zero and is in $\mathbb{F}_q[T]$. Therefore, we obtain

$$
\begin{aligned}
1 &\le |\,\mathrm{Disc}(Q)| \le |\alpha^f-\beta^f|^2|A|^{2m-2}\prod_{\substack{1\le i<j\le m\\(i,j)\ne(1,2)}}\max(1,|\alpha_i^f|)^2\max(1,|\alpha_j^f|)^2 \\
&\le |\alpha-\beta|^{2f}H(Q)^{2m-2}.
\end{aligned}
$$

Hence, we have (3.3) by $H(P) = H(Q)$ and $n = mf$. □

The following proposition is an analogue of Corollary A.2 in [7] and Lemma 2.5 in [22], and is an extension of Theorem 1 in [18].

Proposition 3.4. *Let $\alpha, \beta \in \overline{\mathbb{F}_q(T)}$ be distinct algebraic numbers of degree m, n and inseparable degree f, g, respectively. Then we have*

$$
|\alpha-\beta| \ge \max(1,|\alpha|)\max(1,|\beta|)H(\alpha)^{-n/fg}H(\beta)^{-m/fg}. \tag{3.4}
$$

Proof. From (3.2) and (3.3), the above inequality immediately holds. □

Let $\alpha \in \overline{\mathbb{F}_q(T)}$ be a quadratic number. Then we denote by α' the Galois conjugate of α which is different from α if $\operatorname{insep}\alpha = 1$, and itself if $\operatorname{insep}\alpha = 2$. The lemma below is an analogue of Lemma 3.2 in [22].

Lemma 3.5. *Let $\alpha \in \overline{\mathbb{F}_q(T)}$ be a quadratic number. If $\alpha \ne \alpha'$, then we have*

$$
H(\alpha)^{-1} \le |\alpha-\alpha'| \le H(\alpha). \tag{3.5}
$$

Proof. Let $P_\alpha(X) = AX^2 + BX + C$ be the minimal polynomial of α. Then we have

$$|\alpha - \alpha'| = \frac{|B^2 - 4AC|^{1/2}}{|A|} \leq \max(|B|, |AC|^{1/2}) \leq H(\alpha)$$

and

$$|\alpha - \alpha'| \geq \frac{1}{|A|} \geq H(\alpha)^{-1}.$$

□

We give a better estimate than Proposition 3.4 in some cases, which is an analogue of Lemma 7.1 in [8] and Lemma 4 in [9].

Proposition 3.6. *Let $\alpha, \beta \in \overline{\mathbb{F}_q(T)}$ be quadratic numbers. We denote by $P_\alpha(X) = A(X - \alpha)(X - \alpha')$, $P_\beta(X) = B(X - \beta)(X - \beta')$ the minimal polynomials of α, β, respectively. If $\alpha \neq \alpha'$ and $P_\alpha(X) \neq P_\beta(X)$, then we have*

$$|\alpha - \beta| \geq \max(1, |\alpha - \alpha'|^{-1}) H(\alpha)^{-2} H(\beta)^{-2}. \tag{3.6}$$

Proof. By Proposition 3.4, we may assume that $|\alpha - \alpha'| < 1$. Since $P_\alpha(X)$ and $P_\beta(X)$ does not have common roots, we have

$$\begin{aligned} 1 &\leq |\operatorname{Res}(P_\alpha, P_\beta)| = |B|^2 |P_\alpha(\beta)||P_\alpha(\beta')| \\ &\leq |AB^2||\alpha - \beta||\alpha' - \beta| H(\alpha) \max(1, |\beta'|)^2 \\ &\leq |\alpha - \beta||\alpha' - \beta| H(\alpha)^2 H(\beta)^2. \end{aligned}$$

In the case of $|\alpha' - \beta| > |\alpha - \beta|$, we have $|\alpha - \alpha'| = |\alpha' - \beta|$. Hence, we get (3.6). In other case, using Lemma 3.5, we obtain

$$|\alpha - \beta|^2 \geq |\alpha - \beta||\alpha' - \beta| \geq H(\alpha)^{-2} H(\beta)^{-2} \geq |\alpha - \alpha'|^{-2} H(\alpha)^{-4} H(\beta)^{-4}.$$

Therefore, we have (3.6). □

4. Continued fractions

We collect fundamental properties of continued fractions for Laurent series over a finite field. The lemma below is immediate by induction on n.

Lemma 4.1. *Consider a continued fraction $\xi = [a_0, a_1, a_2, \ldots] \in \mathbb{F}_q((T^{-1}))$. Let $(p_n/q_n)_{n \geq 0}$ be the convergent sequence of ξ. Then the following hold: for any $n \geq 0$,*

(i) $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$,
(ii) $(p_n, q_n) = 1$,
(iii) $|q_n| = |a_1||a_2| \cdots |a_n|$,
(iv) $\xi = \frac{\xi_{n+1} p_n + p_{n-1}}{\xi_{n+1} q_n + q_{n-1}}$, *where* $\xi = [a_0, \ldots, a_n, \xi_{n+1}]$,
(v) $|\xi - p_n/q_n| = |q_n|^{-1}|q_{n+1}|^{-1} = |a_{n+1}|^{-1}|q_n|^{-2}$,
(vi) $q_n/q_{n-1} = [a_n, a_{n-1}, \ldots a_1]$.

We recall an analogue of Lagrange's theorem for Laurent series over a finite field.

Theorem 4.2. *Let ξ be in $\mathbb{F}_q((T^{-1}))$. Then ξ is quadratic if and only if its continued fraction expansion is ultimately periodic.*

Proof. See e.g. [11, Theorems 3 and 4]. □

The lemma below is immediate by Lemma 4.1.

Lemma 4.3. *Consider an ultimately periodic continued fraction*

$$\xi = [0, a_1, \ldots, a_r, \overline{a_{r+1}, \ldots, a_{r+s}}] \in \mathbb{F}_q((T^{-1}))$$

for $r \geq 0, s \geq 1$. Let $(p_n/q_n)_{n\geq 0}$ be the convergent sequence of ξ. Then ξ is a root of the following equation:

$$(q_{r-1}q_{r+s} - q_r q_{r+s-1})X^2 - (q_{r-1}p_{r+s} - q_r p_{r+s-1} + p_{r-1}q_{r+s} - p_r q_{r+s-1})X \\ + p_{r-1}p_{r+s} - p_r p_{r+s-1} = 0,$$

and we have $H(\xi) \leq |q_r q_{r+s}|$. In particular, if $\xi = [0, \overline{a_1, \ldots, a_s}]$, then ξ is a root of the following equation:

$$q_{s-1}X^2 - (p_{s-1} - q_s)X - p_s = 0,$$

and we have $H(\xi) \leq |q_s|$.

Lemma 4.4. *Let $M \geq q$ be an integer and $\xi = [0, a_1, a_2, \ldots], \zeta = [0, b_1, b_2, \ldots] \in \mathbb{F}_q((T^{-1}))$ be continued fractions with $|a_n|, |b_n| \leq M$ for all $n \geq 1$. Assume that there exists an integer $n_0 \geq 1$ such that $a_n = b_n$ for all $1 \leq n \leq n_0$ and $a_{n_0+1} \neq b_{n_0+1}$. Then we have*

$$|\xi - \zeta| \geq \frac{1}{M^2|q_{n_0}|^2}.$$

Proof. See [3, Lemma 3]. □

Lemma 4.5. *For $n \geq 0$, consider an ultimately periodic continued fraction $\xi = [\overline{a_0, a_1, \ldots, a_n}] \in \mathbb{F}_q((T^{-1}))$ with $\deg a_0 \geq 1$. Then we have*

$$-\frac{1}{\xi'} = [\overline{a_n, a_{n-1}, \ldots, a_0}].$$

Proof. See the proof of Lemma 2 in [13]. □

The following lemma is an analogue of Lemma 6.1 in [8].

Lemma 4.6. *For $r, s \geq 1$, consider an ultimately periodic continued fraction $\xi = [0, a_1, \ldots, a_r, \overline{a_{r+1}, \ldots, a_{r+s}}] \in \mathbb{F}_q((T^{-1}))$ with $a_r \neq a_{r+s}$. Let $(p_n/q_n)_{n\geq 0}$ be the convergent sequence of ξ. Then we have*

$$\frac{\min(|a_r|, |a_{r+s}|)}{|q_r|^2} \leq |\xi - \xi'| \leq \frac{|a_r a_{r+s}|}{|q_r|^2}. \tag{4.1}$$

Proof. Put $\tau := [\overline{a_{r+1}, \ldots, a_{r+s}}]$. By Lemma 4.5, we have $\tau' = -[0, \overline{a_{r+s}, \ldots, a_{r+1}}]$. Since

$$\xi = \frac{p_r\tau + p_{r-1}}{q_r\tau + q_{r-1}}, \quad \xi' = \frac{p_r\tau' + p_{r-1}}{q_r\tau' + q_{r-1}},$$

we obtain

$$|\xi - \xi'| = \frac{|\tau - \tau'|}{|q_r\tau + q_{r-1}||q_r\tau' + q_{r-1}|}$$

by Lemma 4.1 (i). We see $|\tau - \tau'| = |a_{r+1}|$ and $|q_r\tau + q_{r-1}| = |q_r||a_{r+1}|$. It follows from Lemma 4.1 (vi) that

$$\begin{aligned} |q_r\tau' + q_{r-1}| &= |q_r|\left|\tau' + \frac{q_{r-1}}{q_r}\right| = \frac{|q_r||[\overline{a_{r+s}, \dots, a_{r+1}}] - [a_r, \dots, a_1]|}{|[\overline{a_{r+s}, \dots, a_{r+1}}]||[a_r, \dots, a_1]|} \\ &= \frac{|q_r||a_{r+s} - a_r|}{|a_r a_{r+s}|}. \end{aligned}$$

Since $1 \le |a_{r+s} - a_r| \le \max(|a_{r+s}|, |a_r|)$, we obtain (4.1). □

The lemma below is an analogue of Lemma 6.3 in [8].

Lemma 4.7. *Let $b, c, d \in \mathbb{F}_q[T]$ be distinct polynomials of degree at least one, $n \ge 1$ be an integer, and $a_1, \dots, a_{n-1} \in \mathbb{F}_q[T]$ be polynomials of degree at least one. Put*

$$\xi := [0, a_1, \dots, a_{n-1}, c, \overline{b}].$$

Then ξ is quadratic and

$$H(\xi) \asymp_{b,c} |q_n|^2,$$

where $(p_k/q_k)_{k\ge0}$ is the convergent sequence of ξ. Let $m \ge 2$ be an integer. Set

$$\zeta := [0, a_1, \dots, a_{n-1}, c, \overline{b, \dots, b, d}],$$

where the length of period part of ζ is m. Then ζ is quadratic and

$$H(\zeta) \asymp_{b,c,d} |\tilde{q}_n \tilde{q}_{n+m}|,$$

where $(\tilde{p}_k/\tilde{q}_k)_{k\ge0}$ is the convergent sequence of ζ.

Proof. It follows from Theorem 4.2 that ξ and ζ are quadratic. By Lemma 4.3, we have $H(\xi) \ll_{b,c} |q_n|^2$ and $H(\zeta) \ll_{b,c,d} |\tilde{q}_n\tilde{q}_{n+m}|$. Let $P_\xi(X) = A(X-\xi)(X-\xi')$ be the minimal polynomial of ξ. Since $P_\xi(p_n/q_n)$ is non-zero, we obtain $|P_\xi(p_n/q_n)| \ge 1/|q_n|^2$. From Lemma 4.1 (v) and 4.6, it follows that

$$\left|\xi - \frac{p_n}{q_n}\right|, \left|\xi' - \frac{p_n}{q_n}\right| \ll_{b,c} \frac{1}{|q_n|^2}.$$

Therefore, we obtain $|q_n|^2 \ll_{b,c} |A| \ll_{b,c} H(\xi)$. We denote by $P_\zeta(X)$ the minimal polynomial of ζ. Since P_ζ and P_ξ do not have a common root, we have

$$1 \le |\operatorname{Res}(P_\zeta, P_\xi)| \le H(\zeta)^2 H(\xi)^2 |\xi - \zeta||\xi' - \zeta||\xi - \zeta'||\xi' - \zeta'|.$$

Note that $q_n = \tilde{q}_n$. By Lemma 4.5, we obtain

$$|\xi - \zeta| \ll_{b,c,d} |\tilde{q}_{n+m}|^{-2}, \quad |\xi' - \zeta|, |\xi - \zeta'|, |\xi' - \zeta'| \ll_{b,c,d} |\tilde{q}_n|^{-2}.$$

Therefore, it follows that $1 \ll_{b,c,d} H(\zeta)^2 H(\xi)^2 |\tilde{q}_n|^{-6} |\tilde{q}_{n+m}|^{-2}$. Hence, we have the inequality $|\tilde{q}_n \tilde{q}_{n+m}| \ll_{b,c,d} H(\zeta)$. □

The next lemma is a well-known result.

Lemma 4.8. *Consider a continued fraction $\xi = [a_0, a_1, a_2, \ldots] \in \mathbb{F}_q((T^{-1}))$. Let $(p_n/q_n)_{n\geq 0}$ be the convergent sequence of ξ. Then we have*

$$w_1(\xi) = \limsup_{n\to\infty} \frac{\deg q_{n+1}}{\deg q_n}.$$

The lemma below is an analogue of Lemma 5.6 in [4].

Lemma 4.9. *Consider a continued fraction $\xi = [a_0, a_1, a_2, \ldots] \in \mathbb{F}_q((T^{-1}))$. Let $(p_n/q_n)_{n\geq 0}$ be the convergent sequence of ξ. If the sequence $(|q_n|^{1/n})_{n\geq 1}$ is bounded, then ξ is not a U_1-number.*

Proof. By the assumption, there exists an integer A such that $q^n \leq |q_n| \leq A^n$ for all $n \geq 1$. Thus, for all $n \geq 1$, we have

$$\frac{\deg q_{n+1}}{\deg q_n} \leq \left(1 + \frac{1}{n}\right) \frac{\log A}{\log q}.$$

By Lemma 4.8, we obtain $w_1(\xi) \leq \log A / \log q$. □

5. Properties of w_n and w_n^*

Theorem 5.1. *Let $n \geq 1$ be an integer and $\xi \in \mathbb{F}_q((T^{-1}))$ be not algebraic of degree at most n. Then we have*

$$w_n(\xi) \geq n, \quad w_n^*(\xi) \geq \frac{n+1}{2}.$$

Furthermore, if $n = 2$, then $w_2^*(\xi) \geq 2$.

Proof. The former estimate follows from an analogue of Minkowski's theorem for Laurent series over a finite field [17] and the later estimates are Satz.1 and Satz.2 of [16]. □

We give an immediate consequence of Propositions 3.2 and 3.4.

Theorem 5.2. *Let $n \geq 1$ be an integer and $\xi \in \mathbb{F}_q((T^{-1}))$ be an algebraic number of degree d. Then we have*

$$w_n(\xi), w_n^*(\xi) \leq d - 1.$$

Note that if $\xi \in \mathbb{F}_q((T^{-1}))$ be an algebraic number, then $\operatorname{insep} \xi = 1$.

We next show that we can replace the definition of w_n by a weak definition of w_n. Let

$n \geq 1$ be an integer and ξ be in $\mathbb{F}_q((T^{-1}))$. We define a Diophantine exponent $\tilde{w}_n$ at ξ by the supremum of a real number w for which there exist infinitely many $P(X) \in (\mathbb{F}_q[T])[X]_{\min}$ of degree at most n such that

$$0 < |P(\xi)| \leq H(P)^{-w}.$$

The lemma below is a slight improvement of a result in [24, Section 3.4].

Lemma 5.3. *Let $n \geq 1$ be an integer and ξ be in $\mathbb{F}_q((T^{-1}))$. Then we have*

$$w_n(\xi) = \tilde{w}_n(\xi).$$

Proof. It is immediate that $\tilde{w}_n(\xi) \leq w_n(\xi)$ and $\tilde{w}_n(\xi) \geq 0$. Therefore, we may assume that $w_n(\xi) > 0$ and $\tilde{w}_n(\xi)$ is finite. For $0 < w < w_n(\xi)$, there exist infinitely many $P(X) \in (\mathbb{F}_q[T])[X]$ of degree at most n such that

$$0 < |P(\xi)| \leq H(P)^{-w}. \tag{5.1}$$

We can write $P(X) = A \prod_{i=1}^{k} P_i(X)$, where $A \in \mathbb{F}_q[T]$ and $P_i(X) \in (\mathbb{F}_q[T])[X]_{\min}$ for $1 \leq i \leq k$. By the definition, for $\tilde{w} > \tilde{w}_n(\xi)$, there exists a positive number C such that for all $Q(X) \in (\mathbb{F}_q[T])[X]_{\min}$ of degree at most n,

$$|Q(\xi)| \geq CH(Q)^{-\tilde{w}}.$$

Therefore, by Lemma 3.1, we obtain

$$|P(\xi)| \geq \min(1, C^n)H(P)^{-\tilde{w}},$$

which implies $\min(1, C^n)H(P)^{w} \leq H(P)^{\tilde{w}}$. Since there exist infinitely many such polynomials $P(X)$, we have $w \leq \tilde{w}$. This completes the proof. □

The lemma below is an analogue of Lemma A.8 in [7] and Lemma 2.4 in [22].

Lemma 5.4. *Let $P(X) \in (\mathbb{F}_q[T])[X]$ be a non-constant irreducible polynomial of degree n and of inseparable degree f. Let ξ be in $\mathbb{F}_q((T^{-1}))$ and α be a root of $P(X)$ such that $|\xi - \alpha|$ is minimal. If $n \geq 2f$, then we have*

$$|\xi - \alpha| \leq |P(\xi)|^{1/f} H(P)^{n/f^2 - 2/f}. \tag{5.2}$$

Proof. We may assume that ξ and α are distinct. We first consider the case of $f = 1$. Write $P(X) = A \prod_{i=1}^{n}(X - \alpha_i)$, where $\alpha = \alpha_1$ and $|\xi - \alpha_1| \leq |\xi - \alpha_2| \leq \ldots \leq |\xi - \alpha_n|$. Put $Q(X) := A \prod_{i=2}^{n}(X - \alpha_i)$ and $\Delta := \prod_{i=2}^{n} |\alpha - \alpha_i|$. Then we have $|\operatorname{Disc}(P)|^{1/2} = \Delta |A| |\operatorname{Disc}(Q)|^{1/2}$. By the definition of discriminant, we obtain

$$\begin{aligned}
|\operatorname{Disc}(Q)|^{1/2} &= |A|^{n-2} |\det(\alpha_i^j)_{2 \leq i \leq n, 0 \leq j \leq n-2}| \\
&\leq |A|^{n-2} \prod_{i=2}^{n} \max(1, |\alpha_i|)^{n-2} \\
&= H(P)^{n-2} \max(1, |\alpha|)^{-n+2}.
\end{aligned}$$

Since the polynomial P is separable, we get

$$\begin{aligned} 1 &\leq |\operatorname{Disc}(P)|^{1/2} \leq H(P)^{n-2}\max(1,|\alpha|)^{-n+2}|A|\prod_{j=2}^{n}|\xi-\alpha_j| \\ &= H(P)^{n-2}\max(1,|\alpha|)^{-n+2}|\xi-\alpha|^{-1}|P(\xi)|. \end{aligned}$$

Therefore, we have (5.2).

We next consider the case of $f > 1$. We can write $P(X) = R(X^f)$, where a separable polynomial $R(X) \in (\mathbb{F}_q[T])[X]$. Thus, in the same way, it follows that

$$|\xi^f - \alpha^f| \leq |R(\xi^f)|H(R)^{n/f-2}.$$

Since $H(P) = H(R)$ and f is a power of p, we have (5.2). □

Lemma 5.5. *Let ξ be in $\mathbb{F}_q((T^{-1}))$ and $n \geq 1$ be an integer. Then we have*

$$w_1(\xi) = w_1(\xi^{p^n}).$$

Proof. By Theorem 5.2, we may assume that ξ is not in $\mathbb{F}_q(T)$. Therefore, we can write $\xi = [a_0, a_1, \ldots]$. Then we have $\xi^{p^n} = [a_0^{p^n}, a_1^{p^n}, \ldots]$ by the Frobenius endmorphism. Hence, it follows from Lemma 4.1 (iii) and 4.8 that

$$w_1(\xi^{p^n}) = \limsup_{k\to\infty} \frac{\sum_{i=1}^{k+1}\deg a_i^{p^n}}{\sum_{i=1}^{k}\deg a_i^{p^n}} = \limsup_{k\to\infty}\frac{p^n \deg q_{k+1}}{p^n \deg q_k} = w_1(\xi),$$

where $(p_k/q_k)_{k\geq 0}$ is the convergent sequence of ξ. □

Proposition 5.6. *Let $n \geq 1$ be an integer and ξ be in $\mathbb{F}_q((T^{-1}))$. Let $k \geq 0$ be an integer such that $p^k \leq n < p^{k+1}$. Then we have*

$$\frac{w_n(\xi)}{p^k} - n + \frac{2}{p^k} - 1 \leq w_n^*(\xi) \leq w_n(\xi). \tag{5.3}$$

Furthermore, if $1 \leq n < 2p$, then we have

$$w_n(\xi) - n + 1 \leq w_n^*(\xi) \leq w_n(\xi). \tag{5.4}$$

Remark 5.7. We are able to define analogues of Diophantine exponents w_n and w_n^* for real numbers and p-adic numbers (see [7, Sections 3.1, 3.3, and 9.3] for the definition of w_n and w_n^*). It is known that for all n, analogues of (5.4) for real numbers and p-adic numbers hold (see [25, 19]). However, in our framework, we are not able to prove (5.4) for all n. The main difficulty is the existence of inseparable irreducible polynomials in $(\mathbb{F}_q[T])[X]$. Therefore, it seems that Proposition 5.6 describes the difference between approximation properties of characteristic zero and that of positive characteristic. On the other hand, when n is sufficiently small, we prove (5.4) using continued fraction theory and the Frobenius endomorphism.

Proof. It is immediate that $w_n(\xi), w_n^*(\xi) \geq 0$. We first show that $w_n^*(\xi) \leq w_n(\xi)$. We may assume that $w_n^*(\xi) > 0$. For $0 < w^* < w_n^*(\xi)$, there exist infinitely many $\alpha \in \overline{\mathbb{F}_q(T)}$ of degree at most n such that

$$0 < |\xi - \alpha| \leq H(\alpha)^{-w^*-1}.$$

Let $P_\alpha(X) = \sum_{i=0}^d a_i X^i$ be the minimal polynomial of α, where $d = \deg \alpha$. Put

$$\begin{aligned} Q_\alpha(X) := a_d X^{d-1} &+ (a_d\alpha + a_{d-1})X^{d-2} + (a_d\alpha^2 + a_{d-1}\alpha + a_{d-2})X^{d-3} \\ &+ \cdots + (a_d\alpha^{d-1} + a_{d-1}\alpha^{d-2} + \cdots + a_1). \end{aligned}$$

Then we have $P_\alpha(X) = (X - \alpha)Q_\alpha(X)$. Since $\max(1, |\alpha|) = \max(1, |\xi|)$, we obtain $|Q_\alpha(\xi)| \leq H(P_\alpha)\max(1, |\xi|)^{2n}$. Hence, it follows that

$$|P_\alpha(\xi)| \leq H(\alpha)^{-w^*} \max(1, |\xi|)^{2n},$$

which gives $w^* \leq w_n(\xi)$. Consequently, we have $w_n^*(\xi) \leq w_n(\xi)$.

Our next claim is that $w_n(\xi)/p^k - n + 2/p^k - 1 \leq w_n^*(\xi)$. We may assume that $w_n(\xi) > 0$. For $0 < w < w_n(\xi)$, there exist infinitely many $P(X) \in (\mathbb{F}_q[T])[X]_{\min}$ of degree at most n such that

$$0 < |P(\xi)| \leq H(P)^{-w}$$

by Lemma 5.3. Let m denote the degree of P and f denote the inseparable degree of P. We first consider the case of $m \geq 2f$. By Lemma 5.4, there exists α of root of P such that

$$|\xi - \alpha| \leq H(\alpha)^{-w/f+m/f^2-2/f} \leq H(\alpha)^{-w/p^k+n-2/p^k}.$$

Now, assume that $m < 2f$. Then we have $m = f$ by $f|m$. Therefore, we can write $P(X) = A(X^m - \alpha^m)$, where $A \in \mathbb{F}_q[T]$ and $\alpha \in \overline{\mathbb{F}_q(T)}$. Thus, we get $|\xi - \alpha| < |A|^{-1/f} H(\alpha)^{-w/f}$. Since $\max(1, |\xi|) = \max(1, |\alpha|)$, we have

$$|\xi - \alpha| \leq \max(1, |\xi|)H(\alpha)^{-w/f-1/f} \leq \max(1, |\xi|)H(\alpha)^{-w/p^k+n-2/p^k}$$

by Lemma 3.1. This is our claim.

Finally, we assume $1 \leq n < 2p$ and show (5.4). Let $0 < w < w_n(\xi)$. If there exist infinitely many separable polynomials $P(X) \in (\mathbb{F}_q[T])[X]_{\min}$ of degree at most n such that

$$0 < |P(\xi)| \leq H(P)^{-w},$$

then we have $w - n + 1 \leq w_n^*(\xi)$ as in the same line of the above proof. Therefore, we may assume that there exist infinitely many inseparable polynomials $P(X) \in (\mathbb{F}_q[T])[X]_{\min}$ of degree at most n such that

$$0 < |P(\xi)| \leq H(P)^{-w}.$$

Then we can write such polynomials $P(X) = AX^p + B$, where $A, B \in \mathbb{F}_q[T]$. By Lemma 5.5 and the definition of w_n, we have $w \leq w_1(\xi)$. Therefore, we obtain $w - n + 1 \leq w_n^*(\xi)$ by $w_1(\xi) = w_1^*(\xi)$. Hence, we have (5.4). □

It follows from Proposition 5.6 that for an integer $n \geq 1$ and $\xi \in \mathbb{F}_q((T^{-1}))$

- $w_n(\xi)$ is finite if and only if $w_n^*(\xi)$ is finite,
- if $w(\xi)$ is finite, then $w^*(\xi)$ is finite.

Consequently, we obtain

- ξ is a U_n-number if and only if it is a U_n^*-number,
- if ξ is an S-number, then it is an S^*-number.

We address the following questions in the last of this section.

Problem 5.8. *Does (5.4) hold for all $n \geq 1$ and $\xi \in \mathbb{F}_q((T^{-1}))$?*

Problem 5.9. *Does Mahler's classification coincide Koksma's classification?*

Note that an analogue of Problem 5.9 for real numbers and p-adic numbers holds. The detail is found in [7, Sections 3.4 and 9.3], [22, Chapter 6], and [23].

6. Applications of Liouville inequalities

The following proposition is an analogue of Lemma 7.3 in [8] and Lemma 5 in [9].

Proposition 6.1. *Let ξ be in $\mathbb{F}_q((T^{-1}))$ and $c_1, c_2, c_3, c_4, \theta, \rho, \delta$ be positive numbers. Let ε be a non-negative number. Assume that there exists a sequence of distinct terms $(\alpha_j)_{j\geq 1}$ with $\alpha_j \in \overline{\mathbb{F}_q(T)}$ is quadratic for $j \geq 1$ such that for all $j \geq 1$*

$$\frac{c_1}{H(\alpha_j)^{1+\rho}} \leq |\xi - \alpha_j| \leq \frac{c_2}{H(\alpha_j)^{1+\delta}},$$
$$H(\alpha_j) \leq H(\alpha_{j+1}) \leq c_3 H(\alpha_j)^{\theta},$$
$$0 < |\alpha_j - \alpha_j'| \leq \frac{c_4}{H(\alpha_j)^{\varepsilon}}.$$

If $(\rho-1)(\delta-1+\varepsilon) \geq 2\theta(2-\varepsilon)$, then we have $\delta \leq w_2^*(\xi) \leq \rho$. Furthermore, if $(\delta-1)(\delta-1+\varepsilon) \geq 2\theta(2-\varepsilon)$, then we have

$$\delta \leq w_2^*(\xi) \leq \rho, \quad w_2(\xi) \geq w_2^*(\xi) + \varepsilon.$$

Finally, assume that there exists a non-negative number χ such that

$$\limsup_{j\to\infty} \frac{-\log|\alpha_j - \alpha_j'|}{\log H(\alpha_j)} \leq \chi.$$

If $(\delta-2+\chi)(\delta-1+\varepsilon) \geq 2\theta(2-\varepsilon)$ when $p \neq 2$ and $(\delta-4+\chi)(\delta-1+\varepsilon) \geq 4\theta(2-\varepsilon)$ when $p = 2$, then we have

$$\delta \leq w_2^*(\xi) \leq \rho, \quad \varepsilon \leq w_2(\xi) - w_2^*(\xi) \leq \chi.$$

Proof. Assume that $(\rho-1)(\delta-1+\varepsilon) \geq 2\theta(2-\varepsilon)$. By the assumption, we have $\theta \geq 1, \rho > 1$, and $\delta + \varepsilon > 1$. Let $\alpha \in \overline{\mathbb{F}_q(T)}$ be an algebraic number of degree at most two with sufficiently large height and $\alpha \notin \{\alpha_j \mid j \geq 1\}$. We define an integer $j_0 \geq 1$ by $H(\alpha_{j_0}) \leq c_3\{(c_2c_4)^{\frac{1}{2}} H(\alpha)\}^{\frac{2\theta}{\delta+\varepsilon-1}} < H(\alpha_{j_0+1})$. Then, by the assumption, we have

$$H(\alpha) < c_3^{-\frac{\delta+\varepsilon-1}{2\theta}} (c_2c_4)^{-\frac{1}{2}} H(\alpha_{j_0+1})^{\frac{\delta+\varepsilon-1}{2\theta}} \leq (c_2c_4)^{-\frac{1}{2}} H(\alpha_{j_0})^{\frac{\delta+\varepsilon-1}{2}}.$$

Hence, it follows from Propositions 3.4, 3.6, and Lemma 3.5 that

$$\begin{aligned}|\alpha - \alpha_{j_0}| &\geq |\alpha_{j_0} - \alpha'_{j_0}|^{-1} H(\alpha_{j_0})^{-2} H(\alpha)^{-2} \\ &> c_2 H(\alpha_{j_0})^{-\delta-1} \geq |\xi - \alpha_{j_0}|.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}|\xi - \alpha| &= |\alpha - \alpha_{j_0}| > c_4^{-1} H(\alpha_{j_0})^{-2+\varepsilon} H(\alpha)^{-2} \\ &\geq c_2^{-\frac{\theta(2-\varepsilon)}{\delta+\varepsilon-1}} c_3^{-2+\varepsilon} c_4^{-1-\frac{2\theta(2-\varepsilon)}{\delta+\varepsilon-1}} H(\alpha)^{-2-\frac{2\theta(2-\varepsilon)}{\delta+\varepsilon-1}}.\end{aligned} \tag{6.1}$$

By the assumption, we have

$$\delta \leq w_2^*(\xi) \leq \max\left(\rho, 1 + \frac{2\theta(2-\varepsilon)}{\delta+\varepsilon-1}\right) = \rho.$$

We next assume that $(\delta - 1)(\delta - 1 + \varepsilon) \geq 2\theta(2 - \varepsilon)$. By (6.1), it follows that

$$|\xi - \alpha| \geq c_2^{-\frac{\theta(2-\varepsilon)}{\delta+\varepsilon-1}} c_3^{-2+\varepsilon} c_4^{-1-\frac{2\theta(2-\varepsilon)}{\delta+\varepsilon-1}} H(\alpha)^{-\delta-1}.$$

Therefore, the sequence $(\alpha_j)_{j\geq 1}$ is the best algebraic approximation to ξ of degree at most two, that is,

$$w_2^*(\xi) = \limsup_{j\to\infty} \frac{-\log|\xi - \alpha_j|}{\log H(\alpha_j)} - 1.$$

We denote by $P_j(X) = A_j(X - \alpha_j)(X - \alpha'_j)$ the minimal polynomial of α_j. By the assumption, we have

$$|P_j(\xi)| \leq \max(c_2, c_4) H(\alpha_j)^{-\varepsilon+1} |\xi - \alpha_j|.$$

Therefore, we obtain

$$w_2^*(\xi) + \varepsilon \leq \limsup_{n\to\infty} \frac{-\log|P_j(\xi)|}{H(P_j)} \leq w_2(\xi).$$

Finally, assume that

$$\limsup_{j\to\infty} \frac{\log|\alpha_j - \alpha'_j|}{\log H(\alpha_j)} = -\chi.$$

We also assume that $(\delta - 2 + \chi)(\delta - 1 + \varepsilon) \geq 2\theta(2 - \varepsilon)$ when $p \neq 2$ and $(\delta - 4 + \chi)(\delta - 1 + \varepsilon) \geq 4\theta(2 - \varepsilon)$ when $p = 2$. Since $|\xi - \alpha_j| \leq c_2$ and $|\alpha_j - \alpha'_j| \leq c_4$, we have

$$\max(1, |\alpha_j|), \max(1, |\alpha'_j|) \leq \max(1, c_2, c_4, |\xi|),$$

which implies $H(P_j) \leq |A_j| \max(1, c_2, c_4, |\xi|)^2$. By the assumption, we get $|\alpha_j - \alpha'_j| < |\xi - \alpha'_j|$ for sufficiently large j. Therefore, we obtain for sufficiently large j

$$|P_j(\xi)| \geq \max(1, c_2, c_4, |\xi|)^{-2} H(P_j) |\xi - \alpha_j| |\alpha_j - \alpha'_j|.$$

Taking a logarithm and a limit superior, we have

$$\limsup_{j\to\infty} \frac{-\log|P_j(\xi)|}{\log H(P_j)} \leq w_2^*(\xi) + \chi.$$

Let $P(X) \in (\mathbb{F}_q[T])[X]_{\min}$ be a polynomial of degree at most two with sufficiently large height such that $P(\alpha_j) \neq 0$ for all $j \geq 1$. When $\deg_X P = 1$, we can write $P(X) = A(X - \alpha)$ and have

$$|P(\xi)| = |A||\xi - \alpha| \geq c_2^{-\frac{\theta(2-\varepsilon)}{\delta+\varepsilon-1}} c_3^{-2+\varepsilon} c_4^{-1-\frac{2\theta(2-\varepsilon)}{\delta+\varepsilon-1}} H(P)^{-\delta-\chi}$$

by (6.1). When $\deg_X P = 2$, we can write $P(X) = A(X - \alpha)(X - \alpha')$ and assume that $|\xi - \alpha| \leq |\xi - \alpha'|$. we first consider the case of $p \neq 2$. Then we obtain

$$\begin{aligned} |P(\xi)| &\geq |A(\alpha - \alpha')||\xi - \alpha| = |\operatorname{Disc}(P)|^{1/2}|\xi - \alpha| \\ &\geq c_2^{-\frac{\theta(2-\varepsilon)}{\delta+\varepsilon-1}} c_3^{-2+\varepsilon} c_4^{-1-\frac{2\theta(2-\varepsilon)}{\delta+\varepsilon-1}} H(P)^{-\delta-\chi} \end{aligned} \tag{6.2}$$

by (6.1). We next consider the case of $p = 2$. If $\alpha \neq \alpha'$, then we have (6.2) and if otherwise, then we have

$$\begin{aligned} |P(\xi)| &\geq |A||\xi - \alpha|^2 \geq c_2^{-\frac{2\theta(2-\varepsilon)}{\delta+\varepsilon-1}} c_3^{-4+2\varepsilon} c_4^{-2-\frac{4\theta(2-\varepsilon)}{\delta+\varepsilon-1}} H(P)^{-4-\frac{4\theta(2-\varepsilon)}{\delta+\varepsilon-1}} \\ &\geq c_2^{-\frac{2\theta(2-\varepsilon)}{\delta+\varepsilon-1}} c_3^{-4+2\varepsilon} c_4^{-2-\frac{4\theta(2-\varepsilon)}{\delta+\varepsilon-1}} H(P)^{-\delta-\chi} \end{aligned}$$

by (6.1). Thus, it follows that $w_2(\xi) \leq w_2^*(\xi) + \chi$ by Lemma 5.3. □

The following proposition is an analogue of Lemma 7.2 in [8].

Proposition 6.2. *Let ξ be in $\mathbb{F}_q((T^{-1}))$. Let $c_0, c_1, c_2, c_3, \theta, \rho, \delta$ be positive numbers and $(\beta_j)_{j\geq 1}$ be a sequence of positive integers such that $\beta_j < \beta_{j+1} \leq c_0\beta_j^\theta$ for all $j \geq 1$. Assume that there exists a sequence of distinct terms $(\alpha_j)_{j\geq 1}$ with $\alpha_j \in \overline{\mathbb{F}_q(T)}$ is quadratic for $j \geq 1$ such that for all $j \geq 1$*

$$\frac{c_1}{\beta_j^{2+\rho}} \leq |\xi - \alpha_j| \leq \frac{c_2 \max(1, |\alpha_j - \alpha_j'|^{-1})}{\beta_j^{2+\delta}},$$

$$H(\alpha_j) \leq c_3\beta_j, \quad \alpha_j \neq \alpha_j'.$$

Then we have

$$1 + \delta \leq w_2^*(\xi) \leq (2 + \rho)\frac{2\theta}{\delta} - 1.$$

Proof. Let $\alpha \in \overline{\mathbb{F}_q(T)}$ be an algebraic number of degree at most two with sufficiently large height. We define an integer $j_0 \geq 1$ by $\beta_{j_0} \leq c_0 c_2^{\frac{\theta}{\delta}} (c_3 H(\alpha))^{\frac{2\theta}{\delta}} < \beta_{j_0+1}$. We first consider the case of $\alpha = \alpha_{j_0}$. By the assumption, we have

$$|\xi - \alpha| \geq c_1\beta_{j_0}^{-2-\rho} \geq c_0^{-2-\rho} c_1 c_2^{-(2+\rho)\frac{\theta}{\delta}} c_3^{-(2+\rho)\frac{2\theta}{\delta}} H(\alpha)^{-(2+\rho)\frac{2\theta}{\delta}}.$$

We next consider the other case. Then, by the assumption, we have

$$H(\alpha) < c_2^{-\frac{1}{2}} c_3^{-1} (c_0^{-1}\beta_{j_0+1})^{\frac{\delta}{2\theta}} \leq c_2^{-\frac{1}{2}} c_3^{-1} \beta_{j_0}^{\frac{\delta}{2}}.$$

Hence, it follows from Propositions 3.4, 3.6, and Lemma 3.5 that

$$\begin{aligned} |\alpha - \alpha_{j_0}| &\geq \max(1, |\alpha_{j_0} - \alpha'_{j_0}|^{-1}) H(\alpha_{j_0})^{-2} H(\alpha)^{-2} \\ &> c_2 \max(1, |\alpha_{j_0} - \alpha'_{j_0}|^{-1}) \beta_{j_0}^{-2-\delta} \geq |\xi - \alpha_{j_0}|. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} |\xi - \alpha| &= |\alpha - \alpha_{j_0}| \geq \max(1, |\alpha_{j_0} - \alpha'_{j_0}|^{-1}) H(\alpha_{j_0})^{-2} H(\alpha)^{-2} \\ &\geq c_3^{-2} \beta_{j_0}^{-2} H(\alpha)^{-2} \geq c_0^{-2} c_2^{-\frac{2\theta}{\delta}} c_3^{-2-\frac{4\theta}{\delta}} H(\alpha)^{-2-\frac{4\theta}{\delta}}. \end{aligned}$$

By the assumption, we have

$$1 + \delta \leq w_2^*(\xi) \leq \max\left(1 + \frac{4\theta}{\delta}, (2+\rho)\frac{2\theta}{\delta} - 1\right) = (2+\rho)\frac{2\theta}{\delta} - 1.$$

□

7. Combinational lemma

The lemma below is a slight improvement of [8, Lemma 9.1].

Lemma 7.1. *Let* $\mathbf{a} = (a_n)_{n\geq 1}$ *be a sequence on a finite set* $\mathcal{A}$. *Assume that there exist integers* $\kappa \geq 2$ *and* $n_0 \geq 1$ *such that for all* $n \geq n_0$,

$$p(\mathbf{a}, n) \leq \kappa n.$$

Then, for each $n \geq n_0$, *there exist finite words* U_n, V_n *and a positive rational number* w_n *such that the following hold:*

(i) $U_n V_n^{w_n}$ *is a prefix of* $\mathbf{a}$,
(ii) $|U_n| \leq 2\kappa |V_n|$,
(iii) $n/2 \leq |V_n| \leq \kappa n$,
(iv) *if* U_n *is not an empty word, then the last letters of* U_n *and* V_n *are different,*
(v) $|U_n V_n^{w_n}|/|U_n V_n| \geq 1 + 1/(4\kappa + 2)$,
(vi) $|U_n V_n| \leq (\kappa + 1)n - 1$,
(vii) $|U_n^2 V_n| \leq (2\kappa + 1)n - 2$.

Proof. For $n \geq 1$, we denote by $A(n)$ the prefix of $\mathbf{a}$ of length n. By Pigeonhole principle, for each $n \geq n_0$, there exists a finite word W_n of length n such that the word appears to $A((\kappa+1)n)$ at least twice. Therefore, for each $n \geq n_0$, there exist finite words $B_n, D_n, E_n \in \mathcal{A}^*$ and $C_n \in \mathcal{A}^+$ such that

$$A((\kappa + 1)n) = B_n W_n D_n E_n = B_n C_n W_n E_n.$$

We can take these words in such way that if B_n is not empty, then the last letter of B_n is different from that of C_n. Firstly, we consider the case of $|C_n| \geq |W_n|$. Then, there exists $F_n \in \mathcal{A}^*$ such that

$$A((\kappa+1)n) = B_n W_n F_n W_n E_n.$$

Put $U_n := B_n, V_n := W_n F_n$, and $w_n := |W_n F_n W_n|/|W_n F_n|$. Since $U_n V_n^{w_n} = B_n W_n F_n W_n$, the word $U_n V_n^{w_n}$ is a prefix of $\mathbf{a}$. It is obvious that $|U_n| \leq (\kappa-1)|V_n|$ and $n \leq |V_n| \leq \kappa n$. By the definition, we have (iv) and (vi). Furthermore, we see that

$$\frac{|U_n V_n^{w_n}|}{|U_n V_n|} = 1 + \frac{n}{|U_n V_n|} \geq 1 + \frac{1}{\kappa},$$
$$|U_n^2 V_n| \leq |U_n V_n| + |U_n| \leq \kappa n + (\kappa-1)n = (2\kappa-1)n.$$

We next consider the case of $|C_n| < |W_n|$. Since the two occurrences of W_n do overlap, there exists a rational number $d_n > 1$ such that $W_n = C_n^{d_n}$. Put $U_n := B_n, V_n := C_n^{\lceil d_n/2 \rceil}$, and $w_n := (d_n+1)/\lceil d_n/2 \rceil$. Obviously, we have (i) and (iv). Since $\lceil d_n/2 \rceil \leq d_n$ and $d_n|C_n| \leq 2\lceil d_n/2 \rceil |C_n|$, we get $n/2 \leq |V_n| \leq n$. Using (iii) and $|U_n| \leq \kappa n - 1$, we can see (ii), (vi), and (vii). It is immediate that $w_n \geq 3/2$. Hence, we obtain

$$\begin{aligned}
\frac{|U_n V_n^{w_n}|}{|U_n V_n|} &= 1 + \frac{\lceil (w_n-1)|V_n| \rceil}{|U_n V_n|} \geq 1 + \frac{w_n - 1}{|U_n|/|V_n| + 1} \\
&\geq 1 + \frac{1/2}{2\kappa+1} = 1 + \frac{1}{4\kappa+2}.
\end{aligned}$$

□

8. Proof of the main results

Proof of Theorem 1.1. Put

$$\xi_{w,j} := [0, a_{1,w}, \ldots, a_{\lfloor w^j \rfloor, w}, \overline{b}] \quad \text{for } j \geq 1.$$

Since ξ_w and $\xi_{w,j}$ have the same first $(\lfloor w^{j+1} \rfloor - 1)$-th partial quotients, while $\lfloor w^{j+1} \rfloor$-th partial quotients are different, we have

$$|\xi_w - \xi_{w,j}| \asymp |q_{\lfloor w^{j+1} \rfloor}|^{-2}$$

by Lemmas 4.1 (v) and 4.4. Let $0 < \iota < w$ be a real number such that $(w-\iota-2)(w-\iota-1) \geq 2(w+\iota)$ when $p \neq 2$, and $(w-\iota-4)(w-\iota-1) \geq 4(w+\iota)$ when $p = 2$. It is obvious that

$$|q_{\lfloor w^j \rfloor}|^{w-\iota} \ll |q_{\lfloor w^{j+1} \rfloor}| \ll |q_{\lfloor w^j \rfloor}|^{w+\iota}$$

for sufficiently large j by Lemma 4.1 (iii). Thus, we have

$$H(\xi_{w,j})^{-w-\iota} \ll |\xi_w - \xi_{w,j}| \ll H(\xi_{w,j})^{-w+\iota}$$

for sufficiently large j by Lemma 4.7. It follows from Lemmas 4.6 and 4.7 that

$$|\xi_{w,j} - \xi'_{w,j}| \asymp H(\xi_{w,j})^{-1}.$$

For sufficiently large j, we see that

$$H(\xi_{w,j}) \leq H(\xi_{w,j+1}) \ll H(\xi_{w,j})^{w+\iota}.$$

It follows from Proposition 6.1 that $w_2^*(\xi_w) \in [w-\iota-1, w+\iota-1]$ and $w_2(\xi_w) - w_2^*(\xi) = 1$. Since ι is arbitrary, we have $w_2^*(\xi) = w-1$ and $w_2(\xi) = w$. □

Proof of Theorem 1.2. Put

$$\xi_{w,\eta,j} := [0, a_{1,w,\eta}, \dots, a_{\lfloor w^j \rfloor, w,\eta}, \overline{b, \dots, b, d}] \quad \text{for } j \geq 1,$$

where the length of period part is $\lfloor \eta w^j \rfloor$. Since $\lfloor w^j \rfloor + (m_j+1)\lfloor \eta w^j \rfloor > \lfloor w^{j+1} \rfloor$, it follows that $\xi_{w,\eta}$ and $\xi_{w,\eta,j}$ have the same first $(\lfloor w^{j+1} \rfloor - 1)$-th partial quotients, while $\lfloor w^{j+1} \rfloor$-th partial quotients are different. Thus, we have

$$|\xi_{w,\eta} - \xi_{w,\eta,j}| \asymp |q_{\lfloor w^{j+1} \rfloor}|^{-2} \quad \text{for } j \geq 1$$

by Lemmas 4.1 (v) and 4.4. We see that for $j \geq 1$

$$|\xi_{w,\eta,j} - \xi'_{w,\eta,j}| \asymp |q_{\lfloor w^j \rfloor}|^{-2}, \quad H(\xi_{w,\eta,j}) \asymp |q_{\lfloor w^j \rfloor} q_{\lfloor w^j \rfloor + \lfloor \eta w^j \rfloor}|$$

by Lemmas 4.6 and 4.7. Let $0 < \iota < \min\{w, 2+\eta\}$ be a real number such that

$$\begin{aligned}\left(\frac{2(w-\iota)}{2+\eta+\iota} - 3 + \frac{2}{2+\eta}\right)&\left(\frac{2(w-\iota)}{2+\eta+\iota} - 2 + \frac{2}{2+\eta+\iota}\right)\\ &\geq 2(w+\iota)\frac{2+\eta+\iota}{2+\eta-\iota}\left(2 - \frac{2}{2+\eta+\iota}\right)\end{aligned}$$

when $p \neq 2$, and

$$\begin{aligned}\left(\frac{2(w-\iota)}{2+\eta+\iota} - 5 + \frac{2}{2+\eta}\right)&\left(\frac{2(w-\iota)}{2+\eta+\iota} - 2 + \frac{2}{2+\eta+\iota}\right)\\ &\geq 4(w+\iota)\frac{2+\eta+\iota}{2+\eta-\iota}\left(2 - \frac{2}{2+\eta+\iota}\right)\end{aligned}$$

when $p = 2$. It is obvious that

$$\begin{aligned}|q_{\lfloor w^j \rfloor}|^{w-\iota} \ll |q_{\lfloor w^{j+1} \rfloor}| \ll |q_{\lfloor w^j \rfloor}|^{w+\iota},\\ |q_{\lfloor w^j \rfloor}|^{1+\eta-\iota} \ll |q_{\lfloor w^j \rfloor + \lfloor \eta w^j \rfloor}| \ll |q_{\lfloor w^j \rfloor}|^{1+\eta+\iota}\end{aligned}$$

for sufficiently large j. Hence, we obtain

$$\begin{aligned}H(\xi_{w,\eta,j})^{-\frac{2(w+\iota)}{2+\eta-\iota}} \ll |\xi_{w,\eta} - \xi_{w,\eta,j}| \ll H(\xi_{w,\eta,j})^{-\frac{2(w-\iota)}{2+\eta+\iota}},\\ H(\xi_{w,\eta,j})^{-\frac{2}{2+\eta-\iota}} \ll |\xi_{w,\eta,j} - \xi'_{w,\eta,j}| \ll H(\xi_{w,\eta,j})^{-\frac{2}{2+\eta+\iota}},\\ H(\xi_{w,\eta,j}) \leq H(\xi_{w,\eta,j+1}) \ll H(\xi_{w,\eta,j})^{(w+\iota)\frac{2+\eta+\iota}{2+\eta-\iota}}\end{aligned}$$

for sufficiently large j. It follows from Proposition 6.1 that

$$w_2^*(\xi_{w,\eta}) \in \left[\frac{2w-2-\eta-3\iota}{2+\eta+\iota}, \frac{2w-2-\eta+3\iota}{2+\eta-\iota}\right],$$

$$w_2(\xi_{w,\eta}) - w_2^*(\xi_{w,\eta}) \in \left[\frac{2}{2+\eta+\iota}, \frac{2}{2+\eta}\right].$$

Since ι is arbitrary, we have

$$w_2^*(\xi_{w,\eta}) = \frac{2w-2-\eta}{2+\eta}, \quad w_2(\xi_{w,\eta}) = \frac{2w-\eta}{2+\eta}.$$

□

Proof of Theorem 1.4. Applying Lemma 7.1, for $n \geq n_0$, we take finite words U_n, V_n and a rational number w_n satisfying Lemma 7.1 (i)-(v) and (vii). We define a positive integer sequence $(n_j)_{j\geq 0}$ by $n_{j+1} = 2(2\kappa+1)\lceil \log A/\log q\rceil n_j$ for $j \geq 0$. Put $r_j := |U_{n_j}|, s_j := |V_{n_j}|$, and $\tilde{w}_j := w_{n_j}$ for $j \geq 0$. By Lemma 7.1 (iv), we have $a_{r_j} \neq a_{r_j+s_j}$ for all $j \geq 0$. By the assumption and Lemma 4.1 (iii), we get $q^n \leq |q_n| \leq A^n$ for all $n \geq 1$. Therefore, it follows from Lemma 7.1 (iii) and (vi) that for $j \geq 0$

$$|q_{r_j}q_{r_j+s_j}| < |q_{r_{j+1}}q_{r_{j+1}s_{j+1}}| \leq |q_{r_j}q_{r_j+s_j}|^{4(2\kappa+1)^2\lceil\frac{\log A}{\log q}\rceil\frac{\log A}{\log q}}. \tag{8.1}$$

Put $\alpha_j := [0, a_1, \ldots, a_{r_j}, \overline{a_{r_j+1}, \ldots, a_{r_j+s_j}}]$ for $j \geq 1$. By Lemma 4.3, we obtain

$$H(\alpha_j) \leq |q_{r_j}q_{r_j+s_j}| \tag{8.2}$$

for $j \geq 0$. Since ξ and α_j have the same first $r_j + \lceil \tilde{w}_j s_j\rceil$-th partial quotients, we have

$$\begin{aligned} |\xi - \alpha_j| &\leq \max\left(\left|\xi - \frac{p_{r_j+\lceil\tilde{w}_js_j\rceil}}{q_{r_j+\lceil\tilde{w}_js_j\rceil}}\right|, \left|\alpha_j - \frac{p_{r_j+\lceil\tilde{w}_js_j\rceil}}{q_{r_j+\lceil\tilde{w}_js_j\rceil}}\right|\right) \\ &\leq |q_{r_j+\lceil\tilde{w}_js_j\rceil}|^{-2}q^{-1} \leq |q_{r_j+s_j}|^{-2}q^{-2(\lceil\tilde{w}_js_j\rceil-s_j)-1} \end{aligned}$$

for $j \geq 0$ by Lemma 4.1 (iii) and (v). By Lemma 7.1 (v), we have

$$q^{2(\lceil\tilde{w}_js_j\rceil-s_j)+1} \gg q^{\frac{r_j+s_j}{2\kappa+1}}|q_{r_j}q_{r_j+s_j}|^{\frac{\log q}{(4\kappa+2)\log A}}$$

for $j \geq 0$. From Lemma 4.6, we deduce that

$$|q_{r_j}|^2 \leq A^2 \max(|\alpha_j - \alpha_j'|^{-1}, 1)$$

for $j \geq 0$. Hence, we obtain

$$|\xi - \alpha_j| \ll A^2 \max(|\alpha_j - \alpha_j'|^{-1}, 1)|q_{r_j}q_{r_j+s_j}|^{-2-\frac{\log q}{(4\kappa+2)\log A}} \tag{8.3}$$

for $j \geq 0$. Take a real number δ which is greater than $\mathrm{Dio}(\mathbf{a})$. Then ξ and α_j have the same at most $\lceil\delta(r_j+s_j)\rceil$-th partial quotients for sufficiently large j. By Lemma 4.4, we have

$$\begin{aligned} |\xi - \alpha_j| &\geq A^{-2}|q_{\lceil\delta(r_j+s_j)\rceil}|^{-2} \gg |q_{r_j}q_{r_j+s_j}|^{-4\delta\frac{(r_j+s_j)\log A}{(2r_j+s_j)\log q}} \\ &\gg |q_{r_j}q_{r_j+s_j}|^{-4\delta\frac{\log A}{\log q}} \end{aligned} \tag{8.4}$$

for sufficiently large j. Applying Proposition 6.2 with (8.1), (8.2), (8.3), and (8.4), we obtain

$$w_2^*(\xi) \leq 128(2\kappa+1)^3 \operatorname{Dio}(\mathbf{a}) \left(\frac{\log A}{\log q} \right)^4 - 1.$$

Thus, we have (1.1) by (5.4).

Assume that the sequence $(|q_n|^{1/n})_{n\geq 1}$ converges. Let M be a limit of the sequence $(|q_n|^{1/n})_{n\geq 1}$. For any $\varepsilon > 0$, there exists an integer n_1 such that for all $n \geq n_1$,

$$(M-\varepsilon)^n < |q_n| < (M+\varepsilon)^n.$$

In the same matter as above, we see that

$$|q_{r_j} q_{r_j+s_j}| < |q_{r_{j+1}} q_{r_{j+1} s_{j+1}}| \leq |q_{r_j} q_{r_j+s_j}|^{4(2\kappa+1)^2 \left\lceil \frac{\log(M+\varepsilon)}{\log(M-\varepsilon)} \right\rceil \frac{\log(M+\varepsilon)}{\log(M-\varepsilon)}},$$
$$|q_{r_j} q_{r_j+s_j}|^{-4\delta \frac{\log(M+\varepsilon)}{\log(M-\varepsilon)}} \ll |\xi - \alpha_j| \ll \max(|\alpha_j - \alpha'_j|^{-1}, 1) |q_{r_j} q_{r_j+s_j}|^{-2-\frac{\log(M-\varepsilon)}{(4\kappa+2)\log(M+\varepsilon)}},$$

for sufficiently large j. Applying Proposition 6.2, we have

$$w_2^*(\xi) \leq 64(2\kappa+1)^3 \operatorname{Dio}(\mathbf{a}) - 1.$$

Thus, we have (1.2) by (5.4). □

Proof of Theorem 1.5. From Theorems 4.2, 5.1, and Proposition 5.6, we have $w_2(\xi) \geq w_2^*(\xi) \geq 2$. Without loss of generality, we may assume that $\operatorname{Dio}(\mathbf{a}) > 1$. Take a real number δ such that $1 < \delta < \operatorname{Dio}(\mathbf{a})$. For $n \geq 1$, there exist finite words U_n, V_n and a real number w_n such that $U_n V_n^{w_n}$ is the prefix of $\mathbf{a}$, the sequence $(|V_n^{w_n}|)_{n\geq 1}$ is strictly increasing, and $|U_n V_n^{w_n}| \geq \delta |U_n V_n|$. Set $r_n := |U_n|$, $s_n := |V_n|$, and $\alpha_n := [0, a_1, \dots, a_{r_n}, \overline{a_{r_n+1}, \dots, a_{r_n+s_n}}]$. Let $\tilde{M}$ denote an upper bound of $(|q_n|^{1/n})_{n\geq 1}$. For any $\varepsilon > 0$, there exists an integer n_0 such that for all $n \geq n_0$,

$$(m-\varepsilon)^n < |q_n| < (M+\varepsilon)^n.$$

Since ξ and α_n have the same first $(r_n + \lceil w_n s_n \rceil)$-th partial quotients, we obtain

$$|\xi - \alpha_n| \leq |q_{r_n + \lceil w_n s_n \rceil}|^{-2} < (M+\varepsilon)^{-2(r_n + \lceil w_n s_n \rceil) \frac{\log(m-\varepsilon)}{\log(M+\varepsilon)}}.$$

Assume that the sequences $(r_n)_{n\geq 1}$ and $(s_n)_{n\geq 1}$ are bounded. Then, for all $n \geq 1$, we have

$$H(\alpha_n) \leq |q_{r_n} q_{r_n+s_n}| \leq \tilde{M}^{2r_n+s_n} \leq C,$$

where C is some constant, by Lemma 4.3. Therefore, the set $\{\alpha_n \mid n \geq 1\}$ is finite. Take a positive integer sequence $(n_i)_{i\geq 1}$ such that $n_i \to \infty$ as $i \to \infty$ and $\alpha_{n_1} = \alpha_{n_2} = \cdots$. Since $(s_n)_{n\geq 1}$ is bounded, we have $w_n \to \infty$ as $n \to \infty$. Hence, we obtain $\mathbf{a} = U_{n_i} \overline{V_{n_i}}$, which is a contradiction.

We next consider the case that $(r_n)_{n\geq 1}$ is unbounded. Here, if necessary, taking a subsequence of $(r_n)_{n\geq 1}$, we assume that $(r_n)_{n\geq 1}$ is increasing and $r_1 \geq n_0$. Since $H(\alpha_n) \leq (M+\varepsilon)^{2r_n+s_n}$ by Lemma 4.3, we have

$$|\xi - \alpha_n| \leq H(\alpha_n)^{-\frac{r_n + \lceil w_n s_n \rceil}{r_n + s_n} \frac{\log(m-\varepsilon)}{\log(M+\varepsilon)}} \leq H(\alpha_n)^{-\delta \frac{\log(m-\varepsilon)}{\log(M+\varepsilon)}}.$$

Hence, we obtain (1.3).

We consider the case that $(r_n)_{n\geq 1}$ is bounded, $(s_n)_{n\geq 1}$ is unbounded, and $\mathrm{Dio}(\mathbf{a})$ is finite. Here, if necessary, taking a subsequence of $(s_n)_{n\geq 1}$, we assume that $(s_n)_{n\geq 1}$ is increasing and $s_1 \geq n_0$. Then, for all $n \geq 1$, we have

$$H(\alpha_n) \leq \tilde{M}^{r_n}(M+\varepsilon)^{r_n+s_n} < C_1(M+\varepsilon)^{r_n+s_n},$$

where C_1 is some constant. Therefore, we obtain

$$|\xi - \alpha_n| \quad \leq \quad (C_1 H(\alpha_n)^{-1})^{2\frac{r_n+\lceil w_n s_n\rceil}{r_n+s_n}\frac{\log(m-\varepsilon)}{\log(M+\varepsilon)}} \leq C_1^{2\,\mathrm{Dio}(\mathbf{a})} H(\alpha_n)^{-2\delta\frac{\log(m-\varepsilon)}{\log(M+\varepsilon)}}.$$

Hence, we obtain (1.3).

We consider the case that $(r_n)_{n\geq 1}$ is bounded, $(s_n)_{n\geq 1}$ is unbounded, and $\mathrm{Dio}(\mathbf{a})$ is infinite. Then, for all $n \geq 1$, we have $q^n \leq |q_n| \leq \tilde{M}^n$, which implies $H(\alpha_n) \leq \tilde{M}^{2r_n+s_n}$. Therefore, in the same matter, we obtain

$$|\xi - \alpha_n| \leq H(\alpha_n)^{-\delta\frac{\log q}{\log \tilde{M}}}.$$

Hence, we have $w_2^*(\xi) = +\infty$.

Assume that the sequence $(|a_n|)_{n\geq 1}$ is bounded. We denote by A its upper bound. We consider the case that $(r_n)_{n\geq 1}$ is unbounded. Here, if necessary, taking a subsequence of $(r_n)_{n\geq 1}$, we assume that $(r_n)_{n\geq 1}$ is increasing and $r_1 \geq n_0$. Let $P_n(X)$ be the minimal polynomial of α_n. From Lemma 4.6, we obtain

$$\begin{aligned} |P_n(\xi)| \quad &\leq \quad H(\alpha_n)|\xi - \alpha_n||\xi - \alpha_n'| \leq A^2 H(\alpha_n) q_{r_n+\lceil w_n s_n\rceil}^{-2} q_{r_n}^{-2} \\ &\leq \quad A^2 H(\alpha_n)^{-2\frac{2r_n+\lceil w_n s_n\rceil}{2r_n+s_n}\frac{\log(m-\varepsilon)}{\log(M+\varepsilon)}+1}. \end{aligned}$$

Since

$$\frac{2r_n + \lceil w_n s_n\rceil}{2r_n + s_n} \quad \geq \quad \frac{r_n + \delta(r_n+s_n)}{2r_n+s_n} \geq \frac{r_n + s_n/2 + \delta(r_n + s_n/2)}{2r_n + s_n} \geq \frac{1+\delta}{2},$$

we obtain (1.4). For the remaining case, we have (1.4) in the same line of proof of (1.3). □

Appendix A. Rational approximation in $\mathbb{F}_q((T^{-1}))$

Lemma A.1. *Let* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a non-ultimately periodic sequence over* $\mathbb{F}_q$. *Set* $\xi := \sum_{n=0}^{\infty} a_n T^{-n}$. *Then we have*

$$w_1(\xi) \geq \max(1, \mathrm{Dio}(\mathbf{a}) - 1). \tag{A.1}$$

Proof. From Theorem 5.1, we have $w_1(\xi) \geq 1$. Without loss of generality, we may assume that $\mathrm{Dio}(\mathbf{a}) > 1$. Take a real number δ such that $1 < \delta < \mathrm{Dio}(\mathbf{a})$. For $n \geq 1$, there exist finite words U_n, V_n and a real number w_n such that $U_n V_n^{w_n}$ is the prefix of $\mathbf{a}$, the sequence $(|V_n^{w_n}|)_{n\geq 1}$ is strictly increasing, and $|U_n V_n^{w_n}| \geq \delta |U_n V_n|$. Put $q_n := T^{|U_n|}(T^{|V_n|} - 1)$. Then there exists $p_n \in \mathbb{F}_q[T]$ such that

$$\frac{p_n}{q_n} = \sum_{k=0}^{\infty} b_k^{(n)} T^{-k},$$

where $(b_k^{(n)})_{k\geq 0}$ is the infinite word $U_n\overline{V_n}$ by Lemma 3.4 in [15]. Since ξ and p_n/q_n have the same first $|U_nV_n^{w_n}|$-th digits, we obtain

$$\left|\xi - \frac{p_n}{q_n}\right| \leq |q_n|^{-\delta}.$$

Hence, we have (A.1). □

The following theorem is an analogue of Théorème 2.1 in [5] and Theorem 1.3 in [21], and is an extension of Theorem 1.2 in [15].

Theorem A.2. *Let* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a non-ultimately periodic sequence over* $\mathbb{F}_q$. *Set* $\xi := \sum_{n=0}^{\infty} a_n T^{-n}$. *Assume that there exist integers* $n_0 \geq 1$ *and* $\kappa \geq 2$ *such that for all* $n \geq n_0$,

$$p(\mathbf{a}, n) \leq \kappa n.$$

If the Diophantine exponent of $\mathbf{a}$ *is finite, then we have*

$$w_1(\xi) \leq 8(\kappa+1)^2(2\kappa+1)\,\mathrm{Dio}(\mathbf{a}) - 1. \tag{A.2}$$

Proof. For $n \geq n_0$, take finite words U_n, V_n and a rational number w_n satisfying Lemma 7.1 (i)-(vi). Put $q_n := T^{|U_n|}(T^{|V_n|} - 1)$. Then there exists $p_n \in \mathbb{F}_q[T]$ such that

$$\frac{p_n}{q_n} = \sum_{k=0}^{\infty} b_k^{(n)} T^{-k},$$

where $(b_k^{(n)})_{k\geq 0}$ is the infinite word $U_n\overline{V_n}$ by Lemma 3.4 in [15]. Since ξ and p_n/q_n have the same first $|U_nV_n^{w_n}|$-th digits, we obtain

$$\left|\xi - \frac{p_n}{q_n}\right| \leq |q_n|^{-1-\frac{1}{4\kappa+2}}.$$

Take a real number δ which is greater than $\mathrm{Dio}(\mathbf{a})$. Note that $\delta > 1$. By the definition of Diophantine exponent, there exists an integer $n_1 \geq n_0$ such that for all $n \geq n_1$

$$\left|\xi - \frac{p_n}{q_n}\right| \geq |q_n|^{-\delta}.$$

We define a positive integer sequence $(n_j)_{j\geq 1}$ by $n_{j+1} = 2(\kappa+1)n_j$ for $j \geq 1$. It follows from Lemma 7.1 (iii) and (vi) that for $j \geq 1$

$$|q_{n_j}| < |q_{n_{j+1}}| \leq |q_{n_j}|^{4(\kappa+1)^2}.$$

Thus, by Lemma 3.2 in [15], we obtain (A.2). □

Consequently, the following result holds.

Corollary A.3. *Let* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a non-ultimately periodic sequence over* $\mathbb{F}_q$. *Set* $\xi := \sum_{n=0}^{\infty} a_n T^{-n}$. *Then the Diophantine exponent of* $\mathbf{a}$ *is finite if and only if* ξ *is not a* U_1*-number.*

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