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A GENERALIZATION OF VECTOR VALUED JACOBI FORMS

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Abstract

The Fourier–Jacobi coefficients of vector valued Siegel modular forms of degree n are more general functions than vector valued Jacobi forms defined by Ziegler [9] even when $n = 2$. We define generalized vector valued Jacobi forms corresponding to the above coefficients when $n = 2$ and prove that such a space is isomorphic to a certain product of spaces of usual scalar valued Jacobi forms of various weights. This isomorphism is realized by certain linear holomorphic differential operators. The half-integral weight case is also treated.

1. Introduction

The usual scalar valued Jacobi forms are defined as functions which have the natural automorphic properties that the Fourier–Jacobi coefficients of scalar valued Siegel modular forms have. When we take a vector valued Siegel modular form F instead, then each Fourier–Jacobi coefficient of F is a vector of holomorphic functions which satisfies more complicated relations. Here we call this kind of functions *vector valued Jacobi forms*. In this paper, we restrict ourselves to the case of functions on $\mathbb{H} \times \mathbb{C}$ where $\mathbb{H}$ is the complex upper half plane, and we study their relation to the usual scalar valued Jacobi forms. Main results are Theorem 1.1 and 1.2 in §1.

By the way, some general definition of vector valued Jacobi forms of general degree was given in Ziegler [9]. But his definition is fairly different from ours. His definition is obtained by changing the action of the semi-simple part of the Jacobi group to the one using vector valued automorphy factor of the symplectic group. This is of course interesting object since it has a good connection with vector valued Siegel modular forms of "half-integral" weight (i.e. the determinant part has a half-integral power). But in the present context, a Jacobi form defined by him can give only some components of our general vector valued Jacobi forms. For example, for functions on $\mathbb{H} \times \mathbb{C}$, his definition can give only scalar valued automorphy factors but ours give vector valued functions even in this case.

Although we confine ourselves to the case of functions on $\mathbb{H} \times \mathbb{C}$ in this paper, some of definitions have obvious generalization to the general degree, which will be omitted here.

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Now we give a precise definition of the vector valued Jacobi forms in this paper. The half-integral weight case will be shortly explained in the last section and here we explain only the case when the weight is integral. Let $\mathbb{H}_n$ be the Siegel upper half space of degree n defined as usual by

$$\mathbb{H}_n = \{Z = {}^tZ \in M_n(\mathbb{C});\ \mathrm{Im}(Z) > 0\}.$$

We write $\mathbb{H} = \mathbb{H}_1$. For any non-negative integer s, we denote by ρ_s the symmetric tensor representation of $GL_2(\mathbb{C})$ of degree s. For any integer k, we write $\rho_{k,s} = \det{}^k \otimes \rho_s$. This exhausts all the rational irreducible representations of $GL_2(\mathbb{C})$. Let $Sp(2, \mathbb{R})$ be the usual symplectic group of size 4, i.e.

$$Sp(2, \mathbb{R}) = \{g \in M_4(\mathbb{R});\ {}^tgJg = J\},$$

where $J = \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix}$ and we put $\Gamma_2 = Sp(2, \mathbb{Z}) = Sp(2, \mathbb{R}) \cap M_4(\mathbb{Z})$. For any $\mathbb{C}^{s+1}$-valued function $F(Z)$ of $Z \in H_2$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \mathbb{R})$ we write

$$(F|_{(k,s)}M)(Z) = \rho_{k,s}(CZ + D)^{-1}F(MZ).$$

For any $k \in \mathbb{Z}$, we say that a $\mathbb{C}^{s+1}$-valued holomorphic function F is a Siegel modular forms of weight $\rho_{k,s}$ of Γ_2 if $F|_{k,s}M = F$ for all $M \in \Gamma_2$. We denote by $A_{k,s}(\Gamma_2)$ the vector space of these Siegel modular forms. For any $Z \in \mathbb{H}_2$, we write $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ and $F(Z) = F(\tau, z, \tau')$. If $F \in A_{k,s}(\Gamma_2)$, then we have $F(\tau, z, \tau') = F(\tau, z, \tau' + l)$ for any integer l, so we have the following Fourier expansion

$$F(Z) = \sum_{m=0}^{\infty} \Phi_m(\tau, z)\, e(m\tau'),$$

where we write $e(x) = \exp(2\pi i m x)$ for any $x \in \mathbb{C}$ and we also write $e^m(x) = e(mx)$ sometimes. This is called the Fourier–Jacobi expansion of F and each $\mathbb{C}^{s+1}$-valued function $\Phi_m(\tau, z)$ is called a Fourier–Jacobi coefficient of F. The real Jacobi group $J(\mathbb{R})$ is (isomorphic to) the subgroup of $Sp(2, \mathbb{R})$ generated by the matrices

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $\lambda, \mu, \kappa \in \mathbb{R}$. For any subgroup Γ of $SL_2(\mathbb{Z})$ with finite index, we define the Jacobi modular group Γ^J by a subgroup generated by the above

two kinds of matrices with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\lambda, \mu, \kappa \in \mathbb{Z}$. By the action of $SL_2(\mathbb{Z})^J$ on Siegel modular forms $F \in A_{k,s}(\Gamma_2)$, we obtain the action of $SL_2(\mathbb{Z})$ and $\mathbb{Z}^2$ on the Fourier–Jacobi coefficients $\Phi_m(\tau, z)$ of F. For any integer k, for any $\mathbb{C}^{s+1}$ valued function Φ on $\mathbb{H} \times \mathbb{C}$ and for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $(X, \kappa) = ([\lambda, \mu], \kappa) \in \mathbb{R}^3$, we write

$$(1)\quad (\Phi|_{(k,s),m} M)(\tau, z) = \rho_{k,s}\begin{pmatrix} c\tau + d & cz \\ 0 & 1 \end{pmatrix}^{-1} e^m\left(-\frac{cz^2}{c\tau + d}\right)\Phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right),$$

$$(2)\quad (\Phi|_{s,m}(X, \kappa))(\tau, z) = \rho_s\begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}^{-1} e^m(\lambda^2\tau + 2\lambda z + \lambda\mu + \kappa)\Phi(\tau, z + \lambda\tau + \mu).$$

Then this is a group action of $J(\mathbb{R})$. When $\kappa \in \mathbb{Z}$, we sometimes write $\Phi|_{s,m}(X, \kappa) = \Phi|_{s,m} X$. Also if $s = 0$, we write $\Phi|_{(k,0),m} M = \Phi|_{k,m} M$ and $\Phi|_{0,m} X = \Phi|_m X$. The Fourier–Jacobi coefficients Φ_m of a Siegel modular form in $A_{k,s}(\Gamma_2)$ satisfy

$$(\Phi_m|_{(k,s),m} M)(\tau, z) = \Phi_m(\tau, z),$$
$$(\Phi_m|_{m,s} X)(\tau, z) = \Phi_m(\tau, z)$$

for any $M \in SL_2(\mathbb{Z})$ and $X \in \mathbb{Z}^2$. Therefore more generally we give the definition below. Let m be a non-negative integer and $\rho_{k,s} = \det^k \otimes \rho_s$ be the irreducible representation of $GL_2(\mathbb{C})$ of dimension $s + 1$ defined as before. Let Γ be a subgroup of $SL_2(\mathbb{Z})$ with finite index.

DEFINITION 1.1. A Jacobi form of weight $\rho_{k,s}$ and index m belonging to Γ is a holomorphic function $\Phi\colon \mathbb{H} \times \mathbb{C} \to \mathbb{C}^{s+1}$ satisfying
(1) $\Phi|_{(k,s),m} M = \Phi$ for all $M \in \Gamma$,
(2) $\Phi|_{s,m} X = \Phi$ for all $X \in \mathbb{Z}^2$,
(3) for each $M \in SL_2(\mathbb{Z})$, the function $\Phi|_{(k,s),m} M$ has the Fourier expansion of the form

$$(\Phi|_{(k,s),m} M)(\tau, z) = \sum_{\substack{r^2 \le 4nm \\ n, r \in N_M^{-1}\mathbb{Z}}} C_M(n, r) q^n \zeta^r,$$

where we write $q = e(\tau)$, $\zeta = e(z)$ for $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$, and N_M is a rational number depending on M.

We denote by $J_{(k,s),m}(\Gamma^J)$ the set of all such functions. If $\Phi|_{(k,s),m} M$ has the Fourier expansion as above with $C_M(n, r) = 0$ unless $r^2 < 4nm$ for every $M \in SL_2(\mathbb{Z})$, then Φ is called a vector valued Jacobi *cusp* form. We denote by $J^{\text{cusp}}_{(k,s),m}(\Gamma^J)$ the subspace of all cusp forms in $J_{(k,s),m}(\Gamma^J)$. Our main theorem of this paper is that this space is essentially described by using only scalar valued Jacobi forms. Indeed, we prove the following theorem in Section 2.

Theorem 1.1. *For any integer m with $m \geq 1$, there exists a linear isomorphism between vector valued Jacobi forms and a direct sum of scalar valued Jacobi forms given by*

$$J_{(k,s),m}(\Gamma^J) \cong J_{k,m}(\Gamma^J) \times J_{k+1,m}(\Gamma^J) \times \cdots \times J_{k+s,m}(\Gamma^J),$$
$$J^{\mathrm{cusp}}_{(k,s),m}(\Gamma^J) \cong J^{\mathrm{cusp}}_{k,m}(\Gamma^J) \times J^{\mathrm{cusp}}_{k+1,m}(\Gamma^J) \times \cdots \times J^{\mathrm{cusp}}_{k+s,m}(\Gamma^J),$$

where $J_{k+\nu,m}(\Gamma^J)$ is a space of scalar valued Jacobi forms of weight $k+\nu$ and index m and $J^{\mathrm{cusp}}_{k+\nu,m}(\Gamma^J)$ is the subspace of cusp forms.

This linear isomorphism is given by a holomorphic differential operator with constant coefficients. This operator is compatible with the action of $J(\mathbb{R})$ to the both sides of the isomorphism, where the action of $J(\mathbb{R})$ to the right hand side is defined as a natural componentwise action. This fact is given in the following theorem and the above theorem is its corollary. We denote by W the vector space of $s+1$ numbers of holomorphic functions on $\mathbb{H} \times \mathbb{C}$. For $\Phi \in W$, we write the l-th component by φ_{s+1-l}, namely $\Phi = {}^t(\varphi_s, \varphi_{s-1}, \ldots, \varphi_0)$.

Theorem 1.2. *There exists a linear holomorphic differential operator $D_{(k,s),m}$ with constant coefficients from W to W such that*

$$\left(D_{(k,s),m}(\Phi|_{(k,s),m}g)\right)_\mu = (D_{(k,s),m}\Phi)_\mu|_{k+\mu,m}g.$$

for any $g \in J(\mathbb{R})$.

Here for any $v \in W$, we denote by v_μ the $(s+1-\mu)$-th component of v. In Section 2, we prove this theorem. In Section 3 we give a short remark on Eisenstein series and inner metric. In Section 4, we give an analogous result for the half-integral weight case.

2. Vector valued Jacobi forms and scalar valued Jacobi forms

In this section we prove Theorem 1.1 and 1.2. Most part of the proof is devoted to the explicit construction of $D_{(k,s),m}$ in Theorem 1.2.

In order to fix the coordinate, first we review the definition of the symmetric tensor representation. For variables u_1, u_2, we define v_1, v_2 by

$$(v_1, v_2) = (u_1, u_2)A$$

where $A \in GL_2(\mathbb{C})$. We put

$$(v_1^s, v_1^{s-1}v_2, \ldots, v_1v_2^{s-1}, v_2^s) = (u_1^s, u_1^{s-1}u_2, \ldots, u_1u_2^{s-1}, u_2^s)\rho_s(A).$$

We write the component of $\Phi(\tau, z) \in J_{(k,s),m}(\Gamma^J)$ as

$$\Phi(\tau, z) = \begin{pmatrix} \varphi_s(\tau, z) \\ \vdots \\ \varphi_0(\tau, z) \end{pmatrix} \in J_{(k,s),m}(\Gamma^J).$$

The automorphic property with respect to the action of Γ and $\mathbb{Z}^2$ is described explicitly as follows.

$$(c\tau+d)^{-k} e^m\left(-\frac{cz^2}{c\tau+d}\right) \begin{pmatrix} \varphi_s\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) \\ \vdots \\ \varphi_1\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) \\ \varphi_0\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) \end{pmatrix}$$

$$= \begin{pmatrix} (c\tau+d)^s & \cdots & \binom{s-2}{s-2}(c\tau+d)^2(cz)^{s-2} & \binom{s-1}{s-1}(c\tau+d)(cz)^{s-1} & \binom{s}{s}(cz)^s \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & \binom{s-2}{1}(c\tau+d)^2(cz) & \binom{s-1}{2}(c\tau+d)(cz)^2 & \binom{s}{3}(cz)^3 \\ 0 & \cdots & (c\tau+d)^2 & \binom{s-1}{1}(c\tau+d)(cz) & \binom{s}{2}(cz)^2 \\ 0 & \cdots & 0 & (c\tau+d) & \binom{s}{1}(cz) \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi_s(\tau,z) \\ \vdots \\ \varphi_1(\tau,z) \\ \varphi_0(\tau,z) \end{pmatrix},$$

$$e^m(\lambda^2\tau + 2\lambda z) \begin{pmatrix} \varphi_s(\tau, z+\lambda\tau+\mu) \\ \vdots \\ \varphi_1(\tau, z+\lambda\tau+\mu) \\ \varphi_0(\tau, z+\lambda\tau+\mu) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \cdots & \binom{s-2}{s-2}(-\lambda)^{s-2} & \binom{s-1}{s-1}(-\lambda)^{s-1} & \binom{s}{s}(-\lambda)^s \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & \binom{s-2}{1}(-\lambda) & \binom{s-1}{2}(-\lambda)^2 & \binom{s}{3}(-\lambda)^3 \\ 0 & \cdots & 1 & \binom{s-1}{1}(-\lambda) & \binom{s}{2}(-\lambda)^2 \\ 0 & \cdots & 0 & 1 & \binom{s}{1}(-\lambda) \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi_s(\tau,z) \\ \vdots \\ \varphi_1(\tau,z) \\ \varphi_0(\tau,z) \end{pmatrix},$$

where $\binom{\alpha}{r}$ are the usual binomial coefficients.

More generally, for any function $\Phi = {}^t(\varphi_\mu) = {}^t(\varphi_s, \varphi_{s-1}, \ldots, \varphi_0) \in W$, we have

$$(\varphi_\mu|_{k,m}M)(\tau, z) = \sum_{\nu=0}^{\mu} \binom{s-\nu}{\mu-\nu}(c\tau+d)^\nu (cz)^{\mu-\nu}(\Phi|_{(k,s),m}M)_\nu(\tau, z), \tag{3}$$

$$(\varphi_\mu|_m X)(\tau, z) = \sum_{\nu=0}^{\mu} \binom{s-\nu}{\mu-\nu}(-\lambda)^{\mu-\nu}(\Phi|_{s,m}X)_\nu(\tau, z), \tag{4}$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $X = [\lambda, \mu] \in \mathbb{R}^2$. By dividing the first expression (3) by $(c\tau + d)^\mu$, we get

$$(\varphi_\mu|_{k+\mu,m}M)(\tau, z) = \sum_{\nu=0}^{\mu} \binom{s-\nu}{\mu-\nu}\left(\frac{cz}{c\tau+d}\right)^{\mu-\nu}(\Phi|_{(k,s),m}M)_\nu(\tau, z). \tag{5}$$

First we construct some operator from W to W compatible with the action of $\mathbb{R}^2$, since this is easier than the action of $SL_2(\mathbb{R})$.

Lemma 2.1. *We fix $X = [\lambda, \mu] \in \mathbb{R}^2$ and put $\partial_z = (1/(2\pi i))\partial/\partial z$. For any pair of scalar valued functions φ and ψ on $\mathbb{H} \times \mathbb{C}$ such that $\varphi|_m X = \psi$, and any non-negative integer t, we have*

$$((\partial_z^t\varphi)|_m X)(\tau, z) = \sum_{j=0}^{t} \binom{t}{j}(2m)^j(-\lambda)^j\, \partial_z^{t-j}\psi(\tau, z).$$

Proof. We prove this by induction on t. By operating ∂_z on both sides of the definition of $\varphi|_m X$, we have

$$\partial_z(\varphi|_m X) = 2\lambda m(\varphi|_m X) + (\partial_z\varphi)|_m X.$$

This is nothing but the relation for $t = 1$. Now assume that the lemma is true for t. By operating ∂_z on both sides of the relation in the lemma for t, we have

$$\partial_z(\partial_z^t\varphi|_m X) = \sum_{j=0}^{t} \binom{t}{j}(2m)^j(-\lambda)^j\, \partial_z^{t+1-j}\psi.$$

Replacing φ by $\partial_z^t\varphi$ in the relation for $t = 1$, we get

$$\partial_z(\partial_z^t\varphi|_m X) = 2\lambda m((\partial_z^t\varphi)|_m X) + (\partial_z^{t+1}\varphi)|_m X.$$

Since $\binom{t}{j}(-2m\lambda)^j - (2\lambda m)\binom{t}{j-1}(-2m\lambda)^{j-1} = \binom{t+1}{j}(-2m\lambda)^j$, we get the relation for $t + 1$. □

We fix m and s. For any vector $\Phi = {}^t(\varphi_s, \varphi_{s-1}, \ldots, \varphi_0) \in W$ and any integer t with $0 \le t \le s$, we define a holomorphic function $g_t(\Phi)$ on $\mathbb{H} \times \mathbb{C}$ by

$$g_t(\Phi)(\tau, z) = \sum_{\mu=0}^{t} (-2m)^{\mu-t} \binom{s-\mu}{t-\mu} (\partial_z^{t-\mu} \varphi_\mu)(\tau, z).$$

Lemma 2.2. *Notation being as above, we have*

$$g_t(\Phi)|_m X = g_t(\Phi|_{s,m} X)$$

for any $X \in \mathbb{R}^2$.

Proof. We fix t throughout the proof. We fix $X \in \mathbb{R}^2$ and put $\psi_\mu = \varphi_\mu|_m X$. Then by Lemma 2.1, we have

$$(\partial_z^{t-\mu} \varphi_\mu)|_m X = \sum_{j=0}^{t-\mu} \binom{t-\mu}{j} (-2m\lambda)^j \, \partial_z^{t-\mu-j} \psi_\mu.$$

Since we have

$$\psi_\mu = \varphi_\mu|_m X = \sum_{\nu=0}^{\mu} \binom{s-\nu}{\mu-\nu} (-\lambda)^{\mu-\nu} (\Phi|_{s,m} X)_\nu,$$

we get

$$g_t(\Phi)|_m X = \sum_{\mu=0}^{t} \sum_{j=0}^{t-\mu} \sum_{\nu=0}^{\mu} \frac{(-1)^{t-\mu} (-\lambda)^{j+\mu-\nu}}{(2m)^{t-\mu-j}} \binom{s-\mu}{t-\mu} \binom{t-\mu}{j} \binom{s-\nu}{\mu-\nu} \partial_z^{t-\mu-j} (\Phi|_{s,m} X)_\nu.$$

We fix $\mu + j$ and put $l = \mu + j$. We also fix ν. Then $0 \le j \le l - \nu$. In the above expression the coefficient of $\partial_z^{t-\mu-j} \varphi_\nu$ for fixed t, ν and $t - l$ is given by

$$\frac{(-1)^{t-l} (-\lambda)^{l-\nu}}{(2m)^{t-l}} \times (-1)^j \binom{s-l+j}{t-l+j} \binom{t-l+j}{j} \binom{s-\nu}{l-\nu-j}.$$

We have

$$\binom{s-l+j}{t-l+j} \binom{t-l+j}{j} \binom{s-\nu}{l-\nu-j} = \binom{s-\nu}{t-\nu} \binom{l-\nu}{j} \binom{t-\nu}{l-\nu}$$

and

$$\sum_{j=0}^{l-\nu} (-1)^j \binom{l-\nu}{j} = \begin{cases} 1 & \text{if} \quad l = \nu, \\ 0 & \text{otherwise.} \end{cases}$$

If $l = \nu$ then $j = 0$ and $\mu = \nu$. So the right hand side becomes $g_t(\Phi|_{s,m} X)$. □

As a corollary of this lemma we see that if $\Phi|_{s,m}X = \Phi$ for any $X \in \mathbb{Z}^2$, then $g_t(\Phi)$ is also invariant by the action of $\mathbb{Z}^2$. But this is not invariant by the action of Γ even if $\Phi|_{(k,s),m}M = \Phi$ for $M \in \Gamma$. So we consider the behaviour under the action of $SL_2(\mathbb{R})$ next.

We prepare some notation. We write $(2n-1)!! = 1 \cdot 3 \cdots (2n-1) = (2n)!/2^n n!$ and $0! = (-1)!! = 1$. We define a heat operator L_m by

$$L_m = 4m\,\partial_\tau - \partial_z^2,$$

where we put $\partial_\tau = (1/(2\pi i))\,\partial/\partial\tau$.

Lemma 2.3. *We fix a holomorphic function φ on $\mathbb{H} \times \mathbb{C}$, a matrix $M \in SL_2(\mathbb{R})$, and put $\psi = \varphi|_{k,m}M$. Notation being as above, we have*

$$\begin{aligned}(6)\qquad &(\partial_z^t\varphi)|_{k+t,m}M \\ &= \sum_{l=0}^{[t/2]} \binom{t}{2l} \left(\frac{1}{2\pi i}\frac{2mc}{c\tau+d}\right)^l (2l-1)!! \sum_{\nu=0}^{t-2l} \binom{t-2l}{\nu} \left(\frac{2mcz}{c\tau+d}\right)^\nu (\partial_z^{t-2l-\nu}\psi).\end{aligned}$$

Proof. We prove this by induction on t. When $t = 0$ the relation is just the definition of ψ. We assume that the relation holds for t. First by differentiating both sides of the definition $\partial_z^t\varphi|_{k+t,m}M$ of the action of M on $\partial_z^t\varphi$, we see

$$\partial_z(\partial_z^t\varphi|_{k+t,m}M) = (\partial_z^{t+1}\varphi)|_{k+t+1,m}M - \left(\frac{2mcz}{c\tau+d}\right)(\partial_z^t\varphi|_{k+t,m}M).$$

In this equality we evaluate $\partial_z^t\varphi|_{k+t,m}M$ by the inductive assumption and calculate the action of ∂_z on that. We note that

$$\begin{aligned}&\partial_z\left((\partial_z^{t-2l-\nu}\psi)\left(\frac{2mcz}{c\tau+d}\right)^\nu\right) \\ &= (\partial_z^{t+1-2l-\nu}\psi)\left(\frac{2mcz}{c\tau+d}\right)^\nu + \nu\left(\frac{1}{2\pi i}\frac{2mc}{c\tau+d}\right)\left(\frac{2mcz}{c\tau+d}\right)^{\nu-1}(\partial_z^{t-2l-\nu}\psi).\end{aligned}$$

We calculate the coefficient of $\partial_z^\rho(2mcz/(c\tau+d))^\mu$ in $(\partial_z^{t+1}\varphi)|_{k+t+1,m}M$. The coefficient is zero unless $\rho + \mu \equiv t+1 \bmod 2$. We put $2\kappa = t+1-\rho-\mu$. Then the coefficient is a sum of three terms corresponding $((1/2\pi i)(2mc/(c\tau+d)))^\kappa$ times the following quantities.

$$(2\kappa-1)!!\binom{t}{2\kappa}\binom{t-2\kappa}{\mu}, \quad \text{if } \nu = \mu,\ t+1-2l-\nu = t+1-\mu-2\kappa\ (\kappa = l),$$

$$(2\kappa-1)!!\binom{t}{2\kappa}\binom{t-2\kappa}{\mu-1}, \quad \text{if } \nu+1 = \mu,\ t-2l-\nu = t+1-\mu-2\kappa\ (\kappa = l),$$

$$(2\kappa-3)!!\binom{t}{2\kappa-2}(\mu+1)\binom{t-2\kappa+2}{\mu+1},$$
$$\text{if } \nu-1=\mu,\ t-2l-\nu=t+1-\mu-2\kappa\ (\kappa=l+1).$$

We have

$$(2\kappa-3)!!\binom{t}{2\kappa-2}\binom{t-2\kappa+2}{\mu+1}(\mu+1)=(2\kappa-1)!!\binom{t}{2\kappa-1}\binom{t-2\kappa+1}{\mu}$$

and

$$\begin{aligned}&\binom{t}{2\kappa}\binom{t-2\kappa}{\mu}+\binom{t}{2\kappa-1}\binom{t-2\kappa+1}{\mu}+\binom{t}{2\kappa}\binom{t-2\kappa}{\mu-1}\\&=\binom{t}{2\kappa}\binom{t-2\kappa+1}{\mu}+\binom{t}{2\kappa-1}\binom{t-2\kappa+1}{\mu}\\&=\binom{t+1}{2\kappa}\binom{t+1-2\kappa}{\mu}.\end{aligned}$$

Hence we get

$$\begin{aligned}&(\partial_z^{t+1}\varphi)|_{k+t+1,m}M\\&=\sum_{\kappa=0}^{[(t+1)/2]}\sum_{\mu=0}^{t+1-2\kappa}(2\kappa-1)!!\binom{t+1}{2\kappa}\left(\frac{1}{2\pi i}\frac{2mc}{c\tau+d}\right)^{\kappa}\binom{t+1-2\kappa}{\mu}\left(\frac{2mcz}{c\tau+d}\right)^{\mu}\partial_z^{t+1-2\kappa-\mu}\psi.\end{aligned}$$

This is nothing but the formula for $t+1$. □

For $L_m=4m\,\partial_\tau-\partial_z^2$, we have $L_m(\varphi|_m X)=(L_m\varphi)|_m X$ for any $X\in\mathbb{R}^2$. Indeed by definition we have $\varphi|_m X=e^m(\lambda^2\tau+2\lambda z)\varphi(\tau,z+\lambda\tau+\mu)$, and if we put $\psi=\varphi|_m X$, then we get

$$\begin{aligned}(\partial_z\varphi)|_m X&=\partial_z\psi-2m\lambda\psi,\\(\partial_z^2\varphi)|_m X&=\partial_z^2\psi-4m\lambda\,\partial_z\psi+(2m\lambda)^2\psi,\\\partial_\tau\psi&=m\lambda^2\psi+(\partial_\tau\varphi)|_m X+\lambda(\partial_z\varphi)|_m X.\end{aligned}$$

So we get $((4m\partial_\tau-\partial_z^2)\varphi)|_m X=L_m\psi$.

Next we see the relation of the action of L_m and $M\in SL_2(\mathbb{R})$.

Lemma 2.4. *We fix $M\in SL_2(\mathbb{R})$ and a holomorphic function φ on $\mathbb{H}\times\mathbb{C}$. We put $\psi=\varphi|_{k,m}M$. Then we have*

$$(L_m^t\varphi)|_{k+2t,m}M=\sum_{l=0}^{t}\binom{t}{l}\frac{(2k+2t-3)!!}{(2k+2t-2l-3)!!}\left(\frac{1}{2\pi i}\frac{2mc}{c\tau+d}\right)^{l}(L_m^{t-l}\psi).\tag{7}$$

Proof. We prove this by induction on t. When $t = 0$ this is just the definition of ψ. When $t = 1$ by direct calculation we see

$$(L_m\varphi)|_{k+2,m}M = L_m(\varphi|_{k,m}M) + \frac{(2k-1)2mc}{2\pi i(c\tau+d)}(\varphi|_{k,m}M). \tag{8}$$

So we have

$$L_m((L_m^t\varphi)|_{k+2t,m}M) = (L_m^{t+1}\varphi)|_{k+2t+2,m}M - \frac{2mc(2k+4t-1)}{2\pi i(c\tau+d)}(L_m^t\varphi)|_{k+2t,m}M.$$

Now we assume that the lemma is true for t. We see

$$\begin{aligned}
&4m\partial_\tau\left(\left(\frac{1}{2\pi i}\frac{2mc}{c\tau+d}\right)^l(L_m^{t-l}\psi)\right)\\
&= -2l\left(\frac{1}{2\pi i}\frac{2mc}{c\tau+d}\right)^{l+1}(L_m^{t-l}\psi) + \left(\frac{1}{2\pi i}\frac{2mc}{c\tau+d}\right)^l(4m\partial_\tau L_m^{t-l}\psi),\\
&\partial_z^2\left(\left(\frac{1}{2\pi i}\frac{2mc}{c\tau+d}\right)^l(L_m^{t-l}\psi)\right) = \left(\frac{1}{2\pi i}\frac{2mc}{c\tau+d}\right)^l(\partial_z^2 L_m^{t-l}\psi).
\end{aligned}$$

Since we have

$$\begin{aligned}
&\binom{t}{l}\frac{(2k+4t-1)(2k+2t-3)!!}{(2k+2t-2l-3)!!} - \binom{t}{l}\frac{2l(2k+2t-3)!!}{(2k+2t-2l-3)!!} + \binom{t}{l+1}\frac{(2k+2t-3)!!}{(2k+2t-2l-5)!!}\\
&= \left(\binom{t}{l+1}(2k+2t-2l-3) + \binom{t}{l}(2k+4t-1) - \binom{t}{l}(2l)\right)\frac{(2k+2t-3)!!}{(2k+2t-2l-3)!!}\\
&= \binom{t+1}{l+1}\frac{(2k+2(t+1)-3)!!}{(2k+2(t+1)-2(l+1)-3)!!},
\end{aligned}$$

we get the relation for $t+1$. (When $l = t$, we understand that $\binom{t}{l+1} = 0$.) □

Lemma 2.5. *For any* $\Phi = {}^t(\varphi_s, \varphi_{s-1}, \ldots, \varphi_0) \in W$, *any* $M \in SL_2(\mathbb{R})$ *and any integer* t *with* $0 \le t \le s$, *we have*

$$\begin{aligned}
&g_t(\Phi)|_{k+t,m}M\\
&= \sum_{\kappa=0}^{[t/2]}(-1)^\kappa(2m)^{-2\kappa}\left(\frac{1}{2\pi i}\frac{2mc}{c\tau+d}\right)^\kappa(2\kappa-1)!!\binom{s-t+2\kappa}{2\kappa}g_{t-2\kappa}(\Phi|_{(k,s),m}M).
\end{aligned}$$

Proof. We fix $M \in SL_2(\mathbb{R})$. For the sake of simplicity, we write $\Phi|_{(k,s),m}M = {}^t(f_s, f_{s-1},\ldots, f_0)$ and for any t with $0 \le t \le s$, we write $g_t = g_t(\Phi)$ and $h_t = g_t(\Phi|_{(k,s),m}M)$.

We put $\psi_\mu = \varphi_\mu|_{k+\mu,m}M$. We have

$$g_t|_{k+t,m}M = \sum_{\mu=0}^{t}(-2m)^{\mu-t}\binom{s-\mu}{t-\mu}(\partial_z^{t-\mu}\varphi_\mu)|_{k+t,m}M.$$

Since

$$(\partial_z^{t-\mu}\varphi_\mu)|_{k+t,m}M = (\partial_z^{t-\mu}\varphi_\mu)|_{k+\mu+(t-\mu),m}M,$$

we have

$$(\partial_z^{t-\mu}\varphi_\mu)|_{k+t,m}M = \sum_{l=0}^{[(t-\mu)/2]}\binom{t-\mu}{2l}\left(\frac{1}{2\pi i}\frac{2mc}{c\tau+d}\right)^l(2l-1)!! \times \sum_{\nu=0}^{t-\mu-2l}\binom{t-\mu-2l}{\nu}\left(\frac{2mcz}{c\tau+d}\right)^\nu(\partial_z^{t-2l-\mu-\nu}\psi_\mu)$$

by Lemma 2.3. By (5), we get

$$\begin{aligned}\partial_z^{t-2l-\mu-\nu}\psi_\mu &= \sum_{j=0}^{\mu}\binom{s-j}{\mu-j}\partial_z^{t-2l-\mu-\nu}\left(\left(\frac{cz}{c\tau+d}\right)^{\mu-j}f_j\right)\\ &= \sum_{j=0}^{\mu}\binom{s-j}{\mu-j}\times\left(\sum_{\gamma=0}^{t-2l-\mu-\nu}\left(\frac{c}{c\tau+d}\right)^{\mu-j}\binom{t-2l-\mu-\nu}{\gamma}\right.\\ &\qquad\left.\times\left(\frac{1}{2\pi i}\right)^\gamma\binom{\mu-j}{\gamma}(\gamma)!\,z^{\mu-j-\gamma}\partial_z^{t-2l-\mu-\nu-\gamma}f_j\right),\end{aligned}$$

where we put $\binom{\mu-j}{\gamma} = 0$ if $\gamma > \mu - j$. As a whole, we have

$$\begin{aligned}&g_t|_{k+t,m}M\\ &= \sum_{\mu=0}^{t}\sum_{l=0}^{[(t-\mu)/2]}\sum_{\nu=0}^{t-\mu-2l}\sum_{j=0}^{\mu}\sum_{\gamma=0}^{t-2l-\mu-\nu}(-1)^{\mu-t}(2m)^{j-t}\left(\frac{1}{2\pi i}\frac{2mc}{c\tau+d}\right)^{l+\gamma}\left(\frac{2mcz}{c\tau+d}\right)^{\nu+\mu-j-\gamma}\\ &\qquad\times(2l-1)!!\binom{s-\mu}{t-\mu}\binom{t-\mu}{2l}\binom{t-\mu-2l}{\nu}\\ &\qquad\times\binom{s-j}{\mu-j}\binom{t-2l-\mu-\nu}{\gamma}\binom{\mu-j}{\gamma}\\ &\qquad\times\gamma!\,(\partial_z^{t-2l-\mu-\nu-\gamma}f_j).\end{aligned}$$

We evaluate the coefficient of $z^\alpha\partial_z^\beta f_j$ for fixed integers $\alpha,\ \beta,\ j \geq 0$. If we put $\alpha = \mu+\nu-j-\gamma$ and $\beta = t-2l-\mu-\nu-\gamma$ for the above l,μ,ν,j,γ in the summation, then

we have $\alpha+\beta+j=t-2l-2\gamma$. Hence if we put $\kappa=(t-\alpha-\beta-j)/2$, then $0\le\kappa$ and κ is an integer. Also we have $l+\gamma=\kappa$ and $\nu+\mu=\kappa+\alpha+j-l$. If $\gamma>\mu-j$ the above sum for this γ is zero, so we may assume $\alpha=\nu+(\mu-j-\gamma)\ge\nu$. Here we checked a kind of necessary condition for parameters. Now we start from any fixed non-negative integers α, β, j such that $\alpha+\beta+j\ge t$ and $t\equiv\alpha+\beta+j \bmod 2$. Then we put $\kappa=(t-\alpha-\beta-j)/2$. We fix any integer ν and l such that $0\le\nu\le\alpha$ and $0\le l\le\kappa$. We put $\mu=j+(\kappa-l)+(\alpha-\nu)$ and $\gamma=\kappa-l$. We now check that ν, μ, l, γ appears in the above summation. It is obvious that $0\le\mu$. We see $\nu+\mu+2l=j+l+\kappa+\alpha\le j+2\kappa+\alpha=t-\beta\le t$. This also implies $\mu\le t$ and $l\le(t-\mu)/2$. We have $0\le\kappa-l=\gamma$. We also have $\mu-j-\gamma=(j+\kappa-l+\alpha-\nu)-j-(\kappa-l)=\alpha-\nu\ge 0$ and $t-2l-\mu-\nu-\gamma=t-2l-(j+\kappa+\alpha-l)-(\kappa-l)=\beta\ge 0$. Under the assumption $\gamma\le\mu-j$, we see

$$\begin{aligned}&\binom{s-\mu}{t-\mu}\binom{t-\mu}{2l}\binom{t-\mu-2l}{\nu}\binom{s-j}{\mu-j}\binom{t-2l-\mu-\nu}{\gamma}\binom{\mu-j}{\gamma}\gamma!\\&=\frac{(s-j)!}{(s-t)!\,\beta!}\times\frac{1}{(2l)!\,\nu!\,(\mu-j-\gamma)!\,\gamma!}\\&=\frac{(s-j)!}{(s-t)!\,\beta!}\times\frac{1}{(2l)!\,(\kappa-l)!\,\nu!(\alpha-\nu)!}.\end{aligned}$$

Since we have $(-1)^{\mu}=(-1)^{\kappa+\alpha+j}(-1)^{l}(-1)^{\nu}$, the coefficient of $z^{\alpha}\partial_z^{\beta}f_j$ in $g_t|_{k+t,m}M$ is given by

$$\begin{aligned}&\frac{(s-j)!}{(s-t)!\beta!}\\&\times\sum_{l,\mu,\nu,\gamma}(2m)^{j-t}(-1)^{\kappa+j+\alpha-t}\left(\frac{1}{2\pi i}\frac{2mc}{c\tau+d}\right)^{\kappa}\left(\frac{1}{2\pi i}\frac{2mc}{c\tau+d}\right)^{\alpha}\frac{(-1)^{l}}{2^{l}l!\,(\kappa-l)!}\times\frac{(-1)^{\nu}}{\nu!(\alpha-\nu)!}.\end{aligned}$$

If $\alpha\ne 0$, then

$$\sum_{\nu=0}^{\alpha}\frac{(-1)^{\nu}}{\nu!(\alpha-\nu)!}=\sum_{\nu=0}^{\alpha}(-1)^{\nu}\binom{\alpha}{\nu}(\alpha!)^{-1}=0.$$

So we can assume $\alpha=0$. We also have

$$\sum_{l=0}^{\kappa}\frac{(-1)^{l}}{2^{l}l!\,(\kappa-l)!}=(\kappa!)^{-1}\left(1-\frac{1}{2}\right)^{\kappa}=\frac{1}{2^{\kappa}\kappa!}=\frac{(2\kappa-1)!!}{(2\kappa)!}.$$

By definition we have

$$h_{\beta+j}=h_{t-2\kappa}=\sum_{j=0}^{t-2\kappa}(-1)^{j-t-2\kappa}(2m)^{j-t+2\kappa}\binom{s-j}{t-2\kappa-j}(\partial_z^{\beta}f_j).$$

Since $\alpha = 0$, we have $\nu = 0$, $t - 2\kappa - j = \beta$, $(-1)^{\kappa+\alpha+j-t} = (-1)^{\kappa}(-1)^{j-t+2\kappa}$, $(2m)^{j-t+2\kappa} = (2m)^{j-t}(2m)^{2\kappa}$ and

$$\binom{s-t+2\kappa}{2\kappa}\binom{s-j}{t-2\kappa-j} = \frac{(s-j)!}{(2\kappa)!\,(s-t)!\,\beta!}.$$

Hence we get our relation. □

For any function $\Phi = {}^t(\varphi_s, \varphi_{s-1}, \ldots, \varphi_0) \in W$ and any integer μ with $0 \le \mu \le s$, we define a scalar-valued function $\iota_\mu(\Phi)$ on $\mathbb{H} \times \mathbb{C}$ by

$$\iota_\mu(\Phi) = \sum_{t=0}^{[\mu/2]} \binom{s-\mu+2t}{2t} \frac{(2k+2\mu-2t-5)!!\,(2t-1)!!}{(2m)^{2t}(2k+2\mu-5)!!} L_m^t(g_{\mu-2t}(\Phi)) \tag{9}$$

and a linear differential operator $D_{(k,s),m}$ from W to W by $D_{(k,s),m}(\Phi) = {}^t(\iota_s(\Phi), \iota_{s-1}(\Phi), \ldots, \iota_0(\Phi))$. We note that $\iota_\mu(\Phi)$ depends only on $\varphi_0, \ldots, \varphi_\mu$ and not on φ_ν with $\nu > \mu$. By using the definition of $g_{\mu-2t}$, we can rewrite the definition of $\iota_\mu(\Phi)$ more simply as

$$\begin{aligned}
&\iota_\mu(\Phi)\\
&= \sum_{t=0}^{[\mu/2]} \sum_{\nu=0}^{\mu-2t} \binom{s-\nu}{\mu-2t-\nu}\binom{s-\mu+2t}{2t} \frac{(-2m)^{\nu-\mu}(2k+2\mu-2t-5)!!\,(2t-1)!!}{(2k+2\mu-5)!!} \partial_z^{\mu-2t-\nu} L_m^t \varphi_\nu\\
&= \sum_{\nu=0}^{\mu} \sum_{t=0}^{[(\mu-\nu)/2]} (-2m)^{\nu-\mu}\binom{s-\nu}{\mu-\nu}\binom{\mu-\nu}{2t} \frac{(2k+2\mu-2t-5)!!\,(2t-1)!!}{(2k+2\mu-5)!!} \partial_z^{\mu-\nu-2t} L_m^t \varphi_\nu.
\end{aligned}$$

Theorem 2.1. *For any $\Phi = {}^t(\varphi_s, \varphi_{s-1}, \ldots, \varphi_0) \in W$ and any elements $M \in SL_2(\mathbb{R})$, we have*

$$(\iota_\mu(\Phi))|_{k+\mu,m} M = \iota_\mu(\Phi|_{(k,s),m} M).$$

In particular, if $\Phi \in J_{(k,s),m}(\Gamma^J)$, then we have $\iota_\mu(\Phi) \in J_{k+\mu,m}(\Gamma^J)$.

Proof. As before we put $g_t = g_t(\Phi)$ and $h_t = g_t(\Phi|_{(k,s),m} M)$. First we calculate $(L_m^t g_{\mu-2t})|_{k+\mu,m} M$. We have

Lemma 2.6.

$$\begin{aligned}
&(L_m^t g_{\mu-2t})|_{k+\mu,m} M\\
&= \sum_{l=0}^{t} \binom{t}{l} \sum_{\kappa=0}^{[\mu/2-t]} (-1)^\kappa (2\kappa-1)!! \binom{s-\mu+2t+2\kappa}{2\kappa}\\
&\qquad \times \frac{(2k+2\mu-2t-2\kappa-3)!!}{(2k+2\mu-2t-2\kappa-2l-3)!!} (2m)^{-\kappa+l} \left(\frac{1}{2\pi i}\frac{c}{c\tau+d}\right)^{\kappa+l} (L_m^{t-l} h_{\mu-2t-2\kappa}).
\end{aligned}$$

Proof. We prove this lemma by induction on μ. When $t = 0$, the lemma is nothing but Lemma 2.5. In particular, when $\mu = 0$, this is just the definition. We assume that the relation is true for $\mu - 2$ and any t with $\mu - 2 \geq 2t$ and under this assumption we show that the relation holds for μ and $t + 1$. By (8), we have

$$(L_m^{t+1} g_{\mu-2t-2})|_{k+\mu,m} M$$
$$= L_m(L_m^t g_{\mu-2t-2}|_{k+\mu-2,m} M) + (2k + 2\mu - 5)\frac{2mc}{2\pi i(c\tau + d)}(L_m^t g_{\mu-2t-2})|_{k+\mu-2,m} M.$$

For any function f on $\mathbb{H} \times \mathbb{C}$, we have

$$L_m((c\tau + d)^{-\kappa-l} f) = (-\kappa - l)\left(\frac{4mc}{c\tau + d}\right)(c\tau + d)^{-\kappa-l} f + (c\tau + d)^{-\kappa-l} L_m f.$$

By the inductive assumption we have

$$\frac{2mc(2k + 2\mu - 5)}{2\pi i(c\tau + d)}(L_m^t g_{\mu-2-2t})|_{k+\mu-2,m} M$$
$$= (2k + 2\mu - 5)\sum_{l=0}^{t}\binom{t}{l}\sum_{\kappa=0}^{[\mu/2-(t+1)]}(-1)^\kappa(2\kappa - 1)!!\binom{s - \mu + 2(t + 1) + 2\kappa}{2\kappa}$$
$$\times \frac{(2k + 2\mu - 2(t + 1) - 2\kappa - 5)!!}{(2k + 2\mu - 2(t + 1) - 2\kappa - 2l - 5)!!}$$
$$\times (2m)^{-k+l+1}\left(\frac{1}{2\pi i}\frac{c}{c\tau + d}\right)^{l+1+\kappa}$$
$$\times (L_m^{t+1-(l+1)} h_{\mu-2(t+1)-2\kappa}),$$

and

$$L_m(L_m^t g_{\mu-2-2t}|_{k+\mu-2,m} M)$$
$$= \sum_{l=0}^{t}\binom{t}{l}\sum_{\kappa=0}^{[\mu/2-(t+1)]}(-2\kappa - 2l)(-1)^\kappa(2\kappa - 1)!!\binom{s - \mu + 2(t + 1) + 2\kappa}{2\kappa}$$
$$\times \frac{(2k + 2\mu - 2(t + 1) - 2\kappa - 5)!!}{(2k + 2\mu - 2(t + 1) - 2\kappa - 2l - 5)!!}$$
$$\times (2m)^{-k+l+1}\left(\frac{1}{2\pi i}\frac{c}{c\tau + d}\right)^{l+1+\kappa}(L_m^{t+1-(l+1)} h_{\mu-2(t+1)-2\kappa})$$
$$+ \sum_{l=0}^{t}\binom{t}{l}\sum_{\kappa=0}^{[\mu/2-(t+1)]}(-1)^\kappa(2\kappa - 1)!!\binom{s - \mu + 2(t + 1) + 2\kappa}{2\kappa}$$
$$\times \frac{(2k + 2\mu - 2(t + 1) - 2\kappa - 5)!!}{(2k + 2\mu - 2(t + 1) - 2\kappa - 2l - 5)!!}(2m)^{-k+l}$$
$$\times \left(\frac{1}{2\pi i}\frac{c}{c\tau + d}\right)^{l+\kappa}(L_m^{t+1-l} h_{\mu-2(t+1)-2\kappa}).$$

Now we see the coefficient of $L_m^{t+1-(l+1)}h_{\mu-2(t+1)-2\kappa}$ for $0 \le l \le t$. Since

$$\binom{t}{l} = \frac{l+1}{t+1}\binom{t+1}{l+1}, \quad \text{and} \quad \binom{t}{l+1} = \frac{t-l}{t+1}\binom{t+l}{l+1},$$

we get

$$\frac{(l+1)(2k+2\mu-5)}{(t+1)(2k+2\mu-2(t+1)-2\kappa-2l-5)} + \frac{(l+1)(-2\kappa-2l)}{(t+1)(2k+2\mu-2(t+1)-2\kappa-2l-5)} + \frac{(t-l)}{t+1}$$
$$= \frac{2k+2\mu-2(t+1)-2\kappa-3}{2k+2\mu-2(t+1)-2\kappa-2l-5},$$

and we see that the coefficient is as desired. This is the same for $L_m^{t+1}h_{\mu-2(t+1)-2\kappa}$. Hence we proved Lemma 2.6. □

Proof of Theorem 2.1. By the above lemma, we have

$$\begin{aligned}
&\iota_\mu(\Phi)|_{k+\mu,m}M \\
&= \sum_{t=0}^{[\mu/2]} \binom{s-\mu+2t}{2t} \frac{(2k+2\mu-2t-5)!!\,(2t-1)!!}{(2m)^{2t}(2k+2\mu-5)!!} \\
&\qquad \times \sum_{l=0}^{t} \binom{t}{l} \sum_{\kappa=0}^{[\mu/2-t]} (-1)^\kappa (2m)^{-\kappa+l} \left(\frac{1}{2\pi i}\frac{c}{c\tau+d}\right)^{\kappa+l} (2\kappa-1)!! \binom{s-\mu+2t+2\kappa}{2\kappa} \\
&\qquad\qquad \times \frac{(2k+2\mu-2t-2\kappa-3)!!}{(2k+2\mu-2t-2\kappa-2l-3)!!} L_m^{t-l} h_{\mu-2t-2\kappa}.
\end{aligned}$$

We rewrite this. Since we have

$$(2t-1)!!\,(2\kappa-1)!! = \frac{(2t)!\,(2\kappa)!\,(2t+2\kappa-1)!!\,(t+\kappa)!}{\kappa!\,t!\,(2t+2\kappa)!},$$

we have

$$\begin{aligned}
&(2t-1)!!\,(2\kappa-1)!! \binom{s-\mu+2t}{2t}\binom{s-\mu+2(t+\kappa)}{2\kappa} \\
&= (2t+2\kappa-1)!! \binom{s-\mu+2t+2\kappa}{2t+2\kappa}\binom{t+\kappa}{\kappa}.
\end{aligned}$$

Now we fix non-negative integers α, β with $0 \le \beta \le \alpha \le [\mu/2]$ and calculate the coefficient of $L_m^\beta h_{\mu-2\alpha}$ in the above. For any l with $0 \le l \le \alpha-\beta$, we put $\kappa = \alpha - t$

and $t = \beta + l$. Then the coefficient of $L_m^\beta h_{\mu-2\alpha}$ for fixed μ, α, β is given by

$$(2m)^{-\alpha-\beta}\left(\frac{1}{2\pi i}\frac{c}{c\tau+d}\right)^{\alpha-\beta} \times (2\alpha-1)!! \binom{s-\mu+2\alpha}{2\alpha} \frac{(2k+2\mu-2\alpha-3)!!}{(2k+2\mu-5)!!} \binom{\alpha}{\beta}$$
$$\times (-1)^{\alpha-\beta} \sum_{l=0}^{\alpha-\beta} \binom{\alpha-\beta}{l} (-1)^l \frac{(2k+2\mu-2\beta-2l-5)!!}{(2k+2\mu-2\alpha-2l-3)!!}.$$

The last term is 0 for $\alpha > \beta$. We see this as follows. For any fixed enough big odd integer A, we put $f(x) = \sum_{l=0}^{t} (-1)^l \binom{t}{l} x^{A/2+t-l}$. Then we have

$$\frac{d^{t-1}f}{dx^{t-1}}(1) = \sum_{l=0}^{t} (-1)^l \binom{t}{l} \binom{A/2+t-l}{t-1} (t-1)!$$
$$= \frac{1}{2^{t-1}} \sum_{l=0}^{t} (-1)^l \binom{t}{l} \frac{(A+2t-2l)!!}{(A-2l+2)!!}.$$

On the other hand, since $f(x) = x^{A/2}(x-1)^t$, the $(t-1)$-th derivative $f^{(t-1)}(x)$ is divisible by $x-1$ and hence $f^{(t-1)}(1) = 0$. Thus we can assume $\alpha = \beta$ and then we have $\kappa = l = 0$, $\alpha = \beta = t$, and the last sum is given by $(2k+2\mu-2\alpha-3)^{-1}$. So the coefficient of $L_m^\alpha g_{\mu-2\alpha}$ is given by

$$\binom{s-\mu+2\alpha}{2\alpha} \frac{(2k+2\mu-2\alpha-5)!!(2\alpha-1)!!}{(2k+2\mu-5)!!}.$$

This is exactly the same as the coefficient of $L_m^\alpha g_{\mu-2\alpha}$ in the definition of $\iota_\mu(\Phi)$. Summing up over α, we have $\iota_\mu(\Phi)|_{k+\mu,m} M = \iota_\mu(\Phi|_{(k,s),m} M)$. □

For any integer l, we denote by $M_l(\Gamma)$ the space of holomorphic modular forms on $\mathbb{H}$ of weight k w.r.t. Γ.

Theorem 2.2. (1) *The mapping $D_{(k,s),m}$ is a bijection from W to W. For any natural integers k, s and $m \geq 1$, this induces an linear isomorphism of $J_{(k,s),m}(\Gamma^J)$ onto $J_{k+s,m}(\Gamma^J) \times J_{k+s-1,m}(\Gamma^J) \times \cdots \times J_{k,m}(\Gamma^J) \subset W$.*
(2) *For $m = 0$, we have $J_{(k,s),0}(\Gamma^J) \cong M_{k+s}(\Gamma)$.*
In particular, $J_{(k,s),m}(\Gamma^J)$ is always finite dimensional.

Proof. First we prove (1). By definition, we see that $L_m^t g_{\mu-2t} = L_m^t \varphi_{\mu-2t} +$ (terms determined by φ_ν with $0 \leq \nu < \mu - 2t$). So $\iota_\mu(\Phi) = \varphi_\mu +$ (terms determined by φ_ν with $0 \leq \nu < \mu - 1$). This means that the transformation matrix from $W \ni \Phi$ to $D_{(k,s),m}(\Phi) \in W$ is "upper triangular" whose diagonal components are 1, and hence the mapping is bijective on W. If $\Phi \in J_{(k,s),m}(\Gamma^J)$, then $\iota_\mu(\Phi) \in J_{k+\mu,m}(\Gamma^J)$ by Theorem 2.1 since the conditions on Fourier coefficients are obviously satisfied. The restriction of this mapping on

$J_{(k,s),m}(\Gamma^J)$ is of course also injective. We see that this is surjective to $J_{k+s,m}(\Gamma^J) \times \cdots \times J_{k,m}(\Gamma^J)$. Take $f_l \in J_{k+l,m}(\Gamma^J)$ for l with $0 \le l \le s$. Then since $D_{(k,s),m}$ is surjective on W, there exists $\Phi \in W$ such that $D_{(k,s),m}(\Phi) = {}^t(f_s, f_{s-1}, \ldots, f_0)$. By Theorem 2.1, for any $M \in \Gamma^J$, we have $\iota_\mu(\Phi|_{(k,s),m}M) = \iota_\mu(\Phi)|_{k+\mu,m}M = f_\mu|_{k+\mu}M = f_\mu = \iota_\mu(\Phi)$. By injectivity of $D_{(k,s),m}$ on W, we have $\Phi|_{(k,s),m} = \Phi$. The conditions on the Fourier coefficients are also easily seen. Hence this mapping is surjective. Now we show the assertion (2) for index zero. When $\Phi = {}^t(\varphi_s, \varphi_{s-1}, \ldots, \varphi_0) \in J_{(k,s),0}(\Gamma^J)$, we now show that $\varphi_l = 0$ for any $l < s$ by induction on l. We assume that $\varphi_j = 0$ for any $j \le l-1$. Then we have $\varphi_l(\tau, z+\tau\lambda+\mu) = \varphi_l(\tau, z)$ for any integers λ and μ. Hence for a fixed τ, the holomorphic function $\varphi_l(\tau, z)$ is a bounded for $z \in \mathbb{C}$ and consequently φ_l is independent of z and we write $\varphi_l(\tau) = \varphi_l(\tau, z)$. Next we show that $\varphi_l = 0$. Since

$$\varphi_{l+1}(\tau, z+\tau\lambda+\mu) = \varphi_{l+1}(\tau, z) - \lambda(s-l)\varphi_l(\tau),$$

we have

$$\frac{\partial\varphi_{l+1}}{\partial z}(\tau, z+\tau\lambda+\mu) = \frac{\partial\varphi_{l+1}}{\partial z}(\tau, z)$$

and hence $\partial\varphi_{l+1}/\partial z$ is also bounded and independent of z. This means that there exist holomorphic functions $c_1(\tau)$ and $c_2(\tau)$ of $\tau \in \mathbb{H}$ such that $\varphi_{l+1}(\tau, z) = c_1(\tau)z + c_2(\tau)$. Hence we have $c_1(\tau)(\tau\lambda+\mu) = -\lambda(s-l)\varphi_l(\tau)$ for any $\lambda, \mu \in \mathbb{Z}$. By taking $\lambda = 1$ and $\mu = 0, 1$, we see that $c_1(\tau) = \varphi_l(\tau) = 0$. By the same reason, $\varphi_s(\tau, z)$ is independent of z and $\varphi_s|_{k+s,m}M = \varphi_s$ for any $M \in \Gamma$. Hence $\varphi_s(\tau, z) \in M_{k+s}(\Gamma)$. □

REMARK. The Fourier–Jacobi coefficient of index 0 of Siegel modular forms F of $Sp(2, \mathbb{Z})$ of weight $\rho_{k,s}$ is ${}^t(f, 0, \ldots, 0)$ where $f \in S_{k+s}(SL_2(\mathbb{Z}))$: the space of cusp forms of weight $k+s$. (cf. Arakawa [1]). Hence an element of $J_{(k,s),m}(SL_2(\mathbb{Z})^J)$ is not necessarily the Fourier–Jacobi coefficient of Siegel modular forms.

EXAMPLES. If we write simply as $D_{(k,s),m}(\Phi) = {}^t(\tilde{\varphi}_s, \tilde{\varphi}_{s-1}, \ldots, \tilde{\varphi}_0)$ for $\Phi = {}^t(\varphi_s, \ldots, \varphi_0)$, then by definition $\tilde{\varphi}_\mu$ depends only on $\varphi_0, \ldots, \varphi_\mu$ and not on $\varphi_{\mu+1}, \ldots, \varphi_s$. For first few μ, they are given explicitly as

$$\begin{aligned}
\tilde{\varphi}_0 &= \varphi_0,\\
\tilde{\varphi}_1 &= \varphi_1 - (2m)^{-1}\binom{s}{1}\partial_z\varphi_0,\\
\tilde{\varphi}_2 &= \varphi_2 - (2m)^{-1}\binom{s-1}{1}\partial_z\varphi_1 + (2m)^{-2}\binom{s}{2}\left(\partial_z^2\varphi_0 + \frac{1}{2k-1}L_m\varphi_0\right),\\
\tilde{\varphi}_3 &= \varphi_3 - (2m)^{-1}\binom{s-2}{1}\partial_z\varphi_2 + (2m)^{-2}\binom{s-1}{2}\left(\partial_z^2\varphi_1 + \frac{1}{2k+1}L_m\varphi_1\right)\\
&\quad - (2m)^{-3}\binom{s}{3}\left(\partial_z^3\varphi_0 + \frac{3}{2k+1}\partial_z L_m\varphi_0\right),
\end{aligned}$$

$$\tilde{\varphi}_4 = \varphi_4 - (2m)^{-1}\binom{s-3}{1}\partial_z\varphi_3 + (2m)^{-2}\binom{s-2}{2}\left(\partial_z^2\varphi_2 + \frac{1}{2k+3}L_m\varphi_2\right)$$
$$- (2m)^{-3}\binom{s-1}{3}\left(\partial_z^3\varphi_1 + \frac{3}{2k+3}\partial_z L_m\varphi_1\right)$$
$$+ (2m)^{-4}\binom{s}{4}\left(\partial_z^4\varphi_0 + \frac{6}{2k+3}\partial_z^2 L_m\varphi_0 + \frac{3}{(2k+1)(2k+3)}L_m^2\varphi_0\right).$$

Next we give the inverse linear mapping of $D_{(k,s),m}$ explicitly. First we note that if $\Phi = {}^t(\varphi_s, \varphi_{s-1}, \ldots, \varphi_\nu, 0, \ldots, 0)$, then we can show that

$$D_{(k,s),m}(\Phi) = (D_{(k+\nu,s-\nu),m}(\varphi_s, \varphi_{s-1}, \ldots, \varphi_\nu), 0, \ldots, 0)$$

by comparing the definition of the both sides. If $\Phi \in J_{(k,s),m}(\Gamma^J)$ besides, then ${}^t(\varphi_s, \varphi_{s-1}, \ldots, \varphi_\nu) \in J_{(k+\nu,s-\nu),m}(\Gamma^J)$. Indeed, if $\Phi|_{(k,s),m}M = \Phi$ and $\Phi|_{s,m}X = \Phi$ for every $M \in \Gamma$ and $X \in \mathbb{Z}^2$, then by the conditions (3) and (4), we have

$$(\varphi_\mu|_{k,m}M) = \sum_{\kappa=0}^{\mu}\binom{s-\kappa}{\mu-\kappa}(c\tau+d)^\kappa(cz)^{\mu-\kappa}\varphi_\kappa,$$
$$\varphi_\mu|_m X = \sum_{\kappa=0}^{\mu}\binom{s-\kappa}{\mu-\kappa}(-\lambda)^{\mu-\kappa}\varphi_\kappa.$$

Since we are assuming that $\varphi_\kappa = 0$ for $\kappa < \nu$, we replace κ by $\kappa+\nu$ in the above and the right hand sides become

$$\sum_{\kappa=0}^{\mu-\nu}\binom{(s-\nu)-\kappa}{(\mu-\nu)-\kappa}(c\tau+d)^{\nu+\kappa}(cz)^{(\mu-\nu)-\kappa}\varphi_{\nu+\kappa}$$

and

$$\sum_{\kappa=0}^{\mu-\nu}\binom{(s-\nu)-\kappa}{(\mu-\nu)-\kappa}(-\lambda)^{(\mu-\nu)-\kappa}\varphi_{\nu+\kappa}.$$

So we get the assertion. For any holomorphic function f on $\mathbb{H}\times\mathbb{C}$ and any integers l, ν, s with $0 \le l \le \nu \le s$ we define a holomorphic function $\eta_{l,\nu}^{(s)}(f)$ on $\mathbb{H}\times\mathbb{C}$ by

$$\eta_{l,\nu}^{(s)}(f) = (2m)^{-\nu+l}\binom{s-l}{\nu-l}\sum_{j=0}^{[(\nu-l)/2]}(-1)^j\binom{\nu-l}{2j}\frac{(2j-1)!!\,(2k-3)!!}{(2k+2j-3)!!}\partial_z^{\nu-l-2j}L_m^j f,$$

and if $0 \le \nu < l$ we define $\eta_{l,\nu}^{(s)}(f) = 0$. We write

$$\eta_l^{(s)}(f) = {}^t(\eta_{l,s}^{(s)}(f), \eta_{l,s-1}^{(s)}(f), \ldots\ldots\ldots\ldots, \eta_{l,0}^{(s)}(f))$$
$$= {}^t(\eta_{l,s}^{(s)}(f), \eta_{l,s-1}^{(s)}(f), \ldots, \eta_{l,l}^{(s)}(f), 0, \ldots, 0).$$

If $l \le \nu$, then we have $\eta_{l,\nu}^{(s)} = \eta_{0,\nu-l}^{(s-l)}$.

Theorem 2.3. *Notation being as above, the mapping* $W \ni F = (f_s, \dots, f_0) \to \eta^{(s)}(F) = \eta_0^{(s)}(f_0) + \cdots + \eta_l^{(s)}(f_l) + \cdots + \eta_s^{(s)}(f_s) \in W$ *gives the inverse of* $D_{(k,s),m}$. *In particular, if* $f_l \in J_{k+l,m}(\Gamma^J)$, *then* $\eta(f_l) \in J_{(k,s),m}(\Gamma^J)$, *and the above mapping gives a linear isomorphism of* $J_{k+s,m}(\Gamma^J) \times \cdots \times J_{k,m}(\Gamma^J)$ *onto* $J_{(k,s),m}(\Gamma^J)$.

Proof. First we prove that $D_{(k,s),m}(\eta_0(f_0)) = (0, \dots, 0, f_0)$. We put $\varphi_\mu = \eta_{0,\mu}(f_0)$ and $\Phi = {}^t(\varphi_s, \dots, \varphi_0)$. We have to show that $\iota_0(\Phi) = f_0$ and $\iota_\mu(\Phi) = 0$ for any $\mu > 0$. By definition (9), we have $\iota_0(\Phi) = \varphi_0 = \eta_{0,0}^{(s)}(f_0) = f_0$. Now we assume that $\mu > 0$. We have

$$\begin{aligned}
&\iota_\mu(\Phi)\\
&= \sum_{\nu=0}^{\mu} \sum_{t=0}^{[(\mu-\nu)/2]} (-2m)^{\nu-\mu} \binom{s-\nu}{\mu-\nu} \binom{\mu-\nu}{2t} \frac{(2k+2\mu-2t-5)!!\,(2t-1)!!}{(2k+2\mu-5)!!} \partial_z^{\mu-\nu-2t} L_m^t \varphi_\nu\\
&= \sum_{\nu=0}^{\mu} \sum_{t=0}^{[(\mu-\nu)/2]} (-2m)^{\nu-\mu} \binom{s-\nu}{\mu-\nu} \binom{\mu-\nu}{2t} \frac{(2k+2\mu-2t-5)!!\,(2t-1)!!}{(2k+2\mu-5)!!}\\
&\qquad \times \sum_{j=0}^{[\nu/2]} (2m)^{-\nu} \binom{s}{\nu} (-1)^j \binom{\nu}{2j} \frac{(2j-1)!!\,(2k-3)!!}{(2k+2j-3)!!} \partial_z^{\mu-2t-2j} L_m^{t+j} f_0.
\end{aligned}$$

We put $\alpha = t + j$ and calculate the coefficient of $\partial_z^{\mu-2\alpha} L_m^\alpha f_0$. We have

$$\binom{s-\nu}{\mu-\nu}\binom{\mu-\nu}{2t}\binom{s}{\nu}\binom{\nu}{2j} = \binom{s}{\mu}\binom{\mu-2\alpha}{\nu-2j}\binom{\mu}{2\alpha}\frac{(2\alpha)!}{(2t)!\,(2j)!}.$$

We also have

$$\frac{(2t-1)!!\,(2j-1)!!\,(2\alpha)!}{(2t)!\,(2j)!} = (2\alpha-1)!!\binom{\alpha}{j}.$$

Hence, fixing μ and α, the coefficient of $\partial_z^{\mu-2\alpha} L_m^\alpha f_0$ is given by

$$\begin{aligned}
&(-2m)^{-\mu}\binom{\mu}{2\alpha}\binom{s}{\mu}\frac{(2k-3)!!\,(2\alpha-1)!!}{(2k+2\mu-5)!!}\\
&\times \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \frac{(2k+2\mu-2\alpha+2j-5)!!}{(2k+2j-3)!!} \sum_{\nu=2j}^{\mu-2\alpha+2j} (-1)^\nu \binom{\mu-2\alpha}{\nu-2j}.
\end{aligned}$$

We have $\sum_{\nu=2j}^{\mu-2\alpha+2j}(-1)^\nu \binom{\mu-2\alpha}{\nu-2j} = 0$ unless $\mu = 2\alpha$. If $\mu = 2\alpha$, then we can show as before by differentiating $x^{k+\alpha-5/2}(x-1)^\alpha$ by x $\alpha - 1$ times that

$$\sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \frac{(2k+2\alpha+2j-5)!!}{(2k+2j-3)!!} = 0$$

unless $\alpha = 0$. But if $\alpha = 0$ then $\mu = 0$, so we have $\iota_\mu(\Phi) = 0$ unless $\mu = 0$. So $\eta_0(f_0)$ gives an inverse image of $(0, \dots, 0, f_0)$ by $D_{(k,s),m}$. Hence by Theorem 2.2 (2), if $f_0 \in J_{k,m}(\Gamma^J)$ besides, then we have $\eta_0^{(s)}(f_0) \in J_{(k,s),m}(\Gamma^J)$. We have

$$\begin{aligned} D_{(s,k),m}(\eta_l^{(s)}(f_l)) &= (D_{(k+l,s-l),m}(\eta_{l,s}^{(s)}(f_l), \dots, \eta_{l,l}^{(s)}(f_l)), 0, \dots, 0) \\ &= (D_{(k+l,s-l),m}(\eta_{0,s-l}^{(s-l)}(f_l), \dots, \eta_{0,0}^{(s-l)}(f_l)), 0, \dots, 0) \\ &= (D_{(k+l,s-l),m}(\eta_0^{(s-l)}(f_l)), 0, \dots, 0). \end{aligned}$$

By the result for $\eta_0^{(s)}(f_0)$, the last expression is equal to $(0, \dots, 0, f_l, 0, \dots, 0)$. So summing up the result for each l, we have

$$\sum_{l=0}^{s} D_{(k,s),m}(\eta_l^{(s)}(f_l)) = (f_s, \dots, f_0).$$

Since $D_{(k,s),m}$ gives a bijection between Jacobi forms by Theorem 2.2 (1), if $f_l \in J_{k+l,m}(\Gamma_J)$, then $\eta_l^{(s)}(f_l) \in J_{(k,s),m}(\Gamma^J)$. □

EXAMPLES. We assume that $f_0 \in J_{k,m}(\Gamma^J)$.

$$\eta_{0,0}^{(s)}(f_0) = f_0,$$

$$\eta_{0,1}^{(s)}(f_0) = (2m)^{-1}\binom{s}{1}\partial_z f_0,$$

$$\eta_{0,2}^{(s)}(f_0) = (2m)^{-2}\binom{s}{2}\left(\partial_z^2 f_0 - \frac{1}{2k-1}L_m f_0\right),$$

$$\eta_{0,3}^{(s)}(f_0) = (2m)^{-3}\binom{s}{3}\left(\partial_z^3 f_0 - \frac{3}{2k-1}\partial_z L_m f_0\right),$$

$$\eta_{0,4}^{(s)}(f_0) = (2m)^{-4}\binom{s}{4}\left(\partial_z^4 f_0 - \frac{6}{2k-1}\partial_z^2 L_m f_0 + \frac{3}{(2k+1)(2k-1)}L_m^2 f_0\right).$$

3. Eisenstein series

As in the case of scalar valued Jacobi forms, we define Eisenstein series of the vector valued Jacobi forms. From now on, for the sake of simplicity we assume that $\Gamma = \Gamma_1 = SL_2(\mathbb{Z})$ and $\Gamma_1^J = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. We define a subgroup Γ_∞^J of Γ_1^J by

$$\Gamma_\infty^J := \{\gamma \in \Gamma_1^J;\ 1|_{k,m}\gamma = 1\} = \left\{\left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0, \mu)\right);\ n, \mu \in \mathbb{Z}\right\}.$$

For any good function $\Phi \in W$ invariant by Γ_∞^J and any natural numbers k, s, and $m \geq 1$, we define an Eisenstein series $E_{(k,s),m}(\tau, z; \Phi) \in J_{(k,s),m}(\Gamma^J)$ by

$$E_{(k,s),m}(\tau, z; \Phi) = \sum_{\gamma \in \Gamma_\infty^J \backslash \Gamma_1^J} \Phi|_{(k,s),m}\gamma.$$

For example, if Φ is a vector of constant functions, then this is written as

$$E_{(k,s),m}(\tau, z; \Phi) = \frac{1}{2} \sum_{(c,d)=1} \sum_{\lambda \in \mathbb{Z}} \rho_{k,s} \begin{pmatrix} c\tau + d & cz - \lambda \\ 0 & 1 \end{pmatrix}^{-1} e^m \left(\lambda^2 \frac{a\tau + b}{c\tau + d} + 2\lambda \frac{z}{c\tau + d} - \frac{cz^2}{c\tau + d} \right) \Phi.$$

We denote by $\mathbf{e}_j$ the unit vector of length $s+1$ whose j-th component is 1 and the others are zero. If $\Phi = q^n \zeta^r \mathbf{e}_{s+1-j}$ for integers $n \geq 0$ and r, then the series $E_{(k,s),m}(\tau, z; \Phi)$ is convergent for $k \geq 4$. For a function ϕ on $\mathbb{H} \times \mathbb{C}$ such that $\phi|_{k,m} M = \phi$ for all $M \in \Gamma_\infty^J$, we also put

$$E_{k,m}(\tau, z; \phi) = \sum_{\gamma \in \Gamma_\infty^J \backslash \Gamma_1^J} \phi|_{k,m} \gamma.$$

By Theorem 2.1, we see

$$D_{(k,s),m}(E_{(k,s),m}(\tau, z; \phi \mathbf{e}_{s+1-j})) = \left(\sum_{\gamma \in \Gamma_\infty^J \backslash \Gamma_1^J} \iota_\mu(\phi \mathbf{e}_{s+1-j})|_{k+\mu,m} \gamma \right)_{0 \leq \mu \leq s}.$$

If we assume that $L_m \phi = 0$ for any $m \geq 1$, then by definition of ι_μ, we have

$$\iota_\mu(\phi \mathbf{e}_{s+1-j}) = \begin{cases} 0 & \text{if} \quad 0 \leq \mu < j, \\ (-2m)^{j-\mu} \binom{s-j}{\mu-j} \partial_z^{\mu-j} \phi & \text{if} \quad j \leq \mu \leq s. \end{cases}$$

Now as in Eichler–Zagier [2], let b be the largest integer such that $b^2 \mid m$ and put $m = ab^2$. For any integer t, we put $\phi_t = q^{at^2} \zeta^{2abt}$. Then this is invariant by Γ_∞^J and we have $L_m \phi_t = (4mat^2 - (2abt)^2)\phi = 0$ and $\partial_z^{\mu-j} \phi_t = (2abt)^{\mu-j} \phi_t$. So we have

$$\iota_\mu(E_{(k,s),m}(\tau, z, \phi_t \mathbf{e}_{s+1-j})) = \begin{cases} 0 & \text{if} \quad 0 \leq \mu < j, \\ (-m^{-1}abt)^{\mu-j} \binom{s-j}{\mu-j} E_{k+\mu,m}(\tau, z, \phi_t) & \text{if} \quad j \leq \mu \leq s. \end{cases}$$

Then for a fixed μ, the series $E_{k+\mu,m,t} = E_{k+\mu,m}(\tau, z, \phi_t)$ are nothing but the Eisenstein series defined in [2] p. 25, which span $J^{\mathrm{Eis}}_{k+\mu,m}$.

We assume that $k \geq 4$. We denote by $J^{\mathrm{Eis}}_{(k,s),m}(\Gamma_1^J)$ the linear space spanned by $E_{(k,s),m}(\tau, z, q^{at^2} \zeta^{2abt} \mathbf{e}_j)$ for any j with $0 \leq j \leq s$, a, b with $m = ab^2$ as above and $t = 0, \ldots, [b/2]$ if k is even and $t = 0, \ldots, [(b-1)/2]$ if k is odd. By the above consideration, we have $D_{(k,s),m}(J^{\mathrm{Eis}}_{(k,s),m}(\Gamma_1^J)) = \prod_{j=0}^s J^{\mathrm{Eis}}_{k+\nu,m}(\Gamma_1^J)$ and also the space $J^{\mathrm{Eis}}_{(k,s),m}(\Gamma_1^J)$ is the space spanned by $\eta^{(s)}(J^{\mathrm{Eis}}_{k+j,m}(\Gamma_1^J)))$ for all integer j with $0 \leq j \leq s$.

The Petersson inner product of $F = (f_s, \dots, f_0)$ and $G = (g_s, \dots, g_0) \in J_{k+s,m}(\Gamma_1^J) \times \cdots \times J_{k,m}(\Gamma_1^J)$ is defined as the sum of the usual inner product of the direct summands and given by

$$\langle F, G \rangle = \sum_{j=0}^{s} \int_{\Gamma^J \backslash \mathbb{H} \times \mathbb{C}} \exp\left(\frac{-4\pi m y^2}{v}\right) v^{k+j-3} f_j(\tau, z) \overline{g_j(\tau, z)}\, dx\, dy\, du\, dv$$

where $\tau = x + iy \in \mathbb{H}$ and $z = u + iv \in \mathbb{C}$. We define the inner product of $J_{(k,s),m}(\Gamma_1^J)$ by this through the isomorphism in Theorem 2.2.

Proposition 3.1. *We assume that $k \geq 4$. We have*

$$J_{(k,s),m}(\Gamma_1^J) = J^{\mathrm{cusp}}_{(k,s),m}(\Gamma_1^J) \oplus J^{\mathrm{Eis}}_{(k,s),m}(\Gamma_1^J).$$

This follows directly from [2] p. 25 Theorem 2.3.

Finally we give a little remark. Imitating [2], we define an operator V_l which shifts the index, mapping $J_{(k,s),m}(\Gamma_1^J)$ to $J_{(k,s),ml}(\Gamma_1^J)$. First we generalize the action of $SL_2(\mathbb{R})$ to the group $GL_2^+(\mathbb{R})$ of 2×2 matrices with positive determinants. For any $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{R})$ with $\det(M) = l$ and any $\Phi \in W$, we write

$$\Phi|_{(k,s),m} M = (c\tau + d)^{-k} e^{ml}\left(-\frac{cz^2}{c\tau + d}\right) \rho_s\left(\begin{matrix} c\tau + d & cz \\ 0 & 1 \end{matrix}\right)^{-1} \Phi\left(\frac{a\tau + b}{c\tau + d}, \frac{lz}{c\tau + d}\right).$$

Then for $M_1, M_2 \in GL_2^+(\mathbb{R})$ with $\det(M_1) = l$, we have

$$\Phi|_{(k,s),m} M_1 M_2 = (\Phi|_{(k,s),m} M_1)|_{(k,s),ml} M_2.$$

Lemma 3.1. *For $M \in GL_2^+(\mathbb{R})$ with $\det(M) = l$, we have*

$$D_{(k,s),ml}(\Phi|_{(k,s),m} M) = (D_{(k,s),m}\Phi)|_{k+s} M,$$

where we write $k + s = (k + s, k + s - 1, \dots, k)$.

Now the operator V_l is defined by

$$\Phi|_{(k,s),m} V_l$$
$$= l^{k-1} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \backslash M(l)} \rho_{k,s}\left(\begin{matrix} c\tau + d & cz \\ 0 & 1 \end{matrix}\right)^{-1} e^{ml}\left(-\frac{cz^2}{c\tau + d}\right) \Phi\left(\frac{a\tau + b}{c\tau + d}, \frac{lz}{c\tau + d}\right),$$

where we put $M(l) = \{M \in M_2(\mathbb{Z});\ \det M = l\}$. We put

$$\tilde{D}_{(k,s),m} = \rho_s\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} D_{(k,s),m}.$$

For any m, we put $E_{(k,s),m} = E_{(k,s),m}(\tau, z; \mathbf{e}_{s+1})$.

Proposition 3.2. *We have*

$$\tilde{D}_{(k,s),ml}(\Phi|_{(k,s),m} V_l) = (\tilde{D}_{(k,s),m}\Phi)|_{k+s} V_l$$

and $E_{(k,s),m} = \sigma_{k-1}(m) E_{(k,s),1}|V_m$.

Proof. By Eichler–Zagier [2] p. 46 Theorem 4.3, it is known that $E_{k,m} = \sigma_{k-1}(m)E_{k,1}|V_m$. Since $\tilde{D}_{(k,s),1}(E_{(k,s),1}) = (0, \ldots, 0, E_{k,1}) \in \prod_{\nu=0}^{s} J_{k+\nu,1}(\Gamma_1^J)$, we have

$$\begin{aligned}\sigma_{k-1}(l)\tilde{D}_{(k,s),l}(E_{k,l}(\tau, z; \mathbf{e}_{s+1})|V_l) &= \sigma_{k-1}(l)(\tilde{D}_{(k,s),l}(E_{(k,s),1}))|_{k+s,1}V_l) = (0, \ldots, 0, E_{k,l}) \\ &= \tilde{D}_{(k,s),l}(E_{(k,s),l}(\tau, z; \mathbf{e}_{s+1})).\end{aligned}$$

Since $\tilde{D}_{(k,s),l}$ is injective, we have the result. □

4. Half-integral weight case

Here we explain how we can modify our results also for half-integral weight case. We put

$$\Gamma_0^{(n)}(4) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Z});\ C \equiv 0 \bmod 4 \right\},$$

and put $\Gamma_0(4)^J = J(\mathbb{R}) \cap \Gamma_0^{(2)}(4)$. For the sake of simplicity, we write $\Gamma_0(4) = \Gamma_0^{(1)}(4)$. In order to define automorphy factors of half-integral weight, we put

$$\theta_n(Z) = \sum_{p \in \mathbb{Z}^n} e(2\pi i^t p Z p)$$

for $Z \in \mathbb{H}_n$. For any $k \in \mathbb{Z}$, we write $\rho_{k+1/2,s}(CZ + D) = (\theta_2(MZ)/\theta_2(Z))^{2k+1} \times \rho_s(CZ + D)$. We define $F|_{k+1/2,s}M$ for $M \in \Gamma_0(4)$ similarly as in the case of integral weight in the introduction by replacing $\rho_{k,s}$ by $\rho_{k+1/2,s}$. We say that a holomorphic function F on $\mathbb{H}_2$ is a Siegel modular form of weight $\rho_{k+1/2,s}$ of $\Gamma_0^{(2)}(4)$, if $F|_{k+1/2,s}M = F$ for any $M \in \Gamma_0^{(2)}(4)$. We denote by $A_{k+1/2,s}(\Gamma_0^{(2)}(4))$ the vector space of such functions. Since F is translation invariant, we have the Fourier–Jacobi expansion $F(Z) = \sum_{m=0}^{\infty} \Phi_m(\tau, z)e(m\tau')$ as before. We denote by χ a character of $\Gamma_0(4)$. Now we define a vector valued Jacobi form of half-integral weight of $\Gamma_0(4)^J$

with character χ. The action $\Phi|_{(k+1/2,s),m}$ of $M \in \Gamma_0(4)$ is defined similarly as in the introduction by replacing $\rho_{k,s}$ by $\rho_{k+1/2,s}$ and the action of $F|_{s,m}(X,\kappa)$ for $(X,\kappa) \in \mathbb{Z}^3$ is just the same as in the introduction. Then a vector valued Jacobi form of weight $\rho_{k+1/2,s}$ is defined to be a $\mathbb{C}^{s+1}$-valued function such that $\Phi|_{(k+1/2,s),m}M = \chi(M)\Phi$ for all $M \in \Gamma_0(4)$, $\Phi|_{s,m}(X,\kappa) = \Phi$ for all $(X,\kappa) \in \mathbb{Z}^3$ and that F satisfies the following condition at each cusps of $\Gamma_0(4)$. For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we put

$$\Phi_M = (c\tau + d)^{-k-1/2}\rho_s\begin{pmatrix} c\tau + d & cz \\ 0 & 1 \end{pmatrix}^{-1} e^m\left(\frac{-cz^2}{c\tau + d}\right)\Phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right)$$

where the branch of $(c\tau + d)^{1/2}$ is fixed for each M. Then the Fourier expansion of the function Φ_M is given by the following form

$$\Phi_M = \sum_{n,r \in N_M^{-1}\mathbb{Z},\, r^2 \le 4nm} C_M(n,r)q^n\zeta^r$$

where N_M is a suitably chosen integer for each M. This condition does not depend on the choice of the branch. We denote by $J_{(k+1/2,s),m}(\Gamma_0(4), \chi)$ the vector space of all such Jacobi forms. When χ is trivial, we write $J_{(k+1/2,s),m}(\Gamma_0(4), \chi) = J_{(k+1/2,s),m}(\Gamma_0(4))$ and when $s = 0$, we write $J_{(k+1/2,s),m} = J_{k+1/2,m}$.

Theorem 4.1. *We have the following linear isomorphism.*

$$J_{(k+1/2,s),m}(\Gamma_0(4)^J) \cong \prod_{l=0}^{s} J_{k+1/2+l,m}(\Gamma_0(4)^J, \chi^l).$$

Here the isomorphism is given by the differential operator $D_{(k+1/2,s),m}$ on $\Phi = {}^t(\varphi_s, \varphi_{s-1}, \ldots, \varphi_0) \in W_s$ given by

$$(D_{(k+1/2,s),m}\Phi)_\mu = \sum_{t=0}^{[\mu/2]} \binom{s-\mu+2t}{2t} \frac{2^{-t}(k+\mu-t-2)!\,(2t-1)!!}{(2m)^{2t}(k+\mu-2)!} L_m^t(g_{\mu-2t}(\Phi)),$$

where $g_{\mu-2t}$ is defined as in Section 2 for a fixed m and s. In particular, for any $\gamma \in \Gamma_0(4)$ and $X \in \mathbb{Z}^2$ and a function $\Phi \in W_s$, we have

$$(D_{(k+1/2,s),m}\Phi)_\mu|_m X = (D_{(k+1/2,s),m}(\Phi|_{s,m}X))_\mu,$$
$$(D_{(k+1/2,s),m}\Phi)_\mu|_{k+\mu+1/2,m}\gamma = \chi(\gamma)^{-\mu}(D_{(k+1/2,s),m}(\Phi|_{(k+1/2,s),m}\gamma))_\mu.$$

We do not give here the details of the proof. Instead we explain which points differ from the case of integral weight. First for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ we put

$$M_\gamma = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we note that $\theta_2(M_\gamma Z)/\theta_2(Z) = \theta_1(\gamma\tau)/\theta_1(\tau)$. So the automorphy factor defined by $\rho_{k+1/2,s}$ is compatible with the usual automorphy factor of degree one. We also have

$$\partial_\tau \left(\frac{\theta(\gamma\tau)}{\theta(\tau)}\right)^{-2k-1} = \frac{-(2k+1)}{2}\left(\frac{d}{2\pi i(c\tau+d)}\right)\left(\frac{\theta(\gamma\tau)}{\theta(\tau)}\right)^{-2k-1}.$$

So the most of the formulas in the previous sections are satisfied in the same way and we have our theorem.

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