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GROWTH PROPERTIES OF p-TH MEANS OF BIHARMONIC GREEN POTENTIALS IN THE UNIT BALL

Dedicated to Professor Masakazu Shiba on the occasion of his sixtieth birthday

TOSHIHIDE FUTAMURA and YOSHIHIRO MIZUTA

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Abstract

Let u be a biharmonic Green potential on the unit ball $\mathbf{B}$ of $\mathbf{R}^n$. We show that

$$\lim_{r\to 1}(1-r)^{n-2-(n-1)/p}\mathcal{M}_p(u,r)=0$$

for p such that $1\le p<(n-1)/(n-4)$ in case $n\ge 5$ and $1\le p<\infty$ in case $n\le 4$. Further, if $n\ge 5$ and $(n-1)/(n-4)\le p<(n-1)/(n-5)$, then it is shown that

$$\liminf_{r\to 1}(1-r)^{n-2-(n-1)/p}\mathcal{M}_p(u,r)=0.$$

Finally we show that these limits characterize biharmonic Green potentials among super-biharmonic functions on $\mathbf{B}$.

1. Introduction and statement of results

A function u on an open set $\Omega\subset\mathbf{R}^n$ $(n\ge 2)$ is called biharmonic if $u\in C^4(\Omega)$ and $\Delta^2 u=0$ on Ω, where Δ denotes the Laplacian and $\Delta^2 u=\Delta(\Delta u)$. We say that a lower semicontinuous and locally integrable function u on Ω is super-biharmonic in Ω if every point of Ω is a Lebesgue point of u and $\Delta^2 u$ is a nonnegative measure on Ω in the weak sense, that is,

$$\int_\Omega u(x)\Delta^2\varphi(x)\,dx\ge 0\quad\text{for all nonnegative}\quad \varphi\in C_0^\infty(\Omega).$$

The open ball and the sphere centered at x with radius r are denoted by $B(x,r)$ and $S(x,r)$. We write $B(r)=B(0,r)$ and $S(r)=S(0,r)$. We also denote by $\mathbf{B}$ and $\mathbf{S}$ the unit ball $B(1)$ and the unit sphere $S(1)$. We write $x^*=|x|^{-2}x$, so that x^* is the inverse point of x relative to the unit sphere $\mathbf{S}$.

Let $G_2(x,y)$ denote the biharmonic Green function in the unit ball $\mathbf{B}$ (cf. [8]), that

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is,

$$G_2(x,y) = \begin{cases} \alpha_n \left(|x-y|^{4-n} - (|y||x-y^*|)^{4-n} - \dfrac{n-4}{2}(1-|x|^2)(1-|y|^2)(|y||x-y^*|)^{2-n} \right) \\ \hfill \text{in case} \quad n \neq 2,4, \\ \alpha_n \left(|x-y|^{4-n} \log\left(\dfrac{|x-y|}{|y||x-y^*|} \right)^2 + (1-|x|^2)(1-|y|^2)(|y||x-y^*|)^{2-n} \right) \\ \hfill \text{in case} \quad n = 2,4, \end{cases}$$

where $\alpha_n^{-1} = 2(4-n)(2-n)\sigma_n$ for $n \neq 2,4$ and $\alpha_n^{-1} = (-1)^{n/2+1} 8\sigma_n$ for $n = 2,4$. Here σ_n denotes the surface measure of the unit sphere $\mathbf{S}$. For a nonnegative measure μ on $\mathbf{B}$, we define

$$G_2\mu(x) = \int_{\mathbf{B}} G_2(x,y)\, d\mu(y).$$

If μ has a density $f \in L^1_{\mathrm{loc}}(\mathbf{B})$, then we write $G_2 f$ instead of $G_2\mu$. The function $G_2\mu$ is called a biharmonic Green potential if $G_2\mu \not\equiv \infty$.

For a Borel measurable function u on $S(r)$, we define the average integral over $S(r)$ by

$$\mathcal{M}(u,r) = \fint_{S(r)} u\, dS = \frac{1}{|S(r)|} \int_{S(r)} u\, dS,$$

where $|S(r)|$ denotes the surface measure of $S(r)$. For $p > 0$ and a Borel measurable function u on $S(r)$, define $\mathcal{M}_p(u,r) = \{\mathcal{M}(|u|^p, r)\}^{1/p}$.

Gardiner [5] and the second author [9] studied the limiting behavior of $\mathcal{M}_p(v,r)$ for (harmonic) Green potentials on $\mathbf{B}$. We also refer to Stoll [13, 14] for invariant potentials in the unit ball of $\mathbf{C}^n$.

In this paper we are concerned with biharmonic Green potentials on $\mathbf{B}$.

Theorem 1.1. *Let $G_2\mu$ be a biharmonic Green potential on* $\mathbf{B}$. *If $1 \leq p < (n-1)/(n-4)$ in case $n \geq 5$ and $1 \leq p < \infty$ in case $n \leq 4$, then*

$$\lim_{r\to 1} (1-r)^{n-2-(n-1)/p} \mathcal{M}_p(G_2\mu, r) = 0.$$

Theorem 1.2. *Let $G_2\mu$ be a biharmonic Green potential on* $\mathbf{B}$. *If $n \geq 5$ and $(n-1)/(n-4) \leq p < (n-1)/(n-5)$, then*

$$\liminf_{r\to 1} (1-r)^{n-2-(n-1)/p} \mathcal{M}_p(G_2\mu, r) = 0.$$

Finally we give a characterization for a super-biharmonic function to be a biharmonic Green potential on $\mathbf{B}$.

Theorem 1.3. *Let u be a super-biharmonic function on* **B**. *If u satisfies*

$$\liminf_{r\to 1}(1-r)^{-1}\mathcal{M}_1(u,r)=0,$$

then it is a biharmonic Green potential on **B**.

2. p-th means of biharmonic Green potentials

Throughout this paper, let M denote various constants independent of the variables in question.

We need the following fundamental estimates for the biharmonic Green function on the unit ball **B** (cf. [1] and [7]).

Lemma 2.1. *There exist positive constants C_i, $1\le i\le 4$, satisfying the following conditions*:

(1) *If $n\ge 5$, then for every $(x,y)\in \mathbf{B}\times\mathbf{B}$*

$$0<C_1\frac{(1-|x|^2)^2(1-|y|^2)^2}{|x-y|^{n-4}(|y|\,|x-y^*|)^4}\le G_2(x,y)\le C_2\frac{(1-|x|^2)^2(1-|y|^2)^2}{|x-y|^{n-4}(|y|\,|x-y^*|)^4}.$$

(2) *If $n=4$, then for every $(x,y)\in \mathbf{B}\times\mathbf{B}$*

$$\begin{aligned}0\;<\;& C_1\frac{(1-|x|^2)^2(1-|y|^2)^2}{(|y|\,|x-y^*|)^4}\log\left(\frac{2|y|\,|x-y^*|}{|x-y|}\right)\\ \le\;& G_2(x,y)\le C_2\frac{(1-|x|^2)^2(1-|y|^2)^2}{(|y|\,|x-y^*|)^4}\log\left(\frac{2|y|\,|x-y^*|}{|x-y|}\right).\end{aligned}$$

(3) *If $n=2,3$, then for every $(x,y)\in \mathbf{B}\times\mathbf{B}$*

$$C_1\frac{(1-|x|^2)^2(1-|y|^2)^2}{(|y|\,|x-y^*|)^n}\le G_2(x,y)\le C_2\frac{(1-|x|^2)^2(1-|y|^2)^2}{(|y|\,|x-y^*|)^n}.$$

Furthe, in all cases,

$$C_3\frac{(1-|x|^2)^2(1-|y|^2)^2}{(|y|\,|x-y^*|)^n}\le G_2(x,y)\le C_4\frac{(1-|x|^2)^2(1-|y|^2)^2}{|x-y|^n}.$$

By Lemma 2.1, we have the following result; see [7] and [10].

Corollary 2.2. *Let μ be a nonnegative measure on* **B**. *Then $G_2\mu$ is a biharmonic Green potential if and only if*

$$\int_{\mathbf{B}}(1-|y|)^2\,d\mu(y)<\infty. \tag{2.1}$$

Lemma 2.3. *If* $(n-1)/n < p < \infty$ *and* $1/2 < r < 1$, *then*

$$\mathcal{M}_p(G_2(\,\cdot\,,y),r) \le M(1-r)^{2-n+(n-1)/p}\left(\frac{1-r}{|r-|y||}\right)^{n-(n-1)/p}(1-|y|)^2.$$

In particular, if $n = 2, 3$ *and* $(n-1)/n < p < \infty$, *then*

$$\mathcal{M}_p(G_2(\,\cdot\,,y),r) \le M(1-r)^{2-n+(n-1)/p}\left(\frac{1-r}{1-r|y|}\right)^{n-(n-1)/p}(1-|y|)^2.$$

This follows from Lemma 2.1 and the fact that, if $\beta < 1-n$, then

$$\fint_{S(r)} |x-y|^\beta \, dS(x) \le M(r+|y|)^{1-n}|r-|y||^{\beta+n-1}, \tag{2.2}$$

where M is a positive constant depending only on n and β.

Set $A(r) = \{y \in \mathbf{B} \colon (5r-1)/4 < |y| < (3r+1)/4\}$ for $0 < r < 1$.

Lemma 2.4. *Let* μ *be a nonnegative measure on* $\mathbf{B}$ *satisfying* (2.1). *If* $(n-1)/n < p < \infty$, *then*

$$\lim_{r\to 1}(1-r)^{n-2-(n-1)/p}\int_{\mathbf{B}\setminus A(r)} \mathcal{M}_p(G_2(\,\cdot\,,y),r)\,d\mu(y) = 0.$$

Proof. By Lemma 2.3, we obtain

$$\begin{aligned}(1-r)^{n-2-(n-1)/p}&\int_{\mathbf{B}\setminus A(r)} \mathcal{M}_p(G_2(\,\cdot\,,y),r)\,d\mu(y)\\ &\le M\int_{\mathbf{B}\setminus A(r)}\left(\frac{1-r}{|r-|y||}\right)^{n-(n-1)/p}(1-|y|)^2\,d\mu(y).\end{aligned}$$

Since $(1-r)/|r-|y|| \le 4$ for $y \in \mathbf{B}\setminus A(r)$, Lebesgue's dominated convergence theorem implies that

$$\lim_{r\to 1}(1-r)^{n-2-(n-1)/p}\int_{\mathbf{B}\setminus A(r)} \mathcal{M}_p(G_2(\,\cdot\,,y),r)\,d\mu(y) = 0.$$

In case $n = 2$ and 3, we can show by Lemma 2.3 that

$$\lim_{r\to 1}(1-r)^{n-2-(n-1)/p}\int_{\mathbf{B}} \mathcal{M}_p(G_2(\,\cdot\,,y),r)\,d\mu(y) = 0. \qquad \square$$

Lemma 2.5. *Let* $1/2 < r < 1$ *and* $y \in A(r)$. *If* $n \geq 5$, *then*

$$\mathcal{M}_p(G_2(\,\cdot\,, y), r) \leq M(1-r)^{2-n+(n-1)/p}(1-|y|)^2$$
$$\times \begin{cases} 1 & \text{if} \quad \dfrac{n-1}{n} < p < \dfrac{n-1}{n-4}, \\ \left(\log \dfrac{1-r}{|r-|y||}\right)^{1/p} & \text{if} \quad p = \dfrac{n-1}{n-4}, \\ \left(\dfrac{1-r}{|r-|y||}\right)^{n-4-(n-1)/p} & \text{if} \quad p > \dfrac{n-1}{n-4}. \end{cases}$$

If $n = 4$, *then*

$$\mathcal{M}_p(G_2(\,\cdot\,, y), r) \leq M(1-r)^{2-n+(n-1)/p}(1-|y|)^2.$$

Proof. For $y \in A(r)$ and $r > 1/2$, setting

$$I_1(y,r) = S(r) \setminus B\left(y, \frac{1-r}{2}\right) \quad \text{and} \quad I_2(y,r) = S(r) \cap B\left(y, \frac{1-r}{2}\right),$$

we write

$$\begin{aligned} \mathcal{M}_p(G_2(\,\cdot\,, y), r)^p &= \frac{1}{\sigma_n r^{n-1}} \left(\int_{I_1(y,r)} G_2(x,y)^p \, dS(x) + \int_{I_2(y,r)} G_2(x,y)^p \, dS(x) \right) \\ &= u_1(y) + u_2(y). \end{aligned}$$

Since $-np + n - 1 < 0$, we have

$$\begin{aligned} u_1(y) &\leq M(1-r)^{2p}(1-|y|)^{2p} \int_{I_1(y,r)} |x-y|^{-np} \, dS(x) \\ &\leq M(1-r)^{2p+n-1-np}(1-|y|)^{2p}. \end{aligned}$$

On the other hand, if $x \in I_2(y,r)$, then $1 - r \leq |y|\,|x - y^*| \leq 3(1-r)$. In case $n \geq 5$ we see that

$$G_2(x,y) \leq M(1-r)^{-2}(1-|y|)^2|x-y|^{4-n},$$

so that

$$
\begin{aligned}
u_2(y) &\le M(1-r)^{-2p}(1-|y|)^{2p}\int_{I_2(y,r)}|x-y|^{(4-n)p}\,dS(x)\\
&\le M(1-r)^{-2p}(1-|y|)^{2p}|\,|y|-r|^{(4-n)p+n-1}\\
&\quad\times\begin{cases}\left(\dfrac{1-r}{|r-|y||}\right)^{n-1+(4-n)p} & \text{if}\quad \dfrac{n-1}{n}<p<\dfrac{n-1}{n-4},\\ \log\dfrac{1-r}{|r-|y||} & \text{if}\quad p=\dfrac{n-1}{n-4},\\ 1 & \text{if}\quad p>\dfrac{n-1}{n-4}.\end{cases}
\end{aligned}
$$

Since $(1-r)/|\,|y|-r|>4$ on $A(r)$, we obtain the required inequality.

Similarly, in case $n=4$, we find

$$
\begin{aligned}
u_2(y) &\le M(1-r)^{(2-n)p}(1-|y|)^{2p}\int_{I_2(y,r)}\left(\log\frac{2(1-r)}{|x-y|}\right)^p dS(x)\\
&\le M(1-r)^{(2-n)p}(1-|y|)^{2p}(1-r)^{n-1}.
\end{aligned}
$$

Hence it follows that

$$\mathcal{M}_p(G_2(\,\cdot\,,y),r)\le M(1-r)^{2-n+(n-1)/p}(1-|y|)^2. \qquad \square$$

3. Proofs of Theorems 1.1 and 1.2

In this section, we give proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let $1\le p<(n-1)/(n-4)$ when $n\ge 5$ and $1\le p<\infty$ when $n\le 4$. By applying Minkowski's inequality for integrals, we have

$$
\begin{aligned}
\mathcal{M}_p(G_2\mu,r) &\le \int_{\mathbf{B}}\mathcal{M}_p(G_2(\,\cdot\,,y),r)\,d\mu(y)\\
&= \int_{\mathbf{B}\setminus A(r)}\mathcal{M}_p(G_2(\,\cdot\,,y),r)\,d\mu(y)+\int_{A(r)}\mathcal{M}_p(G_2(\,\cdot\,,y),r)\,d\mu(y). \qquad (3.1)
\end{aligned}
$$

Thus Theorem 1.1 follows from Lemmas 2.4 and 2.5. $\square$

Proof of Theorem 1.2. First, we give a proof in case $(n-1)/(n-4)<p<(n-1)/(n-5)$. Set $\beta=n-4-(n-1)/p$ and $d\nu(x)=(1-|x|^2)\,d\mu(x)$. Here note that $0<\beta<1$. By Lemmas 2.3, 2.5 and (3.1), we see that

$$(3.2)\qquad (1-r)^{n-2-(n-1)/p}\mathcal{M}_p(G_2\mu,r)\le o(1)+M(1-r)^{\beta}\int_{A(r)}|\,|y|-r|^{-\beta}\,d\nu(y).$$

Hence it suffices to show that

$$\liminf_{r\to 1}(1-r)^{\beta}\int_{A(r)}||y|-r|^{-\beta}\,d\nu(y)=0. \tag{3.3}$$

For this purpose, we see that

$$\int_{1-2^{-j+1}}^{1-2^{-j}}||y|-r|^{-\beta}\,dr\leq M2^{-j(1-\beta)}.$$

Hence it follows that

$$\begin{aligned}
&\int_{1-2^{-j+1}}^{1-2^{-j}}\left((1-r)^{\beta}\int_{A(r)}||y|-r|^{-\beta}d\nu(y)\right)\frac{dr}{1-r}\\
&\leq 2^{-j(\beta-1)}\int_{\{y:2^{-j-1}<1-|y|<2^{-j+2}\}}\left(\int_{1-2^{-j+1}}^{1-2^{-j}}||y|-r|^{-\beta}dr\right)d\nu(y)\\
&\leq M\nu(\{y:2^{-j-1}<1-|y|<2^{-j+2}\}).
\end{aligned}$$

Since $\nu(\mathbf{B})<\infty$, we can find a sequence $\{r_j\}$ such that $2^{-j}<1-r_j<2^{-j+1}$ and

$$\lim_{j\to\infty}(1-r_j)^{\beta}\int_{A(r_j)}||y|-r_j|^{-\beta}\,d\nu(y)=0,$$

which implies (3.3). Thus the case $(n-1)/(n-4)<p<(n-1)/(n-5)$ now follows from (3.2) and (3.3).

Next, we deal with the case $p=(n-1)/(n-4)$. We see that

$$(1-r)^{n-2-(n-1)/p}\mathcal{M}_p(G_2\mu,r)\leq o(1)+M\int_{A(r)}\left(\log\frac{1-r}{||y|-r|}\right)^{1/p}d\nu(y).$$

In the same way as above, we have

$$\begin{aligned}
&\int_{1-2^{-j+1}}^{1-2^{-j}}\left(\int_{A(r)}\left(\log\frac{1-r}{||y|-r|}\right)^{1/p}d\nu(y)\right)\frac{dr}{1-r}\\
&\leq 2^{j}\int_{\{y:2^{-j-1}<1-|y|<2^{-j+2}\}}\left(\int_{1-2^{-j+1}}^{1-2^{-j}}\left(\log\frac{1-r}{||y|-r|}\right)^{1/p}dr\right)d\nu(y)\\
&\leq M\nu(\{y:2^{-j-1}<1-|y|<2^{-j+2}\}),
\end{aligned}$$

which implies that the left hand-side is zero by letting $j\to\infty$. Thus the theorem is established. □

REMARK 3.1. Our theorems are best possible as to the power of $1-r$.

To show this, for $p \geq 1$ and $\delta > 0$, we give an example of biharmonic Green potential v satisfying

$$\lim_{r\to 1}(1-r)^{n-2-(n-1)/p-\delta}\mathcal{M}_p(v,r) = \infty. \tag{3.4}$$

Letting

$$f(x) = (1-|x|)^{-3+\delta/2}|x-e_1|^{1-n},$$

where $e_1 = (1, 0, \ldots, 0)$, we consider the potential

$$v(x) = \int_{\mathbf{B}} G_2(x,y)f(y)\,dy.$$

Then $\int_{\mathbf{B}}(1-|x|)^2 f(x)\,dx < \infty$. Further, in case $n \geq 5$, we have

$$\begin{aligned} v(x) &\geq \int_{B(x,(1-|x|)/2)} G_2(x,y)f(y)\,dy \\ &\geq M(1-|x|)^{-3+\delta/2}|x-e_1|^{1-n}\int_{B(x,(1-|x|)/2)} |x-y|^{4-n}\,dy \\ &= M(1-|x|)^{1+\delta/2}|x-e_1|^{1-n}, \end{aligned}$$

so that

$$\begin{aligned} \mathcal{M}_p(v,r) &\geq M(1-r)^{1+\delta/2}\left(\fint_{S(r)} |x-e_1|^{p(1-n)}\,dS(x)\right)^{1/p} \\ &\geq M(1-r)^{1+\delta/2+1-n+(n-1)/p} \\ &= M(1-r)^{2-n+(n-1)/p+\delta/2}. \end{aligned}$$

Thus (3.4) follows. We can show the remaining case in the same manner.

4. Proof of Theorem 1.3

First we study the spherical means of super-biharmonic functions.

Lemma 4.1 (cf. [6]). *Let u be a super-biharmonic function on* $\mathbf{B}$. *Then*

$$u(x) \geq \frac{1}{r_2^2 - r_1^2}\left(r_2^2 \fint_{S(x,r_1)} u\,dS - r_1^2 \fint_{S(x,r_2)} u\,dS\right)$$

whenever $x \in \mathbf{B}$ and $0 < r_1 < r_2 < 1-|x|$.

Proof. Denote by K_2 the fundamental solution for the operator Δ^2 in $\mathbf{R}^n$, that is,

$$K_2(x) = \begin{cases} \alpha_n|x|^{4-n} & \text{in case} \quad n \neq 2,4, \\ 2\alpha_n|x|^{4-n}\log|x| & \text{in case} \quad n = 2,4. \end{cases}$$

For $x \in \mathbf{B}$ and $0 < r < 1 - |x|$, we can find a biharmonic function h in $B(x,r)$ such that

$$u(y) = \int_{B(x,r)} K_2(z-y)\, d\mu(z) + h(y)$$

for every $y \in B(x,r)$, where $\mu = \Delta^2 u$. By the Almansi expansion, we have

$$h(x) = \frac{r_2^2}{r_2^2 - r_1^2} \fint_{S(x,r_1)} h(y)\, dS(y) - \frac{r_1^2}{r_2^2 - r_1^2} \fint_{S(x,r_2)} h(y)\, dS(y)$$

for every $0 < r_1 < r_2 < r$. Hence we have only to show that

$$K_2(x) \geq \frac{r_2^2}{r_2^2 - r_1^2} \fint_{S(r_1)} K_2(x-y)\, dS(y) - \frac{r_1^2}{r_2^2 - r_1^2} \fint_{S(r_2)} K_2(x-y)\, dS(y) \tag{4.1}$$

for every $x \in \mathbf{R}^n$ and $0 < r_1 < r_2$. We define

$$g_2(x) = K_2(x) - \frac{r_2^2}{r_2^2 - r_1^2} \fint_{S(r_1)} K_2(x-y)\, dS(y) + \frac{r_1^2}{r_2^2 - r_1^2} \fint_{S(r_2)} K_2(x-y)\, dS(y)$$

and

$$g_1(x) = -\Delta g_2(x).$$

We see that

$$g_1(x) = K_1(x) - \frac{r_2^2}{r_2^2 - r_1^2} \fint_{S(r_1)} K_1(x-y)\, dS(y) + \frac{r_1^2}{r_2^2 - r_1^2} \fint_{S(r_2)} K_1(x-y)\, dS(y)$$

where $K_1(x) = (n-2)^{-1}\sigma_n^{-1}|x|^{2-n}$ if $n > 2$ and $K_1(x) = \sigma_2^{-1}\log(1/|x|)$ if $n = 2$. Note that $g_i(x) = g_i(x')$ for $|x| = |x'|$ and $g_i \in C^{2(i-1)}(\mathbf{R}^n \setminus \{0\})$. Further we see that $g_i(x) = 0$ for $|x| \geq r_2$, $g_2(0) = \infty$ if $n \geq 4$ and $g_2(0) > 0$ if $n = 2, 3$. Setting $t = K_1(x)$, we define $f_i(t) = g_i(x)$. Then

$$f_1(t) = \begin{cases} 0 & \text{if} \quad K_1(\infty) < t \leq K_1(r_2), \\ -\dfrac{r_1^2}{r_2^2 - r_1^2}(t - K_1(r_2)) & \text{if} \quad K_1(r_2) < t \leq K_1(r_1), \\ t - t_0 & \text{if} \quad t > K_1(r_1), \end{cases}$$

where $t_0 = (r_2^2 K_1(r_1) - r_1^2 K_1(r_2))/(r_2^2 - r_1^2)$ and $K_1(r) = K_1(x)$ with $|x| = r$. Hence we see that $f_1(t) < 0$ on $K_1(r_2) < t < t_0$ and $f_1(t) > 0$ on $t > t_0$. Since $f_2''(t) = -c(t) f_1(t)$ with $c(t) > 0$ and $\lim_{t\to\infty} f_2(t) > 0$, we obtain $f_2(t) > 0$ for $t > K_1(r_2)$. Thus (4.1) follows. □

Lemma 4.2. *Let u be a super-biharmonic function on* $\mathbf{B}$. *Then*

$$\lim_{r\to 1} \mathcal{M}(u,r) \quad \textit{exists in} \quad (-\infty,\infty].$$

In particular, if $\liminf_{r\to 1} \mathcal{M}(u^+,r) = 0$, *then*

$$\mathcal{M}(u,r) \le M(1-r^2) \quad \textit{for} \quad r_0 < r < 1.$$

Proof. Let u be a super-biharmonic function on $\mathbf{B}$ and $\mu = \Delta^2 u$. For $0 < t < 1$, there exists a biharmonic function h_t on $B(t)$ such that

$$u(x) = \int_{\overline{B}(t)} G_2(x,y)\,d\mu(y) + h_t(x) \quad (x \in B(t)).$$

Set

$$v_t(x) = \begin{cases} h_{(1+t)/2}(x) + \displaystyle\int_{\overline{B}((1+t)/2)\setminus\overline{B}(t)} G_2(x,y)\,d\mu(y) & \text{if} \quad x \in B((1+t)/2), \\ u(x) - \displaystyle\int_{\overline{B}(t)} G_2(x,y)\,d\mu(y) & \text{if} \quad x \in \mathbf{B}\setminus\overline{B}(t). \end{cases}$$

Then v_t is well defined. Further v_t is biharmonic on $B(t)$, super-biharmonic on $\mathbf{B}$, $v_t(0) < \infty$ and

$$u(x) = v_t(x) + \int_{\overline{B}(t)} G_2(x,y)\,d\mu(y) \quad (x \in \mathbf{B}). \tag{4.2}$$

In view of Lemma 2.3, we see that

$$⨍_{S(r)} \left(\int_{\overline{B}(t)} G_2(x,y)\,d\mu(y) \right) dS(x) \le M(1-r)^2 \frac{\mu(\overline{B}(t))}{r-t} \quad \text{for} \quad t < r < 1, \tag{4.3}$$

where M is a positive constant independent of t and r. By Lemma 4.1, we have

$$\mathcal{M}(v_t, r_2) \ge \frac{r_2^2}{r_1^2}\mathcal{M}(v_t, r_1) - \frac{r_2^2 - r_1^2}{r_1^2} v_t(0)$$

for $0 < r_1 < r_2 < 1$, which implies that

$$\liminf_{r_2\to 1} \mathcal{M}(v_t, r_2) \ge \frac{1}{r_1^2}\mathcal{M}(v_t, r_1) - \frac{1-r_1^2}{r_1^2} v_t(0) > -\infty \tag{4.4}$$

for $0 < r_1 < 1$. Hence we have

$$\liminf_{r_2\to 1} \mathcal{M}(v_t, r_2) \ge \limsup_{r_1\to 1} \mathcal{M}(v_t, r_1).$$

In view of (4.4), we see that $\lim_{r\to 1}\mathcal{M}(v_t, r)$ exists in $(-\infty, \infty]$, and so $\lim_{r\to 1}\mathcal{M}(u, r)$ exists in $(-\infty, \infty]$.

Moreover, assume that $\liminf_{r\to 1}\mathcal{M}(u^+, r) = 0$. By Lemma 4.1, we have

$$(r_2^2 - r_1^2)v_t(0) \geq r_2^2\mathcal{M}(v_t, r_1) - r_1^2\mathcal{M}(v_t, r_2) \geq r_2^2\mathcal{M}(v_t, r_1) - r_1^2\mathcal{M}((v_t)^+, r_2)$$

for $0 < r_1 < r_2 < 1$. Since $(v_t)^+ \leq u^+$ and $\liminf_{r_2\to 1}\mathcal{M}(u^+, r_2) = 0$, we have $\liminf_{r_2\to 1}\mathcal{M}((v_t)^+, r_2) = 0$. Hence we obtain

$$\mathcal{M}(v_t, r_1) \leq v_t(0)(1 - r_1^2). \tag{4.5}$$

Combining (4.3) and (4.5) we conclude that

$$\mathcal{M}(u, r) \leq M(1-r)^2 \quad \text{for} \quad t < r < 1,$$

where M is independent of r. The claim follows. □

REMARK 4.3. Let u be a super-biharmonic function on $\mathbf{B}$ satisfying $u(0) < \infty$ and $\liminf_{r\to 1}\mathcal{M}(u^+, r) = 0$. Then

$$\mathcal{M}(u, r) \leq u(0)(1 - r^2) \quad \text{for} \quad 0 < r < 1.$$

Corollary 4.4. *Let u be a super-biharmonic function on $\mathbf{B}$ satisfying*

$$\liminf_{r\to 1}\mathcal{M}(u^+, r) = 0. \tag{4.6}$$

Then the following are equivalent.
(1) $\liminf_{r\to 1}(1-r)^{-1}\mathcal{M}_1(u, r) = 0$ (*resp.* $< \infty$).
(2) $\liminf_{r\to 1}(1-r)^{-1}\mathcal{M}(u^-, r) = 0$ (*resp.* $< \infty$).

Lemma 4.5. *Let u be a super-biharmonic function on $\mathbf{B}$ and $\mu = \Delta^2 u$. Suppose u satisfies*

$$\liminf_{r\to 1}(1-r)^{-1}\mathcal{M}_1(u, r) < \infty.$$

Then (2.1) *is satisfied and u is of the form*

$$u(x) = G_2\mu(x) + (1 - |x|^2)h(x),$$

where h is harmonic on $\mathbf{B}$ satisfying

$$\sup_{r<1}\mathcal{M}_1(h, r) < \infty.$$

Proof. By considering the function $v_{1/2}$ in the proof of Lemma 4.2, we may assume that $u(0) < \infty$. Take a function v_t as in the proof of Lemma 4.2. First we show that

$$\inf_{0<t<1} v_t(0) > -\infty. \tag{4.7}$$

By Corollary 4.4, we see that $\liminf_{r\to 1} \mathcal{M}(u^+, r) = 0$, and so (4.5) holds. Using (4.3) and (4.5), we establish

$$\begin{aligned}
v_t(0) &\geq \frac{1}{1-r^2}\mathcal{M}(v_t, r) \\
&\geq -\frac{1}{1-r^2}\left(\mathcal{M}_1(u, r) + \fint_{S(r)}\left(\int_{B(t)} G_2(x, y)\, d\mu(y)\right) dS(x)\right) \\
&\geq -\frac{1}{1-r^2}\left(\mathcal{M}_1(u, r) + M(1-r)^2\frac{\mu(\overline{B}(t))}{r-t}\right) \\
&= -\frac{1}{1+r}\left((1-r)^{-1}\mathcal{M}_1(u, r) + M(1-r)\frac{\mu(\overline{B}(t))}{r-t}\right)
\end{aligned}$$

for $t < r < 1$. Letting $r \to 1$, we obtain

$$v_t(0) \geq -\frac{1}{2}\liminf_{r\to 1}(1-r)^{-1}\mathcal{M}_1(u, r),$$

which implies (4.7).

By Lemma 2.1, we have

$$\begin{aligned}
C_3\int_{\overline{B}(t)}(1-|y|^2)^2\, d\mu(y) &\leq \int_{\overline{B}(t)} G_2(0, y)\, d\mu(y) \\
&= u(0) - v_t(0) \\
&\leq u(0) - \inf_{0<t<1} v_t(0) < \infty
\end{aligned}$$

with the positive constant C_3 in Lemma 2.1. Hence (2.1) follows. In view of Corollary 2.2, $G_2\mu$ is a biharmonic Green potential, and hence there exists a biharmonic function v on $\mathbf{B}$ such that $u = G_2\mu + v$ on $\mathbf{B}$. By Theorem 1.1, we see that

$$\liminf_{r\to 1}(1-r)^{-1}\mathcal{M}_1(v, r) < \infty. \tag{4.8}$$

We note that v has an Almansi representation (see [3] and [11]):

$$v(x) = g(x) + (1-|x|^2)h(x)$$

where g and h are harmonic on $\mathbf{B}$. If $|x| < r < 1$, then we apply the Poisson integrals of both sides and find that

$$\liminf_{r\to 1}(1-r)^{-1}\left|g(x)+(1-r^2)h(x)\right| < \infty.$$

This shows that g is identically zero in $\mathbf{B}$. □

The above proof also gives the following.

Lemma 4.6. *If u is a biharmonic function on $\mathbf{B}$ satisfying*

$$\liminf_{r\to 1}(1-r)^{-1}\mathcal{M}_1(u,r) = 0,$$

then $u \equiv 0$ on $\mathbf{B}$.

Proof of Theorem 1.3. By Lemma 4.5, we see that u is of the form

$$u(x) = G_2\mu(x) + v(x),$$

where v is biharmonic on $\mathbf{B}$ and

$$\int_{\mathbf{B}}(1-|y|)^2\,d\mu(y) < \infty.$$

By Theorem 1.1 we find

$$\liminf_{r\to 1}(1-r)^{-1}\mathcal{M}_1(v,r) = 0.$$

Hence we see from Lemma 4.6 that $v \equiv 0$. □

5. Remarks

Suppose u is super-biharmonic on $\mathbf{B}$, and set $\mu = \Delta^2 u \geq 0$. Further, suppose there exists a sequence $\{r_j\}$, $0 < r_1 < r_2 < \cdots < s < 1$, tending to 1 such that $\{u(r_j\,\cdot\,)\,dS\}$ converges weak-star to some finite Borel measure ν on $\mathbf{S}$ and

$$\liminf_{r\to 1}(1-r)^{-1}\mathcal{M}((u-F[\nu])^-, r) < \infty.$$

Here $F[\nu](x) = \int_{\mathbf{S}} F(x,y)\,d\nu(y)$ with $F(x,y) = D_{\mathbf{n}(y)}(\Delta_y G_2(x,y))$, where $D_{\mathbf{n}(y)}$ denotes differentiation with respect to the outward unit normal. Then, as in Abkar and Hedenmalm [1], u can be represented as

$$u(x) = G_2\mu(x) + F[\nu](x) + (1-|x|^2)P[\lambda](x),$$

where $P[\lambda]$ denotes the Poisson integral of some finite Borel measure λ on $\mathbf{S}$ and

$$\int_{\mathbf{B}} (1 - |x|^2)^2 \, d\mu(x) < \infty.$$

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Toshihide FUTAMURA
Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima 739-8526, Japan
e-mail: toshi@mis.hiroshima-u.ac.jp

Current adress:
Department of Mathematics
Daido Institute of Technology
Nagoya 457-8530, Japan
e-mail: futamura@daido-it.ac.jp

Yoshihiro Mizuta
The Division of Mathematical and Information Sciences
Faculty of Integrated Arts and Sciences
Hiroshima University
Higashi-Hiroshima 739-8521, Japan
e-mail: mizuta@mis.hiroshima-u.ac.jp