ON THE DIMENSION OF MODULES AND ALGEBRAS, X

A RIGHT HEREDITARY RING WHICH IS NOT LEFT HEREDITARY

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A ring R is said to be right (left) hereditary if every right (left) ideal in R is projective, that is, a direct summand of a free R-module. Cartan and Eilenberg [3, p. 15] ask whether there exists a right hereditary ring which is not left hereditary. The answer: yes.

THEOREM. *Let V be a vector space of countably infinite dimension over a field F. Let C be the algebra of all linear transformations on V with finite-dimensional range. Let B be the algebra obtained by adjoining a unit element to C. Let $A = B \otimes B$ (all tensor products are over F). Then A is right hereditary but not left hereditary.*

The proof that A is right hereditary is broken into four lemmas.

LEMMA 1. *Let R be a regular ring (for any a there exists an x such that $axa = a$). Then every countably generated right ideal I in R is projective.*

Proof. It is known that any finitely generated right ideal in R can be generated by an idempotent. Hence I can be expressed as the union of an ascending sequence of right ideals generated by idempotents. Each of these is projective and is a direct summand of its successor. Hence I is a direct sum of projective ideals and is itself projective.

LEMMA 2. *Let R be a ring, J a two-sided ideal in R. Suppose that in J and R/J every right ideal is countably generated. Then the same is true in R.*

Proof. Take a right ideal I in R. Using $*$ for image mod J, we pick a countable set $\{a_n^*\}$ of generators for I^*. Pick elements $a_n \in I$ mapping on a_n^*.

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Together with a countable set of generators for the right ideal $I \cap J$ in J, these give a countable generation of I.

LEMMA 3. *In the ring A of the Theorem, every right ideal is countably generated.*

Proof. First let us contemplate the ring C. By [4, p. 91, Th. 1] every right ideal in C consists of all linear transformations having range in a certain subspace of V. (We are putting linear transformations on the left of vectors, so Jacobson's "left" is replaced by "right"). Since V has countable dimension, it follows that the right ideals in C are countably generated.

A similar argument applies to $C \otimes C$. First, since C is central simple over F, the same is true of $C \otimes C$ [4, p. 114, Th. 1]. If e is a primitive idempotent of C, then $Ce \otimes Ce$ is a minimal left ideal in $C \otimes C$. But Ce, as a left C-module, is isomorphic to V and hence has countable dimension over F. Therefore $Ce \otimes Ce$ likewise has countable dimension. Now $C \otimes C$ is the algebra of all linear transformations of finite-dimensional range on $Ce \otimes Ce$, continuous relative to a certain dual space [4, p. 75]. By [4, p. 91, Th. 1] again, the right ideals of $C \otimes C$ are countably generated.

Since C is an ideal in B with quotient isomorphic to F, the right ideals of B are countably generated by Lemma 2. Then by two more applications of Lemma 2 we climb from $C \otimes C$ to A; for $C \otimes C$ is an ideal in $B \otimes C$ with quotient isomorphic to C, and $B \otimes C$ is an ideal in $A = A \otimes A$ with quotient isomorphic to B.

LEMMA 4. *The ring A of the Theorem is regular.*

Proof. First, $C \otimes C$ is regular, for this is true for any simple ring with a minimal one-sided ideal [4, p. 90, Th. 3]. Again we climb from $C \otimes C$ to A in several steps, this time making use of the following theorem [2]: if J is a two-sided ideal in a ring R such that J and R/J are regular, then R is regular.

Lemmas 1, 3 and 4 combine to assert that every right ideal in A is projective. It remains for us to exhibit a left ideal in A which is not projective. The underlying idea is that the left ideals in B are not countably generated. This does not stop B from being left hereditary, but it does disturb the tensor product $B \otimes B$. This observation is in essence due to Zelinsky [5].

Select a vector space basis for V over F. Let e_i denote the linear transformation which is identity on the ith coordinate and 0 on all the others. Again, select an uncountable set $\{f_\alpha\}$ of primitive idempotents such that the left ideals Bf_α are independent, that is, their union is their direct sum. (Minimal left ideals in B—or equivalently in C—correspond to one-dimensional subspaces of the dual V^* of V; since V^* has uncountable dimension we are able to pick such a set of f's). Write $g_i = 1 \otimes e_i$, $h_\alpha = f_\alpha \otimes 1$. Let K be the left ideal in A generated by $\{g_i, h_\alpha\}$.

If K is projective there exist [3, Ch. VII, Prop. 3.1] A-homomorphisms ψ_i, ϕ_α of K into A such that for any $k \in K$ only a finite number of $\{\psi_i(k), \phi_\alpha(k)\}$ are non-zero and $\sum \psi_i(k) g_i + \sum \phi_\alpha(k) h_\alpha = k$. There must exist at least one index (indeed uncountably many indices) β such that $\phi_\beta(g_i) = 0$ for all i. To simplify the writing let us simply suppress this index β, writing f, h, ϕ for f_β, h_β, ϕ_β. Now $\phi(h)$ has a unique expression $u + t \otimes 1$ where $u \in B \otimes C$, $t \in B$. We shall argue (1) $t = 0$, (2) $t \neq 0$.

(1) Every element of C is a linear transformation with finite-dimensional range, and hence is left-annihilated by e_i for i sufficiently large. Hence $g_i u = 0$ for large i. Then $\phi(g_i h) = g_i \phi(h) = t \otimes e_i$. On the other hand $\phi(g_i h) = \phi(h g_i) = h\phi(g_i) = 0$. Hence $t = 0$.

(2) For any α we have a unique expression $\phi_\alpha(h) = u_\alpha + t_\alpha \otimes 1$, $u_\alpha \in B \otimes C$, $t_\alpha \in B$. In the equation

$$h = \sum \psi_i(h) g_i + \sum \phi_\alpha(h) h_\alpha + \phi(h) h$$

let us suppress the $(B \otimes C)$-component. The result is

$$\sum t_\alpha f_\alpha \otimes 1 + tf \otimes 1 = f \otimes 1,$$

or $(1-t)f + \sum t_\alpha f_\alpha = 0$. But the left ideals Bf_α, Bf are independent. Hence $(1-t)f = 0$, $t \neq 0$. This completes the proof of the theorem.

Alex Rosenbery showed me that the left global dimension of A is exactly 2. Since the adjunction of an indeterminate lifts dimension by one, we can exhibit the combination n, $n+1$ of right and left global dimensions for any $n \geqq 1$.

It seems unlikely that any minor modification of the ring A can achieve a difference of two or more. It follows from a theorem of M. Auslander [1] that any regular ring of cardinal number $\aleph_1$ has global dimension at most 2; so, if we grant the continuum hypothesis, a regular ring with the cardinal number of the continuum is useless.

BIBLIOGRAPHY

[1] N. Auslander, *On the dimension of modules and algebras.* III. *Global dimension*, Nagoya Math. J., **9** (1955), 67-77.

[2] B. Brown and N. McCoy, *The maximal regular ideal of a ring*, Proc. Amer. Math. Soc. **1** (1950), 165-171.

[3] H. Cartan and S. Eilenberg, Homological Algebra, Princeon, 1956.

[4] N. Jacobson, Structure of Rings, Amer. Math. Soc. Coll. Pub., vol. 37, 1956.

[5] D. Zelinsky, *Raising idempotents*, Duke Math. J., **21** (1954), 315-322.

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