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ON CERTAIN EXAMPLES OF SURFACES WITH $p_g = 0$ DUE TO BURNIAT

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Notation and terminology

Let S be a compact complex projective and smooth variety of $\dim_C 2$, shortly a surface. We employ the following standard notations:
$\mathcal{O}_S$: structure sheaf of S.
K_S: the canonical bundle on S.
$p_g(S) = \dim_C H^0(S, K_S)$, geometric genus.
$q(S) = \dim_C H^1(S, \mathcal{O}_S)$, the irregularity.
$c_1^2(S)$ and $c_2(S)$, the Chern numbers of S.
$\mathscr{X}(S) =$ Euler class of $\mathcal{O}_S = (p_g + 1 - q)(S) = \frac{1}{12}(c_1^2 + c_2)(S)$
For any divisor D on S we let $|D|$ be the linear system corresponding to it, and if $p_1, \cdots, p_m \in S$ we let $|D - p_1 - p_2 - \cdots - p_m|$ be the subsystem of divisors through $p_1, \cdots, p_m$.

If D_1 and D_2 are two divisors, (D_1, D_2) denotes their intersection number, and $D_1 \sim D_2$ means that the divisors are linearly equivalent. We shall write the group action on Pic (S), additively, so, if for instance C is a divisor and F a line bundle such that $[C] \cong F^2$ we simply write $C \sim 2F$. Finally, if $F \in \text{Pic}(S)$, we put $h^p(F) = \dim_C H^p(S, F)$.

Introduction

Recently P. Burniat gave examples of surfaces of general type with $p_g = q = 0$, $c_1^2 = 2, 3, \cdots, 6$, cf. [1]. This paper is written in rather ancient language and somewhat difficult to read. It seems desirable to give his construction in a more up to date way, the more since new interest seems to arise in surfaces of general type with $p_g = q = 0$, see for example [3], [4] and [5].

Burniat constructs his examples by means of what he calls "plans 2^2-uples abéliennes". These 4-fold coverings of the projective plane we study in § 1. Burniats construction is given in § 2 and from general calculations in §1 we find the invariants of the resulting surface. In particular they are all regular. As a special case we find the surfaces with $c_1^2 = 6 - \alpha$ ($\alpha = 0, \cdots, 4$) referred to before, called *Burniat surfaces* $B(\alpha)$. In §3 they are shown to be of general type and we prove there also that the torsion group of $B(\alpha)$ is $\oplus^{6-\alpha} \boldsymbol{Z}_2$. This is a new result and shows the fruitfulness of new techniques.

I also want to mention that, as a corollary, $\pi_1(B(0))$ is not abelian. This also is a new feature: the Godeaux and Campedelli surfaces ([5]) all have abelian fundamental group*. Similar results for $\pi_1(B(\alpha))$, $1 \leq \alpha \leq 4$ are lacking however.

Finally I want to thank Prof. Van de Ven for stimulating interest in this subject.

§1. Abelian 4-fold coverings

Let P be a complex analytic surface and let C_1, C_2 and C_3 be three smooth curves on P, such that for some line bundle F_k on P one has: $[C_i + C_j] \sim 2F_k$, where $\{i, j, k\} = \{1, 2, 3\}$. We assume moreover that the curves C_i intersect each other transversally and $C_1 \cap C_2 \cap C_3 = \phi$. Put $C := C_1 + C_2 + C_3$.

We want to construct a surface Q and a holomorphic map $f: Q \to P$ of degree 4 such that the ramification divisor D of f is of the form $D = D_1 + D_2 + D_3$, with D_i smooth, such that $D_i = f^{-1}(C_i)$ and $f|D_i$ is of degree 2 onto C_i ($i = 1, 2, 3$). We shall call Q the *abelian* 4-*fold covering of* P *branched along* C.

As intermediary steps we construct double coverings, using the next —well known—observation (cf. e.g., Horikawa, [2]):

Let X *be a complex analytic surface,* B *a smooth curve on* X *and* F *a line bundle on* X *such that* $B \sim 2F$. *Then there exists a surface* Y *and holomorphic map* $\phi: Y \to X$ *of degree* 2 *with branch locus* B.

This surface Y is called the *double covering of* X *branched along* B.

We apply this construction as follows: let $p_1, \cdots, p_m$ be the intersection points of C_1 and C_2. One blows up P at these points, obtaining P' and a map $\sigma: P' \to P$. The exceptional curves are $P_i := \sigma^{-1}(p_i)$, $i = 1$,

* M. Reid found a Godeaux type surface with finite non-abelian fundamental group (the quaternion-group of order 8); Kuga also has examples with infinite non-abelian fundamental group. His examples have $c_1^2=8$. Any finite unramified covering of them have $q=0$, unlike our example.

$\cdots, m$. Let the proper transform of a curve Γ on P be denoted by $\tilde{\Gamma}$, then one has:

$$\tilde{C}_j \sim \sigma^* C_j - \sum_{\beta=1}^{s} P_\beta \,, \qquad j = 1, 2 \,.$$

Form the double covering branched along $\tilde{C}_1 + \tilde{C}_2$ (notice that this curve is 2-divisible): $\rho_1: Y' \to P'$; Put $E_\gamma = \rho_1^{-1}(\tilde{C}_\gamma), \gamma = 1, 2, 3$, and $Q_\beta = \rho_1^{-1}(P_\beta)$, $\beta = 1, \cdots, m$. Now, on Y' the curve $H_3 = E_3 + \sum_\beta Q_\beta$ is seen to be smooth and 2-divisible and one may form $\rho_2: Y'' \to Y'$, the double covering of Y' branched along H_3. On Y'' the curves $R_\beta = \rho_2^{-1}(Q_\beta)$ are exceptional, indeed, they are clearly rational and moreover $(R_\beta, R_\beta) = \frac{1}{4}(\rho_2^* Q_\beta, \rho_2^* Q_\beta) = -1$. Blowing down these curves one finally obtains Q. Since the curves R_β map onto points in the composition $g = \sigma \cdot \rho_1 \cdot \rho_2$ we can factor this map g over Q, obtaining a holomorphic map $f: Q \to P$. This map is of degree 4 and ramified along the curves D_γ coming from E_γ, and by construction, $f | D_\gamma$ is of degree 2 onto E_γ. This proves our assertions.

Next, we want to calculate the invariants of Q. The intermediary steps in the construction are blowings up and double coverings and the behaviour of the invariants under these operations is well known, recall e.g.:

LEMMA 1.1 (Cf. Horikawa, [2]). *Let Y be the double covering of X branched along B, then if $B \sim 2F$ we have*

$$\mathscr{X}(Y) = 2\mathscr{X}(X) + \tfrac{1}{2}(F, K_X + F) \,.$$

Using this lemma we can calculate $\mathscr{X}(Q)$, since it is invariant under blowings up. By definition we have $K_Q \sim f^* K_P + R$, with R the ramification locus of the map $f: Q \to P$, and since $R = D_1 + D_2 + D_3$ with $f | D_i: D_i \to C_i$ a 2–1 map, we can compute $c_1^2(Q) = (K_Q, K_Q)$ as well. Carrying out all this we find:

PROPOSITION 1.2. *Let $f: Q \to P$ be the abelian 4-fold covering of P branched along $C = C_1 + C_2 + C_3$, $D = f^{-1}(C)$. Then we have*

(i) $K_Q \sim f^* K_P + D$

(ii) $c_1^2(Q) = 4c_1^2(P) + (C, C) + 4(K_P, C)$

(iii) $\mathscr{X}(Q) = 4\mathscr{X}(P) + \frac{1}{4} \sum_{i \le j} (C_i, C_j) + \frac{1}{2}(K_P, C)$.

Remark 1.3. If, instead, C_3 intersects C_1 and C_2 transversally at α

of their intersectionpoints we can still carry out the above construction, namely, we first blow up at these points and observe that the strict transforms $\tilde{C}_i$ of the curves C_i now full fill the requirements stated at the beginning of this section: $\tilde{C}_i$ are smooth and $\tilde{C}_i \cap \tilde{C}_j$ is a transversal intersection, $\tilde{C}_i \cap \tilde{C}_j \cap \tilde{C}_k = \phi$ and $\tilde{C}_i + \tilde{C}_j$ is 2-divisible

DEFINITION 1.4. In case $C = C_1 + C_2 + C_3$ with C_i as in the beginning of the section except that, instead $C_1 \cdot C_2 \cdot C_3 = p_1 + \cdots + p_\alpha$, $p_i \neq p_j$ if $i \neq j$, the resulting 4-fold abelian covering is called α-modified abelian covering of P with branch locus C.

COROLLARY 1.5. *Let $Q(\alpha)$ be the α-modified 4-fold abelian covering of P branched along C, then we have*

$$c_1^2(Q(\alpha)) = c_1^2(Q(0)) - \alpha \quad \text{and} \quad \mathscr{X}(Q(\alpha)) = \mathscr{X}(Q(0)) .$$

This corollary shows how to lower c_1^2 with constant $\mathscr{X}$.

Next we want to determine the sections of mK_Q. For this, observe that the group $Z = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ acts on Q, hence on the multicanonical forms, i.e. the sections of mK_Q. Under this action $H^0(mK_Q)$ splits into 4 eigenspaces $E_\gamma^{(m)}$, $\gamma = 1, 2, 3, 4$. If Z is generated by z_1 and z_2, we set $z_3 = z_1 z_2$ and:

$$E_\gamma^{(m)} = \{s \in H^0(mK_Q) \mid z_\gamma(s) = s;\, z_\delta(s) = -s, \delta \neq \gamma\} , \qquad \gamma = 1, 2, 3 .$$
$$E_4^{(m)} = \{s \in H^0(mK_Q) \mid z(s) = s, z \in Z\}.$$

How does Z act on the curves D_i? Up to a permutation of indices this action is given by:

$z_\gamma | D_\gamma = \mathrm{id}$, $z_\delta | D_\delta = i_\delta$, $\delta \neq \gamma$, where i_δ is an involution. To determine the spaces $E_\gamma^{(m)}$ one has to study the action of Z on sections of mK_Q in the neighborhood of the curves D_γ ($\gamma = 1, 2, 3$).

PROPOSITION 1.6. *The zeroes of sections in $E_\gamma^{(m)}$ define linear systems $|G_\gamma^{(m)}|$. In case $m = 2n + 1$ we have*

$$|G_4^{(m)}| = D + |f^*(mK_p + nC)|$$

and

$$|G_\gamma^{(m)}| = D_\gamma + |f^*\{n(2K_p + C) + K_p + F_\gamma\}| , \qquad \gamma = 1, 2, 3 .$$

In case $m = 2n$ we have:

$$|G_4^{(m)}| = |n(2K_p + C)|$$

and

$$|G_\nu^m| = D_\mu + D_\sigma + |f^*\{n(2K_p + C) - F_\nu\}|$$

with $\nu = 1, 2, 3$ *and* $\{\nu, \mu, \sigma\} = \{1, 2, 3\}$.

Proof. We shall prove the assertion for $m = 1$, the proof for the remaining m being similar. For simplicity of notation we set $E_\gamma^{(m)} = E_\gamma, \gamma = 1, \cdots, 4$. Since $K_Q \sim f^*K_P + D \sim f^*(K_P + F_\gamma) + D_\gamma$ it suffices to prove the next assertion:

(*) $M \in |G_4|$ if and only if M contains an odd multiple of D

$M \in |G_\gamma|$ if and only if M contains an odd multiple of D_γ but an even multiple of D_δ $(\delta \neq \gamma)$, $\gamma = 1, 2, 3$.

As remarked before 1.6 to prove this assertion one has to study the action of Z on sections $s \in H^0(K_Q)$. We take a Z-invariant coordinate covering $\mathscr{U} = \{U_\alpha\}$ of Q, i.e. $z(U_\alpha) = U_\alpha$ or $z(U_\alpha) \cap U_\alpha = \phi$ if $z \in Z$. We have three types of sets in $\mathscr{U}$:

(*Type* 1) $U_\alpha \cap D = \phi$. Then exactly 4 sets are permuted by Z and these sets are mutually disjoint.

(*Type* 2) $U_\alpha \cap D_i \neq \phi$ for a certain i, say for $i = 1$, but $U_\alpha \cap D_1 \cap D_j = \phi, j \neq 1$. Then $z_1(U_\alpha) = U_\alpha$ and z_1 acts as an involution with D_1 as fixed locus, and $z_2(U_\alpha) \cap U_\alpha = \phi$. Put $U'_\alpha = z_2(U_\alpha)$. Assume coordinates are chosen in U_α and U'_α, say (x_α, y_α), resp. (x'_α, y'_α) such that $x_\alpha = z_2^* x'_\alpha$, $y_\alpha = z_2^* y'_\alpha$ and such that $z_1 | U_\alpha$ is given by $(x_\alpha, y_\alpha) \mapsto (-x_\alpha, y_\alpha)$.

(*Type* 3) $U_\alpha \cap D_i \cap D_j \neq \phi$ for some pair, say $(i, j) = (1, 2)$. Then $z(U_\alpha) = U_\alpha (z \in Z)$ and we may choose coordinates (x_α, y_α) in such a way that $U_\alpha \cap D_1 = \{x_\alpha = 0\}$, $U_\alpha \cap D_2 = \{y_\alpha = 0\}$, $z_1 | U_\alpha$ is given by $(x_\alpha, y_\alpha) \mapsto (-x_\alpha, y_\alpha)$ and $z_2 | U_\alpha$ is given by $(x_\alpha, y_\alpha) \mapsto (x_\alpha, -y_\alpha)$.

Now, take any holomorphic 2-form s on Q. In U_α this can be given as $w(x_\alpha, y_\alpha) dx_\alpha \wedge dy_\alpha$, with w a holomorphic function in U_α. In a coordinate patch of type (2) z_1 and z_2 act as follows:

$$(z_1)^*\{w(x_\alpha, y_\alpha) dx_\alpha \wedge dy_\alpha\} = -w(-x_\alpha, y_\alpha) dx_\alpha \wedge dy_\alpha$$
$$(z_2)^*\{w(x_\alpha, y_\alpha) dx_\alpha \wedge dy_\alpha\} = w(x'_\alpha, y'_\alpha) dx'_\alpha \wedge dy'_\alpha\,.$$

This immediately proves (*) in this case. In a coordinate patch of type (3) z_1 and z_2 act as follows:

$$(z_1)^*\{w(x_\alpha, y_\alpha)dx_\alpha \wedge dy_\alpha\} = -w(-x_\alpha, y_\alpha)dx_\alpha \wedge dy_\alpha$$
$$(z_2)^*\{w(x_\alpha, y_\alpha)dx_\alpha \wedge dy_\alpha\} = -w(x_\alpha, -y_\alpha)dx_\alpha \wedge dy_\alpha$$

and here also $(*)$ is immediately verified.
Since Type–1 neighborhoods do not play a role the assertion $(*)$ is proved, and this establishes the Proposition in case $m = 1$.

§ 2. The examples of Burniat

Choose in $\boldsymbol{P}_2$ three points ξ_1, ξ_2 and ξ_3 which are not collinear. Let C_i^0 be the reducible curve consisting of $2r_i + 1$ lines through ξ_i, such that $\xi_{i+1} \in C_i^0$ and $\xi_{i+2} \notin C_i^0$, $i = 1, 2, 3$, the indices taken modulo 3. Blow up $\boldsymbol{P}_2$ at ξ_1, ξ_2 and ξ_3 and let P be the blown up surface, $\mu: P \to \boldsymbol{P}_2$ the blowing down map, $X_i^0 = \mu^{-1}(\xi_i)$, $i = 1, 2, 3$ the three exceptional curves. Put

$$C_i = \mu^{-1}(C_i^0) + X_{i+2}^0 \qquad (i = 1, 2, 3)$$

where $\mu^{-1}(*)$ denotes the proper transform of $*$. We have that $C_i \sim (2r_i + 1)H - (2r_i + 1)X_i^0 - X_{i+1}^0 + X_{i+2}^0$, hence the curves $C_i + C_j$ are 2-divisible. Assume that C_3^0 passes through α of the intersectionpoints of C_1^0 and C_2^0 different from ξ_1, ξ_2 and ξ_3. Recalling Definition 1.4. We can form the α-modified 4-fold abelian covering of P with $C = C_1 + C_2 + C_3$ as branch locus. Call the resulting surface $B(r_1, r_2, r_3, \alpha)$. Applying (1.2) and (1.5) to it we find

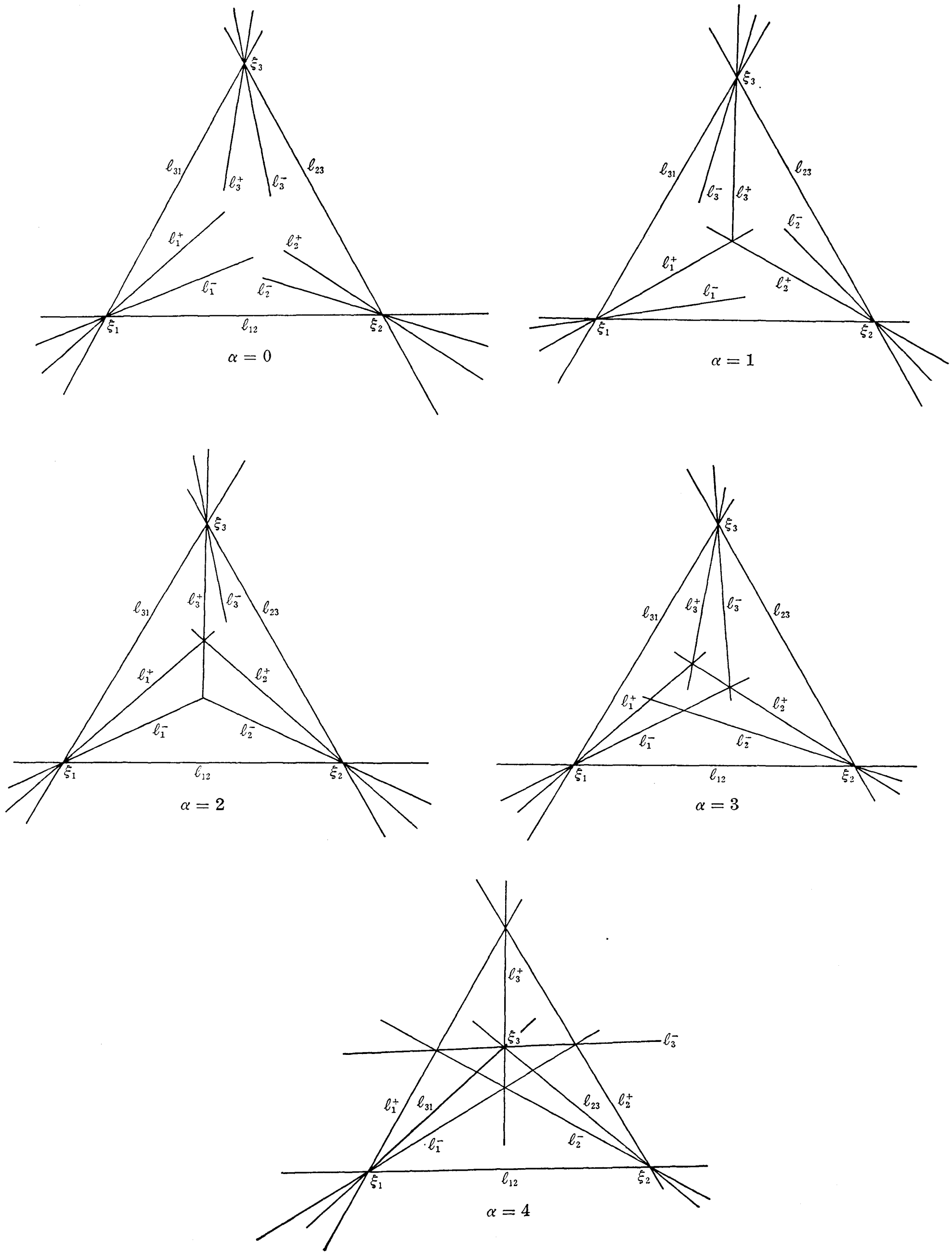
PROPOSITION 2.1. *The values of c_1^2 and $\mathscr{X}$ for $B(r_1, r_2, r_3, \alpha)$ are respectively*

$$(6 - \alpha) + 8[(r_1r_2 + r_2r_3 + r_1r_3 - (r_1 + r_2 + r_3)]$$

and

$$1 + [r_1r_2 + r_2r_3 + r_1r_3 - (r_1 + r_2 + r_3)] \,.$$

EXAMPLE. In case $r_1 = r_2 = r_3 = 1$ we get a surface $B(1, 1, 1, \alpha)$, with c_1^2 equal to $6 - \alpha$ and $\mathscr{X}$ equal to 1. We shall indicate how to construct curves C_3^0 with $\alpha = 0$ up to 4: Let A_i be a curve in $\boldsymbol{P}_2$ consisting of two lines through ξ_i, $i = 1, 2$. There is a point ξ_3 not collinear with ξ_1 and ξ_2 and a curve A_3 consisting of two lines going through ξ_3 such that A_3 goes through any $\alpha \leq 4$ of the intersectionpoints $A_1 \cdot A_2$. Indeed, for $\alpha \leqq 2$ this is trivial and for $\alpha = 3$ one takes a gen-



ξ_3
ℓ_{31}
ℓ_{23}
ℓ_3^+
ℓ_3^-
ℓ_1^+
ℓ_2^+
ℓ_1^-
ℓ_2^-
ξ_1
ℓ_{12}
ξ_2
$\alpha = 0$
ξ_3
ℓ_{31}
ℓ_{23}
ℓ_3^-
ℓ_3^+
ℓ_2^-
ℓ_1^+
ℓ_2^+
ℓ_1^-
ξ_1
ξ_2
$\alpha = 1$
ξ_3
ℓ_{31}
ℓ_3^+
ℓ_3^-
ℓ_{23}
ℓ_1^+
ℓ_2^+
ℓ_1^-
ℓ_2^-
ξ_1
ℓ_{12}
ξ_2
$\alpha = 2$
ξ_3
ℓ_{31}
ℓ_3^+
ℓ_3^-
ℓ_{23}
ℓ_1^+
ℓ_2^+
ℓ_1^-
ℓ_2^-
ξ_1
ℓ_{12}
ξ_2
$\alpha = 3$
ℓ_3^+
ξ_3
ℓ_3^-
ℓ_1^+
ℓ_{31}
ℓ_{23}
ℓ_2^+
ℓ_1^-
ℓ_2^-
ξ_1
ℓ_{12}
ξ_2
$\alpha = 4$

Fig. 1

eral point on a diagonal of the quadrilateral formed by the points $A_1 \cdot A_2$ and for $\alpha = 4$ one takes a diagonal point. Using the curves A_i the construction of the curves C_i^0 is evident. For later use we depict the branch curve C in any of the 5 cases. (Cf. Fig. 1)

PROPOSITION 2.2. *Let* $B = B(r_1, r_2, r_3, \alpha)$ *and* $f: B \to P$ *the defining map. Let* $\{\mu, \nu, \sigma\}$ *be an even permutation of* $\{1, 2, 3\}$ *and put* $F_\mu = (r_\nu + r_\sigma + 1)H - r_\nu E_\nu^0 - (r_\sigma + 1)E_\sigma^0$. *Then one has:*

$|K_B|$ *is the linear combination of the systems*

$$D_\mu + |f^*(-3H + E_1^0 + E_2^0 + E_3^0 + F_\mu)| , \qquad \mu = 1, 2, 3 .$$

Proof. Apply Proposition 1.6 with $m = 1$ in this case, noticing that $|f^* K_P| = \phi$.

COROLLARY 2.3. $q(B) = 0$

Proof. Computation of the dimension of the linear systems $|-3H + E_1^0 + E_2^0 + E_3^0 + F_\mu|$ shows that $p_g(B) = r_1 r_2 + r_1 r_3 + r_2 r_3 - (r_1 + r_2 + r_3)$, hence, by Riemann-Roch

$$q(B) = \mathscr{X}(B) - p_g(B) - 1 = 0$$

For example, if $r_1 = r_2 = r_3 = 1$ we find that the surface has invariants $p_g = q = 0$, $c_1^2 = 6 - \alpha$.

DEFINITION 2.4. We call $B(1, 1, 1, \alpha)$ the *Burniat surface* of type α, notation $B(\alpha)$.

§ 3. The torsion group of $B(\alpha)$

In this section we want to prove that the torsion group of $B(\alpha)$ equals $\oplus^{6-\alpha} \boldsymbol{Z}_2$. For brevity we put $\beta = 6 - \alpha$ and we shall give $2^\beta - 1$ different non-trivial 2-torsion elements. Then we shall prove that this is the only torsion. To find 2-torsion elements we proceed as follows:

Suppose that $|2K_S|$ contains a divisor which is multiple, say the double of a curve Γ. Then $\Gamma \in |K_S + g|$, with g a non-trivial 2-torsion element.

LEMMA 3.1. *Let* $\bar{f}: B(\alpha) \to \boldsymbol{P}_2$ *be the defining map. Then, with* A *the* α *points in* $C_1^0 \cap C_2^0 \cap C_3^0$ *we have that*

$$|2K_{B(\alpha)}| = \bar{f}^* |3H - \xi_1 - \xi_2 - \xi_3 - A|$$

Proof. Apply Proposition 1.6 with $m = 2$. We have that $|2K_P + C - F_\nu| = \phi$ with P the plane $\boldsymbol{P}_2$ blown up at ξ_1, ξ_2, ξ_3 and A. Indeed, $|2K_P + C - F_\nu| = |X_\nu - X_{\nu+1}| = \phi$. (Convention: $\nu + 1 = 1$, if $\nu = 3$). Moreover $|2K_P + C|$ corresponds to $|3H - \xi_1 - \xi_2 - \xi_3 - A|$. This completes the proof.

COROLLARY 3.2. *$B(\alpha)$ is of general type. Moreover it is minimal.*

Proof. Since $c_1^2 > 0$ and the number of curves of degree 3 through $3 + \alpha$ points is at least $7 - \alpha$[1)], hence $P_2(B(\alpha)) \geqslant 3$, since $\alpha \geqslant 4$ and the first assertion follows from classification theory. The second follows from the equality $P_2(B(\alpha)) = 7 - \alpha = c_1^2(\alpha) + 1$.

Now we can exhibit the $2^\beta - 1$ ($\beta = 6 - \alpha$) torsion elements by giving the corresponding curves of degree 3 through ξ_1, ξ_2, ξ_3 and A which give double curves on $B(\alpha)$. To this end we must be careful with curves having multiple points at ξ_i and A. Since, on $B(\alpha)$ the curves corresponding to ξ_i are seen to be divisible by 2 these points are harmless. However, the curves corresponding to points of A are *not* 2-divisible on $B(\alpha)$ and we can only allow curves that have a $2k + 1$-fold point at these points, since otherwise curves from $|3H - \xi_1 - \xi_2 - \xi_3 - A|$ always contain an odd multiple of the curves coming from A. With this in mind we find, using the notation H_k for a line through ξ_k:

$\alpha = 0$: $\ell_1^\pm \ell_2^\pm \ell_3^\pm$, $\ell_{ij}\ell_k^\pm \ell_i^\mp$, $\ell_{ij}\ell_k^\pm \ell_j^\mp$, $\ell_{ij}^+ \ell_k^+ \ell_k^-$, $\ell_{ij} H_k^2$, $\ell_{ij}^2 \ell_k^\pm$, $\ell_{ij}\ell_{jk}\ell_i^\pm$, $\ell_{ij}\ell_{jk}\ell_j^\mp$, $\ell_{ij}\ell_{jk}\ell_k^\pm$, $\ell_{12}\ell_{23}\ell_{31}$.

$\alpha = 1$: $\ell_i^+ \ell_j^- \ell_k^-$, $\ell_1^+ \ell_2^+ \ell_3^+$, $\ell_{ij}\ell_k^+ \ell_i^-$, $\ell_{ij}\ell_k^- \ell_i^+$, $\ell_{ij}\ell_k^+ \ell_j^-$, $\ell_{ij}\ell_k^+ \ell_j^-$, $\ell_{ij}\ell_k^+ \ell_k^-$, $\ell_{ij}\ell_{jk}\ell_i^+$, $\ell_{ij}\ell_{jk}\ell_j^+$, $\ell_{ij}\ell_{jk}\ell_k^+$.

$\alpha = 2$: $\ell_1^+ \ell_2^+ \ell_3^+$, $\ell_1^- \ell_2^- \ell_3^+$, $\ell_1^+ \ell_2^- \ell_3^-$, $\ell_1^- \ell_2^+ \ell_3^-$, $\ell_{12}\ell_{31}\ell_3^+$, $\ell_{12}\ell_{23}\ell_3^+$, $\ell_{23}\ell_{31}\ell_3^+$, $\ell_{31}\ell_3^+ \ell_3^-$, $\ell_{23}\ell_3^+ \ell_3^-$, $\ell_{12}\ell_3^+ \ell_3^-$, $\ell_{23}\ell_1^- \ell_2^+$, $\ell_{23}\ell_1^+ \ell_2^-$, $\ell_{31}\ell_1^+ \ell_2^-$, $\ell_{31}\ell_1^- \ell_2^+$, $\ell_3^+ \ell_{12}^2$

$\alpha = 3$: $\ell_1^- \ell_1^+ \ell_{23}$, $\ell_2^- \ell_2^+ \ell_{31}$, $\ell_3^- \ell_3^+ \ell_{12}$, $\ell_1^+ \ell_2^+ \ell_3^+$, $\ell_1^- \ell_2^- \ell_3^+$, $\ell_1^+ \ell_2^- \ell_3^-$, $\ell_1^- \ell_2^+ \ell_3^-$.

$\alpha = 4$: $\ell_{12}\ell_3^+ \ell_3^-$, $\ell_{23}\ell_1^+ \ell_1^-$, $\ell_{31}\ell_2^+ \ell_2^-$.

(Cf. Fig. 1, convention $\{i, j, k\} = \{1, 2, 3\}$).

LEMMA 3.3. *Let $B = B(\alpha)$. We have*
(i) *If $m = 2n + 1$, $|mK_B|$ is the sum of*

$$D_\nu + |f^*\{n(3H - \xi_1 - \xi_2 - \xi_3 - A) - \xi_\nu + \xi_{\nu+1}\}|$$

and

1) In our case Fig. 1 checks that we have equality.

$$D + |f^*\{(n-1)(3H - \xi_1 - \xi_2 - \xi_3 - A)\}| \qquad (\textit{indices} \bmod 3)$$

(ii) *If* $m = 2n$, $|mK_B|$ *is the sum of*

$$f^*|n(3H - \xi_1 - \xi_2 - \xi_3 - A)|$$

and

$$D_\mu + D_\sigma + f^*|\{(n-1)(3H - \xi_1 - \xi_2 - \xi_3 - A) - \xi_\nu + \xi_{\nu+1}\}|$$

$\{\mu, \nu, \sigma\} = \{1, 2, 3\}$, *and* $\nu + 1 = 1$ *in case* $\nu = 3$).

Proof. Observe that, on P, the plane $\boldsymbol{P}_2$ blown up at ξ_1, ξ_2, ξ_3 and A one has:

$$\begin{aligned} 2K_P + C &= 3H - X_1^0 - X_2^0 - X_3^0 - A \\ 2K_P + C - F_\nu &= -X_\nu^0 + X_{\nu+1}^0 \qquad (\nu = 1, 2, 3) \\ K_P + F_\nu &= -X_\nu^0 + X_{\nu+1}^0 \qquad (\nu = 1, 2, 3) \end{aligned}$$

Then apply Proposition 1.6.

LEMMA 3.4. *There is no odd torsion in* $\operatorname{Pic}(B(\alpha))$.

Proof. Suppose $0 \neq \ell \in \operatorname{Pic}(B)$ is a torsion element. Then, since $q(B) = 0$, $h^0(\ell) = 0$, so dually $h^0(K_B \otimes \ell) = 0$ for any non zero torsion element ℓ. By Riemann-Roch one concludes that $|K_B \otimes \ell|$ contains a divisor Γ and if the order of ℓ equals t, $t\Gamma \in |tK_B|$. Applying this remark in case $t = 2n + 1$ (n some natural number) there has to be a curve in $|(2n+1)K_B|$ which is $2n + 1$-fold. It is sufficient to look for $2n + 1$-fold curves in any of the subsystems $D_\nu + |f^*\{(n(3H - \xi_1 - \xi_2 - \xi_3 - A) - \xi_\nu + \xi_{\nu+1}\}|$, (cf. Lemma 3.3). This subsystem has a $2n + 1$-fold curve if $nC_\nu = n(3H - \xi_\mu - 2\xi_\sigma - A)$ appears in $|n(3H - \xi_1 - \xi_2 - \xi_3 - A) - \xi_\nu + \xi_{\nu+1}|$, where (ν, μ, σ) is an even permutation of $(1, 2, 3)$. The residual system, however is clearly seen to be empty. This completes the proof of the lemma.

LEMMA 3.5. *There is no 4-torsion in* $\operatorname{Pic}(B(\alpha))$.

Proof. From Lemma 3.3., part (ii) with $n = 2$ we have that $|4K_B|$ consists of the subsystems $|2(3H - \sum_\nu \xi_\nu - A)|$ and $I_\nu = D_\mu + D_\sigma + f^*|3H - \sum_\nu \xi_\nu - A|$. The first system consists of the sum of pairs of elements of $|2K_B|$, hence, to prove the assertion, it is sufficient to prove that I_ν does not contain multiple curves. Suppose the contrary, then, since $D_\mu + D_\sigma \sim f^*F_\nu$, the system $|3H - \sum_\nu \xi_\nu - A - \xi_\nu + \xi_{\nu+1}|$ would have to contain $F_\nu = 3H - \xi_\mu - 2\xi_\sigma - A$, but the residual system clearly is empty. This contradiction proves the Lemma.

PROPOSITION 3.6. *The torsion group of $B(\alpha)$ equals $\oplus^{6-\alpha} \boldsymbol{Z}_2$.*

Proof. According to Lemmas 3.4 and 3.5 there is only 2-torsion. We have exhibited $2^{6-\alpha} - 1$ different non-trivial 2-torsion elements, hence the assertion follows.

COROLLARY 3.7 (of proof). *The fundamental group of $B(0)$ is not abelian:*

Proof. Suppose it were and form $\varphi: \tilde{B}(0) \to B(0)$, the unramified 2^6-fold covering with $\tilde{B}(0)$ the universal covering of $B(0)$. Since $0 = q(\tilde{B}(0)) = \oplus_{g \in G} h^1(\mathcal{O}_{B(0)}(g))$ with $G = \oplus^6 \boldsymbol{Z}_2$, we must have that $h^1(\mathcal{O}_{B(0)}(g)) = 0$ for all $g \in G$. By Riemann-Roch it would then follow that $h^2(\mathcal{O}_{B(0)}(g)) = 1$ $g \neq 1, g \in G$. Now the Serre duality gives $h^0(K_{B(0)} + g) = 1$ for those $g \in G$, but clearly the elements $\ell_{ij} H_k^2$ give 2-torsion elements $g, g \neq 1$ with at least 2 independent sections in $K_{B(0)} + g$. This contradiction shows that $q(\tilde{B}(0)) \neq 0$, and hence $\pi_1(B(0))$ is *not* abelian.

Remark 3.8. It is not clear whether this Corollary holds for $B(\alpha)$, $\alpha = 1, \cdots, 4$, since the above proof fails to hold.

Remark 3.9. M. Reid pointed out to me a possible way of describing the Burniat surface with $c_1^2 = 6$, starting with the 3 pencils of curves of genus 3 lying above the pencils $\{H_k\}$. From this construction $B(0)$ has an 8-fold covering lying in the product of 3 elliptic curves as a divisor of type (2.2) invariant under a certain group of order 8. This explains 3.7 geometrically.

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