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ON SASAKIAN MANIFOLDS WITH CONSTANT SCALAR CURVATURE WHOSE C-BOCHNER CURVATURE TENSOR VANISHES

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§1. Introduction

In a Sasakian manifold, a C-Bochner curvature tensor is constructed from the Bochner curvature tensor in a Kaehlerian manifold by the fibering of Boothby-Wang [2]. Many subjects for vanishing C-Bochner curvature tensor have been studied in [4] $\sim$ [9] and so on. One of those, done by Choi, Ki and Takano, asserts that the following theorem :

Theorem A([3]). *Let $M^n (n \geq 5)$ be a Sasakian manifold with vanishing C-Bochner curvature tensor. Then the scalar curvature R is constant if and only if $TrRic^{(m)}$ is constant for an integer $m(\geq 2)$. Furthermore, if $TrRic^{(m)}$ is constant for a positive integer m and the length of the η-Einstein tensor is less than $\frac{\sqrt{2}(R-n+1)}{\sqrt{(n-1)(n-3)}}$, then M is a space of constant ϕ- holomorphic sectional curvature.*

The purpose of the present paper is to investigate a Sasakian manifold with vanishing C-Bochner curvature tensor and with constant scalar curvature. Our main result appeared in §3.

§2. Preliminaries

Let M be an $n(>3)$-dimensional Sasakian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$, where here and in the sequel the indices $h, i, j, \ldots$ run over the range $\{1, 2, \ldots, n\}$ (The summation convention will be used with respect to these indices). If we denote by ∇ the operator of covariant differentiation with respect to the Riemannian connection of M, then there exists a unit Killing vector ξ^h satisfying

$$\text{(2.1)} \qquad \begin{cases} \phi_j{}^r \phi_r{}^h = -\delta_j{}^h + \eta_j \xi^h, \quad \eta_j = g_{jr}\xi^r, \quad \eta_r \phi_j{}^r = 0, \\ \phi_r{}^h \xi^r = 0, \quad g_{rs}\phi_j{}^r \phi_i{}^s = g_{ji} - \eta_j \eta_i, \quad \phi_{ji} + \phi_{ij} = 0, \end{cases}$$

$$\text{(2.2)} \qquad \phi_{ji} = \nabla_j \eta_i, \quad \nabla_k \phi_{ji} = -g_{kj}\eta_i + g_{ki}\eta_j.$$

Because of the Ricci formula for ξ^i, it is clear that

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(2.3) $$R_{kji}{}^{r}\eta_r = \eta_k g_{ji} - \eta_j g_{ki}$$

and hence

(2.4) $$R_{jr}\xi^r = (n-1)\eta_j,$$

where R_{kjih} and R_{ji} denote the components of the Riemannian curvature tensor K and of the Ricci tensor Ric respectively.

It is well known that in a Sasakian manifold the following equations hold :

(2.5) $$H_{ji} + H_{ij} = 0,$$

(2.6) $$R_{ji} = R_{rs}\phi_j{}^r\phi_i{}^s + (n-1)\eta_j\eta_i,$$

(2.7) $$\begin{aligned}\nabla_k R_{ji} - \nabla_j R_{ki} =& (\nabla_s R_{kr})\phi_j{}^r\phi_i{}^s \\ & - \{H_{ki} - (n-1)\phi_{ki}\}\eta_j - 2\{H_{kj} - (n-1)\phi_{kj}\}\eta_i,\end{aligned}$$

(2.8) $$\nabla_k R_{ji} - (\nabla_k R_{rs})\phi_j{}^r\phi_i{}^s = -\{H_{kj} - (n-1)\phi_{kj}\}\eta_i - \{H_{ki} - (n-1)\phi_{ki}\}\eta_j,$$

(2.9) $$\xi^r\nabla_r R_{kji}{}^h = 0,$$

where we put $H_{ji} = \phi_j{}^r R_{ri}$.

We denote a tensor field $W^{(m)}$ with components $W_{ji}{}^{(m)}$ and a function $W_{(m)}$ for any positive integer m as follows :

(2.10) $$W_{ji}{}^{(m)} = W_{ji_1}W_{i_2}{}^{i_1}\dots W_{i}{}^{i_{m-1}}, \quad W_{(m)} = \operatorname{Tr} W^{(m)} = W^{(m)}{}_i{}^i.$$

Also, we define the η-Einstein tensor T_{ji} by

(2.11) $$T_{ji} = R_{ji} - \left(\frac{R}{n-1} - 1\right) g_{ji} + \left(\frac{R}{n-1} - n\right)\eta_j\eta_i.$$

If the η-Einstein tensor vanishes, then M is called an η-Einstein manifold. From (2.4) and (2.5),we have

(2.12) $$\operatorname{Tr} T = T_r{}^r = 0,$$

(2.13) $$T_{jr}\xi^r = 0,$$

$$(2.14) \qquad T_{jr}\phi_i{}^r + T_{ir}\phi_j{}^r = 0.$$

A Sasakian manifold M is called a space of constant ϕ-holomorphic sectional curvature c if the curvature tensor of M has the form :

$$\begin{aligned} R_{kji}{}^h =& \frac{c+3}{4}(g_{ji}\delta_k^h - g_{ki}\delta_j{}^h) \\ &+ \frac{c-1}{4}(g_{ki}\eta_j\xi^h - g_{ji}\eta_k\xi^h + \eta_k\eta_i\delta_j{}^h - \eta_j\eta_i\delta_k^h - \phi_{ki}\phi_j{}^h + \phi_{ji}\phi_k^h - 2\phi_{kj}\phi_i{}^h). \end{aligned}$$

Matsumoto and Chūman ([8]) were introduced the C-Bochner curvature tensor $B_{kji}{}^h$ defined by

$$\begin{aligned} B_{kji}{}^h =& R_{kji}{}^h + \frac{1}{n+3}(R_{ki}\delta_j{}^h - R_{ji}\delta_k^h + g_{ki}R_j{}^h - g_{ji}R_k^h + H_{ki}\phi_j{}^h \\ &- H_{ji}\phi_k^h + \phi_{ki}H_j{}^h - \phi_{ji}H_k^h + 2H_{kj}\phi_i{}^h + 2\phi_{kj}H_i{}^h \\ &- R_{ki}\eta_j\xi^h + R_{ji}\eta_k\xi^h - \eta_k\eta_i R_j{}^h + \eta_j\eta_i R_k^h) \\ &- \frac{k+n-1}{n+3}(\phi_{ki}\phi_j{}^h - \phi_{ji}\phi_k^h + 2\phi_{kj}\phi_i{}^h) - \frac{k-4}{n+3}(g_{ki}\delta_j{}^h - g_{ji}\delta_k^h) \\ &+ \frac{k}{n+3}(g_{ki}\eta_j\xi^h - g_{ji}\eta_k\xi^h + \eta_k\eta_i\delta_j{}^h - \eta_j\eta_i\delta_k^h), \end{aligned} \tag{2.15}$$

where $k = \frac{R+n-1}{n+1}$. It is well-known that if a Sasakian manifold with vanishing C-Bochner curvature tensor is an η-Einstein manifold, then it is a space of constant ϕ-holomorphic sectional curvature $c = \frac{4R-(n-1)(3n-1)}{(n+1)(n-1)}$.

By a straightforward computation, we can prove

$$\begin{aligned} \frac{n+3}{n-1}\nabla_r B_{kji}{}^r =& \nabla_k R_{ji} - \nabla_j R_{ki} - \eta_k\{H_{ji} - (n-1)\phi_{ji}\} \\ &+ \eta_j\{H_{ki} - (n-1)\phi_{ki}\} + 2\eta_i\{H_{kj} - (n-1)\phi_{kj}\} \\ &+ \frac{1}{2(n+1)}\{(g_{ki} - \eta_k\eta_i)\delta_j{}^r - (g_{ji} - \eta_j\eta_i)\delta_k^r \\ &+ \phi_{ki}\phi_j{}^r - \phi_{ji}\phi_k^r + 2\phi_{kj}\phi_i{}^r\}R_r, \end{aligned} \tag{2.16}$$

where we put $R_j = \nabla_j R$.

§3. Vanishing C-Bochner curvature tensor with constant scalar curvature

Let M be an $n(\geq 5)$-dimensional Sasakian manifold with vanishing C-Bochner curvature tensor. By (2.1), (2.4), (2.7) $\sim$ (2.9) and (2.16), we then obtain

$$\begin{aligned} \nabla_k R_{ji} =& \{R_{kr} - (n-1)g_{kr}\}(\phi_j{}^r\eta_i + \phi_i{}^r\eta_j) \\ &+ \frac{1}{2(n+1)}\{2R_k(g_{ji} - \eta_j\eta_i) + R_j(g_{ki} - \eta_k\eta_i) \\ &+ R_i(g_{kj} - \eta_k\eta_j) - \phi_{kj}\phi_i{}^r R_r - \phi_{ki}\phi_j{}^r R_r\}. \end{aligned} \tag{3.1}$$

Now, suppose that the scalar curvature R of M is constant. Then the equation (3.1) is reduced to

$$\nabla_k R_{ji} = \{R_{kr} - (n-1)g_{kr}\}(\phi_j{}^r \eta_i + \phi_i{}^r \eta_j), \tag{3.2}$$

which implies $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$, namely, the Ricci tensor is cyclic parallel and hence the η-Einstein tensor T is also cyclic parallel because of (2.11). Therefore, using the Ricci formula for R_{ji}, we find

$$\nabla^k \nabla_k R_{ji} = 2(R_{rjis}R^{rs} - R_{ji}{}^{(2)}).$$

Applying ∇^k to (3.2) and owing to (2.1) and (2.4), we get

$$\nabla^k \nabla_k R_{ji} = -2[R_{ji} - (n-1)g_{ji} - \{R - n(n-1)\}\eta_j \eta_i].$$

On the other hand, by virtue of (2.1) $\sim$ (2.6) and (2.15), it is clear that

$$\begin{aligned}(n+3)R_{rjis}R^{rs} =& 4R_{ji}{}^{(2)} - (4n - R + 2k)R_{ji} + \{R_{(2)} - (k-4)R + (n-1)k\}g_{ji} \\ &- \{R_{(2)} + (n-1)^2 - (n-1)k - kR\}\eta_j \eta_i.\end{aligned}$$

From the last three equations, we have

$$R_{ji}{}^{(2)} = \beta R_{ji} + \gamma g_{ji} + \{(n-1)^2 - (n-1)\beta - \gamma\}\eta_j \eta_i, \tag{3.3}$$

where β and γ are given by

$$(n+1)\beta = R - 3n - 5, \tag{3.4}$$

$$(n-1)\gamma = R_{(2)} - \frac{1}{n+1}R^2 + 4R - \frac{n-1}{n+1}(n^2 + 3n + 4). \tag{3.5}$$

The scalar curvature assumed to be constant, by Theorem A we see that $R_{(2)}$ is constant. so are β and γ.

Transforming (2.11) by $R_k{}^i$ and making use of (2.4) and (3.3), we find

$$T_{jr}R_k{}^r = \left(\beta + 1 - \frac{R}{n-1}\right) R_{jk} + \gamma g_{jk} + \{R - n + 1 - (n-1)\beta - \gamma\}\eta_j \eta_k,$$

which together with (2.11) and (2.13) implies that

$$\begin{aligned}T_{ji}{}^{(2)} =& \left(\beta + 2 - \frac{2R}{n-1}\right) T_{ji} \\ &+ \left\{\gamma + \left(\frac{R}{n-1} - 1\right)\left(\beta + 1 - \frac{R}{n-1}\right)\right\}(g_{ji} - \eta_j \eta_i).\end{aligned}$$

If we take account of (3.4) and (3.5), then it turns out to be

$$(3.6)\qquad \begin{aligned} T_{ji}{}^{(2)} = & -\frac{n+3}{n^2-1}(R+n-1)T_{ji} \\ & + \left(\frac{R_{(2)}}{n-1} - \frac{R^2}{(n-1)^2} + \frac{2R}{n-1} - n\right)(g_{ji} - \eta_j\eta_i). \end{aligned}$$

Thus, it is seen that

$$(3.7)\qquad T_{(2)} = R_{(2)} - \frac{R^2}{n-1} + 2R - n(n-1).$$

We notice here that the Sasakian manifold is η-Einstein if and only if $T_{(2)} \geq 0$ and $T_{(2)} = 0$. From (3.6) and (3.7) it follows that we get

$$(3.8)\qquad T_{ji}{}^{(2)} = a\,T_{ji} + \frac{T_{(2)}}{n-1}(g_{ji} - \eta_j\eta_i),$$

where we have put

$$(3.9)\qquad a = -\frac{n+3}{n^2-1}(R+n-1).$$

First of all we prove

Lemma 1. *Let M be an $n(\geq 5)$-dimensional Sasakian manifold with vanishing C-Bochner curvature tensor. If $R+n-1=0$, then M is an η-Einstein manifold or M admits a cyclic parallel almost product structure which is not parallel.*

Proof. By the assumption $R+n-1=0$, the equation (2.11) and (3.8) are respectively reduced to

$$(3.10)\qquad T_j{}^i = R_j{}^i + 2\delta_j{}^i - (n+1)\eta_j\xi^i,$$

$$(3.11)\qquad T_{ji}{}^{(2)} = \frac{T_{(2)}}{n-1}(g_{ji} - \eta_j\eta_i).$$

Suppose that M is not η-Einstein, namely $T_{(2)} \neq 0$. Then we can define a tensor field P of type (1,1) by

$$(3.12)\qquad P = \frac{1}{2}\left(I + \eta\otimes\xi - \sqrt{\frac{n-1}{T_{(2)}}}\;T\right),$$

where I denotes the unit tensor field and T is the tensor field of type (1,1) given by (3.10). Thus, by using (2.13) and (3.11), we see that

$$P^2 = P, \quad PQ = 0, \quad Q^2 = Q,$$

where $Q = I - P$. Accordingly P and Q define two complementary almost product structures. Since T is cyclic parallel we see, using (3.12), that P and Q are cyclic parallel. But P is not parallel. In fact, if P is parallel, then by (3.12) we have

$$\nabla_k T_{ji} = \sqrt{\frac{T_{(2)}}{n-1}}\,(\phi_{kj}\eta_i + \phi_{ki}\eta_j).$$

On the other hand, from (3.10) we get

$$\nabla_k T_{ji} = R_{kr}\phi_j{}^r\eta_i + R_{kr}\phi_i{}^r\eta_j - 2(\phi_{kj}\eta_j + \phi_{ki}\eta_j)$$

because of (3.2). From the last two equations, it is seen that $T_{(2)} = 0$, a contradiction. This completes the proof.

Lemma 2. *Let M be an $n(\geq 5)$-dimensional Sasakian manifold with vanishing C-Bochner curvature tensor and constant scalar curvature. If $T_{(2)} \neq 0$ and $R+n-1 \neq 0$, then M admits a cyclic parallel almost product structure which is not parallel.*

Proof. The equations (2.13) and (3.8) tell us that M has at most three constant eigenvalues of $T : 0, x_1$ and x_2, where

$$x_1 = \frac{1}{2}(a+\sqrt{D}), \quad x_2 = \frac{1}{2}(a-\sqrt{D}), \quad D = a^2 + \frac{4}{n-1}T_{(2)}.$$

The multiplicities of x_1 and x_2 are respectively denoted by r and t. Then by (2.12) and (2.13) we have

$$a(r+t) = \sqrt{D}\,(t-r). \tag{3.13}$$

The trace of $T_{ji}{}^{(2)}$ is also given by

$$T_{(2)} = \frac{1}{2}(r+t)a^2 + \frac{r+t}{n-1}T_{(2)} - \frac{a}{2}(t-r)\sqrt{D}.$$

Combining above two equations, we have $(r+t-n+1)T_{(2)} = 0$ and hence $r+t = n-1$. Thus (3.13) turns out to be $T_{(2)} = m(n-1)a^2$, where $m = r(n-1-r)/(2r-n+1)^2$. Therefore equation (3.8) is reduced to

$$T_{ji}{}^{(2)} = aT_{ji} + ma^2(g_{ji} - \eta_j\eta_i). \tag{3.14}$$

By the hypothesis $R+n-1 \neq 0$ it is clear that $m \neq 0$. Thus we can define a tensor field P' of type (1,1) by

$$P' = \frac{1}{a\sqrt{4m+1}}T + \frac{1}{2}(1 - \frac{1}{\sqrt{4m+1}})(I - \eta\otimes\xi),$$

where T is the tensor field of type (1,1) derived from (2.11). Because of (2.13) and (3.14) we see that

$$P'^2 = P', \quad P'Q' = 0, \quad Q'^2 = Q',$$

where $Q' = I - P'$. As in the proof of Lemma 1, we can verify P' is cyclic parallel and not parallel. Thus Lemma 2 is proved.

According to (2.15), Lemma 1 and Lemma 2, we have

Theorem 3. *Let M be an $n(\geq 5)$-dimensional Sasakian manifold with constant scalar curvature R whose C-Bochner curvature tensor vanishes. Then M is a space of constant ϕ- holomorphic sectional curvature $\frac{4R-(n-1)(3n-1)}{(n-1)(n+1)}$ or M admits a cyclic parallel almost product structure which is not integrable.*

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