# TWO CONDITIONS ON THE RICCI TENSOR OF A REAL HYPERSURFACE OF COMPLEX PROJECTIVE SPACE 

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Abstract. We study two conditions on the Ricci tensor of a real hypersurface of complex projective space that allows us to characterize certain real hypersurfaces. We also introduce a new kind of such real hypersurfaces.

## §1. Introduction

Let $C P^{m}$ be a complex projective space of complex dimension $m$ endowed with the Fubini-Study metric $g$ of constant holomorphic sectional curvature 4. Let $M$ be a connected real hypersurface of $C P^{m}$ and $N$ a local unit normal vector field on $M$. Then $\xi=-J N$ is tangent to $M$, where $J$ denotes the almost complex structure of $\boldsymbol{C P}{ }^{m}$.

Let us denote by $A, R$ and $S$ the shape operator, the curvature tensor and the Ricci tensor of $M$, respectively. We put $h=\operatorname{trace} A$ and $H=h A-A^{2}$.

A real hypersurface $M$ of $C P^{m}$ is called pseudo-Einstein if its Ricci tensor satisfies

$$
\begin{equation*}
S X=a X+b g(X, \xi) \xi \tag{1.1}
\end{equation*}
$$

for any vector field $X$ tangent to $M$ and some functions $a, b$ on $M$, where $g$ denotes the induced Riemannian metric on $M$. Pseudo-Einstein real hypersurfaces of $C P^{m}$ are classified by the following theorem:

Theorem A. ([1]) Let $M$ be a complete pseudo-Einstein real hypersurface of $C P^{m}, m \geq 3$. Then $M$ is locally congruent to one of the following spaces:
a) a geodesic hypersphere,
b) a tube of radius $r$ over a totally geodesic $C P^{k} \quad(1 \leq k \leq m-2)$, where

$$
0<r<\frac{\pi}{2} \text { and } \cot ^{2} r=\frac{k}{m-k-1}
$$

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c) a tube over a complex quadric $Q_{m-1}$.

Ruled real hypersurfaces in $C P^{m}$ were introduced by Kimura in [5]. Several authors have studied them ([6],[7],[10]).

Let us consider real hypersurfaces of $C P^{m}$ satisfying

$$
\begin{equation*}
R(X, Y) S Z+R(Y, Z) S X+R(Z, X) S Y=0 \tag{1.2}
\end{equation*}
$$

for any $X, Y, Z$ tangent to $M$. It is clear that any pseudo-Einstein real hypersurface satisfies (1.2). The converse is proved in [3]. Moreover, in [4] we have proved (as another characterization of pseudo-Einstein real hypersurfaces in $C P^{m}$ ) that a real hypersurface in $C P^{m}$ satisfying

$$
\begin{equation*}
R(X, Y) S Z+R(Y, Z) S X+R(Z, X) S Y=0 \text { for any } X, Y, Z \perp \xi \tag{1.3}
\end{equation*}
$$

is pseudo-Einstein, under the assumption that the structure vector $\xi$ is principal. In this paper, first we shall generalize this result as follows.

Theorem 1. Let $M$ be a real hypersurface of $C P^{m}, m \geq 3$, satisfying (1.3). Then $M$ is locally congruent to either
i) a pseudo-Einstein real hypersurface, or
ii) a non-homogeneous real hypersurface such that $H$ has three distinct eigenvalues with respective multiplicities 2m-9, 1 and 1.

Remark. The second kind of real hypersurfaces afpearing in Theorem 1 are nonhomogeneous real hypersurfaces with at most 4 distinct principal curvatures. If the eigenvalue of $H$ with multiplicity $2 m-3$ is equal to 0 , the expression of $H$ is similar to the expression of $A$ in a ruled real hypersurface.

On the other hand, we can also generalize condition (1.2) by considering real hypersurfaces of $C P^{m}$ satisfying

$$
\begin{equation*}
R(X, Y) S \xi+R(Y, \xi) S X+R(\xi, X) S Y=0 \quad \text { for any } X, Y \perp \xi \tag{1.4}
\end{equation*}
$$

It is easy to see that any real hypersurface of $C P^{m}$ such that $\xi$ is principal satisfies (1.4). So it seems to be interesting to study real hypersurfaces of $C P^{m}$ satisfying (1.4) and such that $\xi$ is not principal. In this way we shall obtain

Theorem 2. Let $M$ be a real hypersurface of $C P^{m}, m \geq 3$, satisfying (1.4) and such that $\xi$ is not principal. Then $M$ is a ruled real hypersurface.

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## §2. Preliminaries.

Let $M$ be a real hypersurface of $m(\geq 3)$-dimensional complex projective space $C P^{m}$ and let $N$ be a unit normal field on a neighborhood of a point $x$ in $M$. We denote by $J$ an almost complex structure of $C P^{m}$. For a local vector field $X$ on a neighborhood of $x$ in $M$, the images of $X$ and $N$ under the linear transformation $J$ can be represented as

$$
J X=\phi X+\eta(X) N, \quad J N=-\xi
$$

where $\phi$ defines a skew-symmetric transformation on the tangent bundle $T M$ of $M$, while $\eta$ and $\xi$ denote a 1 -form and a vector field on a neighborhood of $x$ in $M$, respectively. Moreover, it is seen that $g(\xi, X)=\eta(X)$, where $g$ denotes the induced Riemannian metric on $M$. By properties of the almost complex structure $J$, the set $(\phi, \xi, \eta, g)$ of tensors satisfies

$$
\phi^{2}=-I+\eta \otimes \xi, \quad \phi \xi=0, \quad \eta(\phi X)=0, \quad \eta(\xi)=1
$$

where $I$ denotes the identity transformation. Accordingly, the set is so called an almost contact metric structure. Furthermore the covariant derivative of the structure tensors are given by

$$
\begin{equation*}
\left(\nabla_{X} \phi\right) Y=\eta(Y) A X-g(A X, Y) \xi, \quad \nabla_{X} \xi=\phi A X \tag{2.1}
\end{equation*}
$$

for any vector fields $X, Y$ tangent to $M$, where $\nabla$ is the Riemannian connection of $g$ and $A$ denotes the shape operator with respect to the unit normal $N$ on $M$.

Since the ambient space is of constant holomorphic sectional curvature 4, the equations of Gauss is given by

$$
\begin{align*}
R(X, Y) Z= & g(Y, Z) X-g(X, Z) Y+g(\phi Y, Z) \phi X-g(\phi X, Z) \phi Y  \tag{2.2}\\
& -2 g(\phi X, Y) \phi Z+g(A Y, Z) A X-g(A X, Z) A Y
\end{align*}
$$

for any vector fields $X, Y, Z$ on $M$, where $R$ denotes the Riemannian curvature tensor of $M$.

From (2.2) the Ricci tensor of $M$ is given by

$$
\begin{equation*}
S X=(2 m+1) X-3 \eta(X) \xi+H X \tag{2.3}
\end{equation*}
$$

for any $X$ tangent to $M$.
To be used later we recall that the type number $t$ of $M$ at a point $p$ of $M$ is defined as the rank of $A$ at $p$. Then

Proposition. ([10]) Let $M$ be a real hypersurface of $C P^{m}, m \geq 3$, satisfying $t(p) \leq 2$ for any point $p$ in $M$. Then $M$ is a ruled real hypersurface.

## §3. Proof of Theorem 1.

Bearing in mind the expression (2.3) for any vector field $X$ orthogonal to $\xi$ and the first identity of Bianchi (1.3) is equivalent to

$$
\begin{equation*}
R(X, Y) H Z+R(Y, Z) H X+R(Z, X) H Y=0 \tag{3.1}
\end{equation*}
$$

for any $X, Y, Z \perp \xi$.
Let $\left\{E_{j}\right\}_{j=1, \ldots, 2 m-2}$ be a local orthonormal basis of $\xi^{\perp}$. Substituting (2.2) into (3.1) and taking $Y=\phi E_{j}, Z=E_{j}$, we obtain

$$
\begin{align*}
0= & \left\{g\left(\phi X, H E_{j}\right)-g\left(\phi E_{j}, H X\right)\right\} E_{j}+\left\{g\left(\phi X, H \phi E_{j}\right)+g\left(E_{j}, H X\right)\right\} \phi E_{j}  \tag{3.2}\\
& -\left\{g\left(E_{j}, H E_{j}\right)+g\left(\phi E_{j}, H \phi E_{j}\right)\right\} \phi X-2 g\left(X, E_{j}\right) \phi H E_{j} \\
& +2 g\left(E_{j}, \phi X\right) \phi H \phi E_{j}+2 \phi H X
\end{align*}
$$

for any $X \perp \xi$.
Lemma 1. $g(H X, \phi X)=0 \quad$ for any $X \perp \xi$.
Proof. If we take the scalar product of (3.2) and $X$ and add in $j$, we have from the formulas of section 2

$$
\begin{equation*}
0=(8-4 m) g(H X, \phi X) \tag{3.3}
\end{equation*}
$$

for any $X \perp \xi$. As $m \geq 3$ the result follows.

Lemma 2. Let be $X, Z \perp \xi$ such that $Z \perp S p a n\{X, \phi X\}$. Then $g(H X, Z)=0$.
Proof. If we take $X, Y \perp \xi$ from Lemma 1, $g(H(X+Y), \phi(X+Y))=0$. This implies again by Lemma 1

$$
\begin{equation*}
g(H \phi X, Y)=g(\phi H X, Y) \quad \text { for any } X, Y \perp \xi \tag{3.4}
\end{equation*}
$$

Taking an inner product (3.2) with $Z$ and adding in $j$, we have

$$
\begin{align*}
0= & g(H \phi X, Z)+g(\phi H X, Z)+g(H \phi X, Z)-g(H X, \phi Z)+2 g(X, H \phi Z)  \tag{3.5}\\
& +2 g(X, H \phi Z)+2(2 m-2) g(\phi H X, Z) .
\end{align*}
$$

From (3.4) and (3.5) it follows

$$
\begin{equation*}
0=(4 m-4) g(H \phi X, Z) \tag{3.6}
\end{equation*}
$$

Since $m \geq 3$, exchanging $X$ by $\phi X$ in (3.6), then we have the above result.
Lemma 3. $g(H X, X)=g(H Z, Z) \quad$ for any unit $X, Z \perp \xi$.
Proof. Taking an inner product (3.2) with $\phi X$ and adding in $j$, we have

$$
\begin{equation*}
0=g(H \phi X, \phi X)+(2 m-3) g(H X, X)-a \tag{3.7}
\end{equation*}
$$

where $a=\Sigma_{j=1}^{2 m-2} g\left(E_{j}, H E_{j}\right)$.
On the other hand, putting $Y=\phi X$ in (3.4), we know $g(H \phi X, \phi X)=g(H X, X)$.
From this and (3.7) it follows

$$
\begin{equation*}
0=(2 m-2) g(H X, X)-a . \tag{3.8}
\end{equation*}
$$

That is,

$$
\begin{equation*}
g(H X, X)=\frac{a}{2 m-2} \tag{3.9}
\end{equation*}
$$

for any unit $X \perp \xi$.

From the above Lemmas we have two cases: If $\xi$ is an eigenvector of $H$, any $X \perp \xi$ is an eigenvector of $H$ with eigenvalue $d=\frac{a}{i \leq!n-2}$ of multiplicity $2 m-2$. This implies that $M$ is pseudo-Einstein.

If $\xi$ is not an eigenvector of $H$, there exists a unit $U \perp \xi$ such that $H \xi=\nu U+\mu \xi$, for certain functions $\nu, \mu$ on $M$ such that $\nu$ is not identically zero. From the above Lemmas $H U=d U+\nu \xi$ and $H X=d X$ for any $X \perp S p a n\{U, \xi\}$. Thus $M$ is a real hypersurface of type ii) in Theorem 1. This finishes the proof of Theorem 1.

## §4. Proof of Theorem 2.

From (2.2) we know that (1.4) is equivalent to

$$
\begin{align*}
0= & g(\phi Y, S \xi) \phi X-g(\phi X, S \xi) \phi Y-2 g(\phi X, Y) \phi S \xi \\
& +g((S A-A S) Y, \xi) A X+g((A S-S A) X, \xi) A Y  \tag{4.1}\\
& +g((S A-A S) X, Y) A \xi
\end{align*}
$$

for any $X, Y \perp \xi$. Using (2.3) and the definition of $H$, (4.1) becomes

$$
\begin{align*}
0= & -3 g(A Y, \xi) A X+3 g(A X, \xi) A Y+g(\phi Y, H \xi) \phi X  \tag{4.2}\\
& -g(\phi X, H \xi) \phi Y-2 g(\phi X, Y) \phi H \xi
\end{align*}
$$

for any $X, Y \perp \xi$. Notice that any real hypersurface with $\xi$ principal satisfies (4.2). Let us replace $Y$ by $\phi Y$ in (4.2). Then from the almost contact metric structure we get

$$
\begin{align*}
0= & -3 g(A \phi Y, \xi) A X+3 g(A X, \xi) A \phi Y  \tag{4.3}\\
& -g(Y, H \xi) \phi X+g(\phi X, H \xi) Y-2 g(X, Y) \phi H \xi .
\end{align*}
$$

Contracting (4.3) with respect to $X, Y$ we have $6 A \phi A \xi+(2-4 m) \phi H \xi=0$, that is

$$
\begin{equation*}
A \phi A \xi=\frac{2 m-1}{3} \phi H \xi . \tag{4.4}
\end{equation*}
$$

Taking $Y=\phi A \xi$ in (4.2) and using (4.4), we have

$$
\begin{align*}
0= & \{-g(A \xi, H \xi)+g(A \xi, \xi) g(H \xi, \xi)\} \phi X+(2 m-3) g(A X, \xi) \phi H \xi  \tag{4.5}\\
& +g(\phi X, H \xi)\{A \xi-g(A \xi, \xi) \xi\} .
\end{align*}
$$

Let us replace $X$ by $\phi X$ in (4.5). Then

$$
\begin{align*}
0= & \{g(A \xi, H \xi)-g(A \xi, \xi) g(H \xi, \xi)\} X+(2 m-3) g(A \phi X, \xi) \phi H \xi \\
& +g(X, H \xi)\{A \xi-g(A \xi, \xi) \xi\} \tag{4.6}
\end{align*}
$$

for any $X \perp \xi$.

Here we assert two vectors $\phi H \xi$ and $A \xi-g(A \xi, \xi) \xi$ are orthogonal. In fact, by (4.4) and the definition of $H$ we have

$$
\begin{aligned}
g(\phi H \xi, A \xi-g(A \xi, \xi) \xi) & =g(\phi H \xi, A \xi) \\
& =-g\left(h A \xi-A^{2} \xi, \phi A \xi\right) \\
& =\frac{1-2 m}{3} g(h \xi-A \xi, \phi H \xi) \\
& =-\frac{1-2 m}{3} g(A \xi, \phi H \xi)
\end{aligned}
$$

Comparing the second and last quantities in these equations, we have $g(A \xi, \phi H \xi)=$ 0 since $m \geq 3$, which shows our assertion. From this assertion and (4.6) we have for any $X \perp \operatorname{Span}\{\xi, \phi H \xi, A \xi-g(A \xi, \xi) \xi\}$ that
i) $\quad g(A \xi, H \xi)-g(A \xi, \xi) g(H \xi, \xi)=0$,
ii) either $g(H \xi, X)=0$ or $A \xi=g(A \xi, \xi) \xi$ and
iii) either $g(A \phi X, \xi)=0$ or $\phi H \xi=0$.

As $\xi$ is supposed to be not principal, from (ii) we have $g(H \xi, X)=0$ for any $X$ orthogonal to $S p a n\{\xi, \phi H \xi, A \xi-g(A \xi, \xi) \xi\}$, and $g(H \xi, A \xi-g(A \xi, \xi) \xi)=0$ from (i). Thus, since $g(H \xi, \phi H \xi)=0$, if $\phi H \xi \neq 0$, then we have a contradiction that $H \xi$ is parallel to $\xi$. Hence we have $\phi H \xi=0$. By this and (4.2) we have

$$
\begin{equation*}
g(A Y, \xi) A X=g(A X, \xi) A Y \tag{4.7}
\end{equation*}
$$

for any $X, Y \perp \xi$. As $\xi$ is not principal there exists a unit vector field $U \perp \xi$ and two functions $\lambda, \delta$ on $M$ such that $A \xi=\lambda \xi+\delta U$ with $\delta$ non zero. Taking $Y=U$ in (4.7) we have $\delta A X=\delta g(X, U) A U$, for any $X \perp \xi$. Therefore $A X=0$ if $X$ is orthogonal to $\operatorname{Span}\{U, \xi\}$. Thus $t(p) \leq 2$ for any point $p$ in $M$ and the result follows from the Proposition mentioned in paragraph 2.

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