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SURFACES OF EUCLIDEAN 4-SPACE WHOSE GEODESICS ARE W-CURVES*

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0. Abstract

We study complete connected surfaces in 4-dimensional Euclidean space E^4 whose geodesics through a fixed point are W-curves and classify complete connected surfaces in E^4 with geodesic W-curves.

1. Introduction

Surfaces in a Euclidean space are most natural geometric objects which are easily understandable to us. Smooth curves on surfaces, esecially geodesic, are powerful tools to study the given surfaces by examining their curvatures.

In [3], D. Ferus and S. Schirrmacher classified compact connected surfaces in 4-dimensional Euclidean space E^4 with simple geodesics. And, Y.H. Kim([5]) studied such a surface in Euclidean space E^5 and U-H. Ki and Y.H. Kim ([6]) gave classifications of compact connected surfaces in Euclidean space E^4 with a point through which every geodesic is simple.

However, there has been no study on a complete connected surface M in a Euclidean space E^4 with the property that for a fixed point p in M every geodesic passing through p is a W-curve, which will be defined in section 2. From this point of view, we are going to study surfaces in E^4 which have such property.

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2. Preliminaries

Let M_1 be a Riemannian manifold with Levi-Civita connection D. Let $\gamma : I \longrightarrow M_1$ be a curve and let $\gamma'(s) = T_1(s)$ be the unit tangent vector and put $\kappa_1 = \| D_{T_1}T_1 \|$. If κ_1 is identically zero on I, then γ is said to be of *rank 1* . If κ_1 is not identically zero, then one can define T_2 by $D_{T_1}T_1 = \kappa_1 T_2$ on $I_1 = \{s \in I \mid \kappa_1(s) \neq 0\}$. Set $\kappa_2 = \| D_{T_1}T_2 + \kappa_1 T_1 \|$. If κ_2 is identically zero on I_1, then γ is said to be of *rank 2* . If κ_2 is not identically zero on I_1, then we define T_3 by $D_{T_1}T_2 = -\kappa_1 T_1 + \kappa_2 T_3$. Inductively, we can define T_d and $\kappa_d = \| D_{T_1}T_d + \kappa_{d-1}T_{d-1} \|$ and if $\kappa_d = 0$ identically on $I_{d-1} = \{s \in I \mid \kappa_{d-1}(s) \neq 0\}$, then γ is said to be of *rank d*. If γ is of rank d, then we have a matrix equation

$$D_{T_1}(T_1, T_2, \ldots, T_d) = (T_1, T_2, \ldots, T_d)\Lambda$$

on I_{d-1}, where Λ is a $d \times d$-matrix defined by

$$\Lambda = \begin{pmatrix} 0 & -\kappa_1 & 0 & 0 & \ldots & 0 \\ \kappa_1 & 0 & -\kappa_2 & 0 & \ldots & 0 \\ 0 & \kappa_2 & 0 & 0 & \ldots & 0 \\ & \ldots & & & & \\ 0 & 0 & 0 & \ldots & 0 & -\kappa_{d-1} \\ 0 & 0 & 0 & \ldots & \kappa_{d-1} & 0 \end{pmatrix}$$

The matrix Λ, $\{T_1, T_2, \ldots, T_d\}$ and $\kappa_1, \kappa_2, \ldots, \kappa_d$ are called the *Frenet formula*, *Frenet frame*, *Frenet curvatures* of γ respectively.

Let E^m be an m-diemensional Euclidean space with Levi-Civita connection $\tilde{\nabla}$. A regular curve $c : I \subset R \longrightarrow E^m$ is said to be a *W-curve* of rank d if for all $t \in I$, $c'(t) \wedge c''(t) \wedge \ldots \wedge c^{(d)}(t) \neq 0$, $c'(t) \wedge c''(t) \wedge \ldots \wedge c^{(d+1)}(t) = 0$ and the Frenet curvatures $\kappa_1, \kappa_2, \ldots, \kappa_{d-1} : I \longrightarrow R_+$ are constant along c.

By fundamental theorem of curves, if a W-curve is of even rank, then there are positive constants $a_1, a_2, \ldots, a_k$, unique up to order, corresponding positive constants $r_1, r_2, \ldots, r_k$ and orthonormal vectors $f_1, f_2, \ldots, f_{2k} \in E^m$ such that

$$\gamma(t) = (constant) + \sum_{i=1}^{k} \{r_i(\cos a_i t) f_{2i-1} + r_i(\sin a_i t) f_{2i}\}.$$

The rank of unbounded W-curve is odd and the equation for such a curve contains an additional linear term in it.

A surface M in Euclidean space is said to be *helical* at $p \in M$ if all geodesics through p have the same constant curvatures, which are independent of the choice of direction.

Let M be a complete connected surface in E^m with Levi-Civita connection ∇.

Lemma 2.1.([4]) *Let M be helical at $p \in M$ in E^4. Then every geodesic through p is planar .*

For any two vector fields X and Y tangent to M, the second fundamental form h is given by

$$h(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y.$$

For a vector field ξ normal to M and X a vector field tangent to M, we have $\tilde{\nabla}_X \xi$ as

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi,$$

where A_ξ is the Weingarten map associated with ξ and $\nabla^{\perp}$ the normal connection of the normal bundle $T^{\perp}M$. For the second fundamental form h, the covariant derivative of h, denoted by $\bar{\nabla} h$, is defined by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for any vector fields X, Y and Z tangent to M.

The curvature tensor R to M is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for any vector fields X, Y and Z tangent to M.

Let $\tilde{M}$ be a compact Riemannian manifold and let X be an element of the unit tangent space $U_p\tilde{M} = \{X \in T_p\tilde{M} \mid \ \| X \| = 1\}$ and let γ be a geodesic emanating from p with initial velocity X, *i.e.*, $\gamma(s) = exp_p(sX)$, where s is the arclength. Let $Seg(p,q)$ be the set of all minimal geodesics from p to q which are parametrized by arclength. Then for s small enough, $Seg(p = \gamma(0), \gamma(s))$ contains only one element $\gamma \mid_{[0,s]}$. The set

$$A = \{s \in R_+ \mid \quad \gamma \mid_{[0,s]} \in \mathrm{Seg}(p = \gamma(0), \gamma(s))\}$$

is necessarily R_+ or an interval $(0, r]$ for some $r \in R_+$. We say that there is no cut-point on γ if $A = R_+$ and $\gamma(r)$ is the *cut point* of p and r is the *cut value* of γ if $A = (0, r]$. Let $U\tilde{M} = \bigcup_{p \in \tilde{M}} U_p\tilde{M}$ be the unit tangent bundle over $\tilde{M}$. The cut map $\phi : U\tilde{M} \longrightarrow R_+ \cup \{\infty\}$ defined by $\phi(X) = r$ if $A = (0, r]$ and $\phi(X) = \infty$ if $A = R_+$. The cut map is continuous (See[1]).

The *cut-locus* $Cut(p)$ of a point p in $\tilde{M}$ is the set of all cut-points of p, *i.e.*,

$$Cut(p) = \{exp_p(\phi(X)X) \mid X \in U_p\tilde{M}\}.$$

For two distinct points p and q in $\tilde{M}$ we define the *link* from p to q to be

$$\Lambda_{(p,q)} = \{\frac{d\gamma}{ds}(q) \in U_q\tilde{M} \mid \gamma \in Seg(p,q)\}.$$

A subset Θ of the unit sphere of a Euclidean space E^m is said to be a *great sphere* if there exists a subspace W of E^m such that $\Theta = S \cap W$. By definition, the dimension of Θ is $\dim W - 1$.

A compact Riemannian manifold $\tilde{M}$ is said to be a *Blaschke manifold at the point* p in $\tilde{M}$ if for every q in $\mathrm{Cut}(p)$ the link $\Lambda_{(p,q)}$ is a great sphere of $U_q\tilde{M}$. The manifold $\tilde{M}$ is said to be a *Blaschke manifold* if it is a Blaschke manifold at every point in $\tilde{M}$.

For a chracterization of pointed Blaschke manifold, Besse ([1], pp 137) gave

Theorem 2.2. *For a Riemannian manifold M and a point p in M, M is Blaschke manifold at p if and only if $Cut(p)$ is spherical, that is, cut values of geodesics through p are independent of the choice of geodesic.*

3. Complete connected surfaces in E^4 with W-curves through a point

Let M be a complete connected Riemannian surface in 4-dimensional Euclidean space E^4 with an isometric immersion $\mathbf{X} : M \longrightarrow E^4$. Let $\tilde{\nabla}$ be the Riemannian connection on E^4 and ∇ the induced connection on M.

We now define the property $(*)$.

$(*)$: There is a point p in M such that every geodesic through p, which is regarded as a curve in E^4, is a W-curve.

Suppose that M satisfies the property $(*)$. Without loss of generality, we may assume the base point p of $(*)$ as the origin o of E^4.

In the case of compact connected surfaces in E^4 with $(*)$, U-H. Ki and Y.H. Kim ([6]) obtained

Lemma 3.1([6]). *Let M be a compact connected surfaces in E^4. Suppose that there exists a nonperiodic geodesic γ through o. Then M is isometric to a* *standard torus* $S^1(a) \times S^1(b) \subset E^4$.

Lemma 3.2([6]). *Let M be a compact connected surface in E^4. Then M is helical at o if and only if every geodesic through o is a periodic simple curve .*

Remark. *We mean a simple geodesic of submanifold in E^m is a geodesic of the submanifold as a W-curve in E^m .*

Using Theorem 2.2, Theorem 7.23 ([1], pp 186-187) and the classification theorem of in [4], we have

Theorem 3.3([6]). *Let M be a compact connected surface in E^4. Then M satisfies $(*)$ if and only if M is a standard torus $S^1(a) \times S^1(b) \subset E^4$, a standard sphere $\subset E^3$ or a Blaschke surface at $o \in E^4$ diffeomorphic to RP^2 of the form*

$$\mathbf{X}(s,\theta) = (\frac{1}{\kappa}\sin\kappa s\cos\theta, \frac{1}{\kappa}\sin\kappa s\sin\theta, \frac{1}{\kappa}(1-\cos\kappa s)\cos 2\theta, \frac{1}{\kappa}(1-\cos\kappa s)\sin 2\theta),$$

where κ is the Frenet curvature of geodesics through o .

In this case, every geodesic through o on M is of rank 2 (See [6]).

Now we are going to study a complete connected surfaces in E^4.

Suppose that there exists a geodesic γ through o whose rank is 4. But it was proved in [6] that γ is a non-periodic curve and in this case, M is a standard torus $S^1(a) \times S^1(b) \subset E^4$.

We now assume that every geodesic through o has the rank less than or equal to 3. For the geodesic coordinate neighborhood system (s,θ) around o, we may write the isometric immersion $\mathbf{X} : M \longrightarrow E^4$ as

$$\begin{aligned} \mathbf{X}(s,\theta) &= r_1(\theta)(\cos\alpha(\theta)s - 1)f_1(\theta) \\ &\quad + r_1(\theta)\sin\alpha(\theta)s f_2(\theta) + \beta(\theta)s f_3(\theta), \end{aligned} \tag{3.1}$$

where $\mathbf{X}(0,\theta) = o$, $r_1(\theta)$ and $\beta(\theta)$ are nonnegative functions, $\alpha(\theta)$ a positive valued function on $(0,2\pi)$, $f_1(\theta), f_2(\theta)$ and $f_3(\theta)$ orthonormal vectors in E^4 depending on θ in $(0,2\pi)$. If $r_1(\theta) = 0$ for some $\theta \in (0,2\pi)$, then $\mathbf{X}(s,\theta)$ is a straight line. If $\beta(\theta) = 0$ for some $\theta \in (0,2\pi)$, then $\mathbf{X}(s,\theta)$ is a circle. For $r_1(\theta) > 0$ and $\beta(\theta) > 0$, $\mathbf{X}(s,\theta)$ is a helix for such a fixed θ.

Now, let $f_4(\theta)$ be a unit vector in E^4 such that $\{f_1(\theta), f_2(\theta), f_3(\theta),\ f_4(\theta)\}$ forms an orthonormal basis for E^4.

For each $\theta \in (0,2\pi)$, we get

$$\begin{aligned} \mathbf{X}_*(\frac{\partial}{\partial s}) &= -r_1(\theta)\alpha(\theta)\sin\alpha(\theta)s f_1(\theta) \\ &\quad + r_1(\theta)\alpha(\theta)\cos\alpha(\theta)s f_2(\theta) + \beta(\theta)f_3(\theta), \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\mathbf{X}_*(\frac{\partial}{\partial \theta}) \\ &= r_1(\theta)(\cos\alpha(\theta)s - 1)f_1'(\theta) + r_1(\theta)\sin\alpha(\theta)s f_2'(\theta) \\ &+ \{r_1'(\theta)(\cos\alpha(\theta)s - 1) - r_1(\theta)\alpha'(\theta)s\sin\alpha(\theta)s\}f_1(\theta) \\ &+ \{r_1'(\theta)\sin\alpha(\theta)s + r_1(\theta)\alpha'(\theta)s\cos\alpha(\theta)s\}f_2(\theta) \\ &+ \beta(\theta)s f_3'(\theta) + \beta'(\theta)s f_3(\theta), \end{aligned} \tag{3.3}$$

$$\begin{aligned} \tilde{\nabla}_{\mathbf{X}_*(\frac{\partial}{\partial s})}\mathbf{X}_*(\frac{\partial}{\partial s}) &= -r_1(\theta)\alpha^2(\theta)\cos\alpha(\theta)s f_1(\theta) \\ &\quad - r_1(\theta)\alpha^2(\theta)\sin\alpha(\theta)s f_2(\theta). \end{aligned} \tag{3.4}$$

For a fixed θ, $\mathbf{X}(s,\theta)$ is a geodesic and thus

$$< \tilde{\nabla}_{\mathbf{X}_*(\frac{\partial}{\partial s})}\mathbf{X}_*(\tfrac{\partial}{\partial s}), \mathbf{X}_*(\tfrac{\partial}{\partial \theta}) >= 0,$$

which gives

$$\begin{aligned}
(3.5)\qquad & - r_1(\theta)r_1'(\theta)\alpha^2(\theta) \\
& + r_1(\theta)r_1'(\theta)\alpha^2(\theta)\cos\alpha(\theta)s \\
& + r_1^2(\theta)\alpha^2(\theta) < f_2(\theta), f_1'(\theta) > \sin\alpha(\theta)s \\
& - r_1(\theta)\alpha^2(\theta)\beta(\theta) < f_1(\theta), f_3'(\theta) > s\cos\alpha(\theta)s \\
& - r_1(\theta)\alpha^2(\theta)\beta(\theta) < f_2(\theta), f_3'(\theta) > s\sin\alpha(\theta)s \\
& = 0.
\end{aligned}$$

By linearly independence of functions in (3.5), we obtain

$$(3.6)\qquad r_1(\theta) = const.$$

$$\begin{aligned}
(3.7)\qquad & r_1(\theta) < f_2(\theta), f_1'(\theta) >= r_1(\theta)\beta(\theta) < f_1(\theta), f_3'(\theta) > \\
& = r_1(\theta)\beta(\theta) < f_2(\theta), f_3'(\theta) >= 0.
\end{aligned}$$

If $r_1(\theta) = 0$, then the surface is a 2-plane E^2.

We now assume that $r_1(\theta) \neq 0$, which is denoted by r_1.

Case 1) Suppose $f_1'(\theta) = f_2'(\theta) = f_3'(\theta) = 0$ for all $\theta \in (0, 2\pi)$. We may assume that $f_1(\theta) = (1,0,0,0)$, $f_2(\theta) = (0,1,0,0)$, $f_3(\theta) = (0,0,1,0)$ and $f_4(\theta) = (0,0,0,1)$. In this case, $\mathbf{X}(s,\theta)$ can be written as

$$\mathbf{X}(s,\theta) = (r_1(\cos\alpha(\theta)s - 1),\ \ r_1\sin\alpha(\theta)s,\ \ \beta(\theta)s,\ \ 0),$$

which is a circular cylinder in E^3.

Case 2) Let $J = \{\theta \in (0, 2\pi) | f_1'(\theta) \neq 0\} \cup \{\theta \in (0, 2\pi) | f_2'(\theta) \neq 0\} \cup \{\theta \in (0, 2\pi) | f_3'(\theta) \neq 0\}$. Suppose $J \neq \phi$. Let $\beta(\theta_0) \neq 0$ for some $\theta_0 \in J$. Since β is continuous and J is an open subset of $(0, 2\pi)$, there is an open subset J_1 containing θ_0 such that $J_1 \subset J$. Let C_1 be a connected component in J_1. From (3.7), we have

$$< f_1(\theta), f_2'(\theta) >=< f_1(\theta), f_3'(\theta) >=< f_2(\theta), f_3'(\theta) >= 0$$

on C_1. Then, we get

$$f_i'(\theta) = \lambda_i(\theta) f_4(\theta), \qquad f_4(\theta) = -\sum_{i=1}^{3} \lambda_i(\theta) f_i(\theta) \tag{3.8}$$

for i=1,2 and 3, in other words,

$$\begin{pmatrix} f_1'(\theta) \\ f_2'(\theta) \\ f_3'(\theta) \\ f_4'(\theta) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \lambda_1(\theta) \\ 0 & 0 & 0 & \lambda_2(\theta) \\ 0 & 0 & 0 & \lambda_3(\theta) \\ -\lambda_1(\theta) & -\lambda_2(\theta) & -\lambda_3(\theta) & 0 \end{pmatrix} \begin{pmatrix} f_1(\theta) \\ f_2(\theta) \\ f_3(\theta) \\ f_4(\theta) \end{pmatrix}$$

where the λ_i's are some functions on C_1.

We now compute $\tilde{\nabla}_{\mathbf{X}_*(\frac{\partial}{\partial s})} \tilde{\nabla}_{\mathbf{X}_*(\frac{\partial}{\partial s})} \mathbf{X}_*(\frac{\partial}{\partial \theta})$ on C_1.

$$\begin{aligned} (3.9) \qquad & \tilde{\nabla}_{\mathbf{X}_*(\frac{\partial}{\partial s})} \tilde{\nabla}_{\mathbf{X}_*(\frac{\partial}{\partial s})} \mathbf{X}_*(\frac{\partial}{\partial \theta}) \\ &= \{-2r_1 \alpha(\theta)\alpha'(\theta) \cos\alpha(\theta)s + r_1 \alpha^2(\theta)\alpha'(\theta)s \sin\alpha(\theta)s\} f_1(\theta) \\ &\quad - \{2r_1 \alpha(\theta)\alpha'(\theta) \sin\alpha(\theta)s + r_1 \alpha^2(\theta)\alpha'(\theta)s \cos\alpha(\theta)s\} f_2(\theta) \\ &\quad - r_1 \alpha^2(\theta) \cos\alpha(\theta) s f_1'(\theta) - r_1 \alpha^2(\theta) \sin\alpha(\theta) s f_2'(\theta). \end{aligned}$$

We denote $\mathbf{X}_*(\frac{\partial}{\partial s})$ by T and $\mathbf{X}_*(\frac{\partial}{\partial \theta})$ by Q. If we identify T with $\frac{\partial}{\partial s}$ and Q with $\frac{\partial}{\partial \theta}$, then we have

$$
\begin{aligned}
(3.10)\qquad & \tilde{\nabla}_T\tilde{\nabla}_T Q \\
&= \tilde{\nabla}_T(\nabla_T Q + h(T,Q)) \\
&= \nabla_T\nabla_T Q + h(T,\nabla_T Q) - A_{h(T,Q)}T + \nabla_T^{\perp} h(T,Q) \\
&= R(T,Q)T + \nabla_Q\nabla_T T + \nabla_{[T,Q]}T \\
&\quad + h(T,\nabla_T Q) - A_{h(T,Q)}T + \nabla_T^{\perp} h(T,Q) \\
&= R(T,Q)T + \nabla_Q\nabla_T T + \nabla_{[T,Q]}T \\
&\quad + h(T,\nabla_Q T) - A_{h(T,Q)}T + (\bar{\nabla}_T h)(T,Q) \\
&\quad + h(\nabla_T T, Q) + h(T,\nabla_Q T).
\end{aligned}
$$

Together with (3.9) and (3.10), we get $\tilde{\nabla}_T\tilde{\nabla}_T Q \longrightarrow 0$ as $s \longrightarrow 0$ and hence $2r_1\alpha'(\theta)\alpha(\theta)f_1(\theta) + r_1\alpha^2(\theta)f_1'(\theta) = 0$ on C_1. Then, (3.8) implies

$$
\begin{aligned}
(3.11)\qquad & 2r_1\alpha(\theta)\alpha'(\theta)f_1(\theta) + r_1\alpha^2(\theta)f_1'(\theta) \\
&= 2r_1\alpha(\theta)\alpha'(\theta)f_1(\theta) + r_1\alpha^2(\theta)\lambda_1(\theta)f_4(\theta) \\
&= 0.
\end{aligned}
$$

By linearly independence of basis in (3.11), we get

$$
(3.12)\qquad \alpha'(\theta) = 0 \quad \text{and} \quad \lambda_1(\theta) = 0
$$

for all $\theta \in C_1$. It follows that α is constant on C_1. And, $\| \mathbf{X}_*(\frac{\partial}{\partial s}) \|^2 = 1$ gives

$$
(3.13)\qquad r_1^2\alpha^2(\theta) + \beta^2(\theta) = 1.
$$

Thus β is constant on the component C_1. Hence, α, β and r_1 are constant on C_1. Then, the curvatures κ_1 and κ_2 are automatically constant on C_1. By continuity,

C_1 must be $(0,2\pi)$. Therefore, every geodesic through o is of rank 3 and has the same constant Frenet curvatures, that is, M is helical at o. By Lemma 2.1, every geodesic through o must be planar. So this case cannot occur.

Consequently, we have

Theorem 3.4. *Let M be a complete connected surface in E^4. Then M satisfies $(*)$ if and only if M is a plane E^2 in E^3, a standard sphere $\subset E^3$, a circular cylinder in E^3, a standard torus $S^1(a) \times S^1(b) \subset E^4$ or a Blaschke surface at $o \subset E^4$ diffeomorphic to RP^2 of the form*

$$\mathbf{X}(s,\theta) = \Big(\frac{1}{\kappa}\sin\kappa s\cos\theta, \frac{1}{\kappa}\sin\kappa s\sin\theta, \frac{1}{\kappa}(1-\cos\kappa s)\cos 2\theta,$$
$$\frac{1}{\kappa}(1-\cos\kappa s)\sin 2\theta\Big),$$

where κ is the Frenet curvature of geodesics through o.

Using this theorem, we can have

Theorem 3.5. *Let M be a complete connected surface in E^4. Then every geodesic on M is a W-curve if and only if M is a plane E^2 in E^3, a standard sphere $\subset E^3$, a circular cylinder in E^3 or a standard torus $S^1(a) \times S^1(b) \subset E^4$.*

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